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Characterization of Group of Invertible Elements of Six Index Zero Completely Primary Finite Rings of Characteristic p

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ABSTRACT: The study of finite extension of Galois rings in the recent past have given rise to commutative completely primary finite rings that have attracted much attention as they have yielded important results towards classification of finite rings into well-known structures. In this paper, we give a construction of a class of completely primary finite ring R of characteristic p whose subsets of zero divisors $Z(R)$ satisfy the condition $(Z(R))^6=(0); (Z(R))^5 \neq (0)$. The ring R is constructed over its subring $R_0 = GR(p^r, p)$ as an idealization of the R_0 - modules. A thorough determination and classification of the structure of the group of invertible elements using fundamental theorem of finitely generated abelian groups is given.

Keywords: Completely Primary Finite Rings, Group of Invertible Elements, Six Index Zero

1. INTRODUCTION

The significant contribution of completely primary finite rings with identity $1 \neq 0$ in the classification of finite rings cannot be underscored in the field of algebraic structures. Since every finite ring is expressible as a direct sum of completely primary finite rings, several authors have constructed completely primary finite rings whose Jacobson radical $J(R)$ yield particular structures. The results due to Raghavendran [1] on the structure of the unit groups of the Galois ring $R_0 = GR(p^{kr}, p^k)$ stirred up the research on the classification of unit groups in the finite extension of these rings. In [1,6,7], the authors have given a construction of completely primary finite rings satisfying $(J(R))^2 = (0)$ whose structures are well classified for characteristic p and p^2. Chikunji [2], provided a construction of a class of completely primary finite rings R whose Jacobson radical satisfy $(J(R))^3 = (0)$ and $(J(R))^2 \neq (0)$. The structure of the group of units of these rings for characteristic p, p^2 and p^3 were later determined and classified by Chikunji in [3,4,5]. In [8], Oduor and Ojiema gave a construction of completely primary finite rings R satisfying the conditions $(J(R))^4 = (0); (J(R))^3 \neq (0)$ and determined the structure of the unit groups of these rings for characteristic p, p^2, p^3 and p^4. Later in [10], the authors provided a construction of completely primary rings whose Jacobson radical $J(R)$ satisfies the condition $(J(R))^5 = (0); (J(R))^4 \neq (0)$. The classification of the unit groups of these rings for characteristic p, p^2, p^3, p^4 and p^5 based on variant orders of first and second Galois ring module generators were outlined in [11,12]. Authors including Raghavendran [9] and Wilson [14] having extensively studied finite rings, although the classification of these rings into well-known structures is incomplete despite the fact that the tools necessary for describing the completely primary finite rings had been there for some time. This has triggered the construction of a specific class of the general construction of completely primary finite rings in which $(Z(R))^6 = (0); (Z(R))^5 \neq (0)$ and classification of its group of invertible elements for the characteristic p. This construction involved the idealization of R_0 modules whose choice was based on Wilson [14].

Throughout this paper, R shall denote a completely primary finite ring, $Z(R)$ the set of zero divisors of R, $J(R)$ Jacobson radical, and $U(R)$ multiplicative group of units of R.

The rest of this paper is organized as follows; in section 2, we give preliminary results and definitions that are important in this paper. In section 3, we provide a specific construction of the general construction of completely primary finite rings with six nilpotent subsets of zero divisors of characteristic p and investigate as well as determine its structure of the group of invertible elements. This is supplemented using an illustrative example at the end of the main results. Finally in section 4, we give a conclusion of this research and some areas in which this research may focus on in future.

2. PRELIMINARIES

The following definitions and theorems are fundamental and their proofs can be obtained from the cited references.

Theorem 2.1 ([9], Theorem 2): Let R be a finite ring with multiplicative identity $1 \neq 0$, whose set of zero divisors form an additive group $J(R)$. Then:

(i) $J(R)$ is the Jacobson radical of R;

(ii) $|R| = p^{kr}$, and $|J(R)| = p^{(k-1)r}$, for some prime p and some positive integer k, r;

(iii) $(J(R))^n = (0)$;

(iv) The characteristic of the ring R is p^n for some integer n with $1 \leq n \leq k$ and

(v) If the characteristic is p^k, then R will be commutative.

Theorem 2.2 ([14], Proposition 2.2): Let R be a completely primary finite ring of characteristic p^k with radical J such that $R/J \cong GR(F)$. Then there exists an independent generating set $u_1,\ldots,u_h \in J$ of U as an R_0 module such that each R_0u_i, $i=1,\ldots,h$ is an R_0 submodule of U and $R = R_0 \oplus U = R_0 \oplus R_0u_1 \oplus \cdots \oplus R_0u_h$.

Theorem 2.3 ([13]): Let $R = GR\left(p^{kr}, p^k\right)$ be the Galois ring of characteristic p^k and of order p^{kr}, that is $GR(p^{kr}, p^k) = \frac{\mathbb{Z}_{p^k}[x]}{(f)}$ where $f \in \mathbb{Z}_{p^k}[x]$ is a monic polynomial of degree r whose image in $p[x]$ is irreducible. Then R has a coefficient subring R_0 of the form $GR\left(p^{kr}, p^k\right)$ which is clearly a maximal Galois subring of R. The $Char\ R_0 = Char\ R$ and R_0/pR_0 same as $R/J(R)$. Moreover R_0 is a Galois ring of the form $GR\left(p^{kr}, p^k\right)$, $1 \le k \le n$.

Definition 2.4 ([1]): A completely primary finite ring is a ring in which the set of all zero divisors form a unique maximal ideal. For more information on these rings, the reader is referred to [9].

Definition 2.5 ([6], section 4): Let R be a finite ring. Then there is no distinction between left and right zero divisors(units) and every element in R is either a zero divisor or a unit.

3. MAIN RESULTS

Construction 3.1: We give a specific case of the general construction of a completely primary finite ring with six nilpotent subsets of zero divisors.

For a prime integer p and a positive integer r, let R_0 be a Galois ring of order p^r and of characteristic p. Let $u_i, v_j, w_k, x_l, y_m \in Z(R)$ for $1 \le i \le d$, $1 \le j \le e$, $1 \le k \le f$, $1 \le l \le g$, $1 \le m \le h$ and U, V, W, X, Y be finitely generated R_0 - modules by the generators $\{u_i\}$, $\{v_j\}$, $\{w_k\}$, $\{x_l\}$, and $\{y_m\}$ respectively such that $R = R_0 \oplus U \oplus V \oplus W \oplus X \oplus Y$ is an additive abelian group. Additionally, the generators satisfy the conditions $pu_i = 0$, $pv_j = 0$, $pw_k = 0$, $px_l = 0$, $py_m = 0$ and $u_iu_{i'}u_{i''}u_{i'''}u_{i''''}u_{i'''''} = 0$.

Let $r_0 + \sum_{i=1}^{d} r_iu_i + \sum_{j=1}^{e} s_jv_j + \sum_{k=1}^{f} t_kw_k + \sum_{l=1}^{g} b_lx_l + \sum_{m=1}^{h} z_my_m$ and

$r_0' + \sum_{i=1}^{d} r_i'u_i + \sum_{j=1}^{e} s_j'v_j + \sum_{k=1}^{f} t_k'w_k + \sum_{l=1}^{g} b_l'x_l + \sum_{m=1}^{h} z_m'y_m$ be any two elements of R. Then on R, define multiplication by

$$\left(r_0 + \sum_{i=1}^{d} r_iu_i + \sum_{j=1}^{e} s_jv_j + \sum_{k=1}^{f} t_kw_k + \sum_{l=1}^{g} b_lx_l + \sum_{m=1}^{h} z_my_m\right)\left(r_0' + \sum_{i=1}^{d} r_i'u_i + \sum_{j=1}^{e} s_j'v_j + \sum_{k=1}^{f} t_k'w_k + \sum_{l=1}^{g} b_l'x_l + \sum_{m=1}^{h} z_m'y_m\right)$$

$$=\Big(r_0r'_0+\sum_{i=1}^{d}\big[r_0\left(r'_i\right)+r_i\left(r'_0\right)\big]u_i+\sum_{j=1}^{e}\Big[r_0\left(s'_j\right)+s_j\left(r'_0\right)+\sum_{i}r_i\left(r'_i\right)\Big]v_j+\sum_{k=1}^{f}\Big[r_0\left(t'_k\right)+t_k\left(r'_0\right)+\sum_{i,j}r_i\left(s'_j\right)+$$

$$s_j\left(r'_i\right)\Big]w_k+\sum_{l=1}^{g}\Big[r_0\left(b'_l\right)+b_l\left(r'_0\right)+\sum_{i,k}r_i\left(t'_k\right)+t_k\left(r'_i\right)+\sum_{j}s_j\left(s'_j\right)\Big]x_l+\sum_{m=1}^{h}\Big[r_0\left(z'_m\right)+z_m\left(r'_0\right)+$$

$$\sum_{i,l}r_i\left(b'_l\right)+b_l\left(r'_i\right)+\sum_{j,k}s_j\left(t'_k\right)+t_k\left(s'_j\right)\Big]y_m\Big)$$

We verify that the given multiplication turns R into a ring with identity $(1,0,0,0,0,0)$.

First, we demonstrate that this multiplication is associative. Suppose

$$\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right),\left(r'_0+\sum_{i=1}^{d}r'_iu_i+\sum_{j=1}^{e}s'_jv_j+\sum_{k=1}^{f}t'_kw_k+\sum_{l=1}^{g}b'_lx_l+\sum_{m=1}^{h}z'_my_m\right),$$

$\left(r''_0+\sum_{i=1}^{d}r''_iu_i+\sum_{j=1}^{e}s''_jv_j+\sum_{k=1}^{f}t''_kw_k+\sum_{l=1}^{g}b''_lx_l+\sum_{m=1}^{h}z''_my_m\right)$ are elements of R where $r_0,r'_0,r''_0\in R_0$, then

$$\left[\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)\left(r'_0+\sum_{i=1}^{d}r'_iu_i+\sum_{j=1}^{e}s'_jv_j+\sum_{k=1}^{f}t'_kw_k+\sum_{l=1}^{g}b'_lx_l+\sum_{m=1}^{h}z'_my_m\right)\right]$$

$$\left(r''_0+\sum_{i=1}^{d}r''_iu_i+\sum_{j=1}^{e}s''_jv_j+\sum_{k=1}^{f}t''_kw_k+\sum_{l=1}^{g}b''_lx_l+\sum_{m=1}^{h}z''_my_m\right)$$

$$=\Big[r_0r'_0+\sum_{i=1}^{d}\big[r_0\left(r'_i\right)+r_i\left(r'_0\right)\big]u_i+\sum_{j=1}^{e}\Big[r_0\left(s'_j\right)+s_j\left(r'_0\right)+\sum_{i}r_i\left(r'_i\right)\Big]v_j+\sum_{k=1}^{f}\Big[r_0\left(t'_k\right)+t_k\left(r'_0\right)+\sum_{i,j}r_i\left(s'_j\right)+$$

$$s_j\left(r'_i\right)\Big]w_k+\sum_{l=1}^{g}\Big[r_0\left(b'_l\right)+b_l\left(r'_0\right)+\sum_{i,k}r_i\left(t'_k\right)+t_k\left(r'_i\right)+\sum_{j}s_j\left(s'_j\right)\Big]x_l+\sum_{m=1}^{h}\Big[r_0\left(z'_m\right)+z_m\left(r'_0\right)+$$

$$\sum_{i,l}r_i\left(b'_l\right)+b_l\left(r'_i\right)+\sum_{j,k}s_j\left(t'_k\right)+t_k\left(s'_j\right)\Big]y_m\Big]\left(r''_0+\sum_{i=1}^{d}r''_iu_i+\sum_{j=1}^{e}s''_jv_j+\sum_{k=1}^{f}t''_kw_k+\sum_{l=1}^{g}b''_lx_l+\sum_{m=1}^{h}z''_my_m\right)$$

$$=\left(r_0r'_0\right)r''_0+\sum_{i=1}^{d}\Big[\left(r_0r'_0\right)r''_i+\left(r_0r'_i\right)r''_0+\left(r_ir'_0\right)r''_0\Big]u_i+\sum_{j=1}^{e}\Big[\left(r_0r'_0\right)s''_j+\left(r_0s'_j\right)r''_0+\left(s_jr'_0\right)r''_0+$$

$$\sum_{i}\left(r_ir'_i\right)r''_0+\left(r_0r'_i\right)r''_i+\left(r_ir'_0\right)r''_i\Big]v_j+\sum_{k=1}^{f}\Big[\left(r_0r'_0\right)t''_k+\left(r_0t'_k\right)r''_0+\left(t_kr'_0\right)r''_0+\sum_{i,j}\left(r_is'_j\right)r''_0+\left(s_jr'_i\right)r''_0+$$

$$\left(r_0r'_i\right)s''_j+\left(r_ir'_0\right)s''_j+\left(r_0s'_j\right)r''_i+\left(s_jr'_0\right)r''_i+\sum_{i}\left(r_ir'_i\right)r''_i\Big]w_k+\sum_{l=1}^{g}\Big[\left(r_0r'_0\right)b''_l+\left(r_0b'_l\right)r''_0+\left(b_lr'_0\right)r''_0+$$

$$\sum_{i,k}\left(r_it'_k\right)r''_0+\left(t_kr'_i\right)r''_0+\left(r_0r'_i\right)t''_k+\left(r_ir'_0\right)t''_k+\left(r_0t'_k\right)r''_i+\left(t_kr'_0\right)r''_i+\sum_{j}\left(s_js'_j\right)r''_0+\left(r_0s'_j\right)s''_j+\left(s_jr'_0\right)s''_j$$

$$\sum_{i,j}\left(r_is'_j\right)r''_i+\left(s_jr'_i\right)r''_i+\left(r_ir'_i\right)s''_j\Big]x_l+\sum_{m=1}^{h}\Big[\left(r_0r'_0\right)z''_m+\left(r_0z'_m\right)r''_0+\left(z_mr'_0\right)r''_0+\sum_{i,l}\left(r_ib'_l\right)r''_0+\left(b_lr'_i\right)r''_0+$$

$$\left(r_0r'_i\right)b''_l+\left(r_ir'_0\right)b''_l+\left(r_0b'_l\right)r''_i+\left(b_lr'_0\right)r''_i+\sum_{j,k}\left(s_jt'_k\right)r''_0+\left(t_ks'_j\right)r''_0+\left(r_0s'_j\right)t''_k+\left(s_jr'_0\right)t''_k+\left(r_0t'_k\right)s''_j+$$

$$\left(t_kr'_0\right)s''_j+\sum_{i,k}\left(r_it'_k\right)r''_i+\left(t_kr'_i\right)r''_i+\left(r_ir'_i\right)t''_k+\sum_{i,j}\left(s_js'_j\right)r''_i+\left(r_is'_j\right)s''_j+\left(s_jr'_i\right)s''_j\Big]y_m$$

$$
\begin{aligned}
&= r_0\left(r_0'r_0''\right) + \sum_{i=1}^{d}\Big[r_0\left(r_i'r_i''\right) + r_0\left(r_i'r_0''\right) + r_i\left(r_0'r_0''\right)\Big]u_i + \sum_{j=1}^{e}\Big[r_0\left(r_0's_j''\right) + r_0\left(s_j'r_0''\right) + s_j\left(r_0'r_0''\right) + \\
&\quad \sum_{i} r_i\left(r_i'r_0''\right) + r_0\left(r_i'r_i''\right) + r_i\left(r_0'r_i''\right)\Big]v_j + \sum_{k=1}^{f}\Big[r_0\left(r_0't_k''\right) + r_0\left(t_k'r_0''\right) + t_k\left(r_0'r_0''\right) + \sum_{i,j}\Big(r_i\left(s_j'r_0''\right) + s_j\left(r_i'r_0''\right) + \\
&\quad r_0\left(r_i's_j''\right) + r_i\left(r_0's_j''\right) + r_0\left(s_j'r_i''\right) + s_j\left(r_0'r_i''\right) + \sum_{i} r_i\left(r_i'r_i''\right)\Big]w_k + \sum_{l=1}^{g}\Big[r_0\left(r_0'b_l''\right) + r_0\left(b_l'r_0''\right) + b_l\left(r_0'r_0''\right) + \\
&\quad \sum_{i,k} r_i\left(t_k'r_0''\right) + t_k\left(r_i'r_0''\right) + r_0\left(r_i't_k''\right) + r_i\left(r_0't_k''\right) + r_0\left(t_k'r_i''\right) + t_k\left(r_0'r_i''\right) + \sum_{j} s_j\left(s_j'r_0''\right) + r_0\left(s_j's_j''\right) + s_j\left(r_0's_j''\right) \\
&\quad \sum_{i,j} r_i\left(s_j'r_i''\right) + s_j\left(r_i'r_i''\right) + r_i\left(r_i's_j''\right)\Big]x_l + \sum_{m=1}^{h}\Big[r_0\left(r_0'z_m''\right) + r_0\left(z_m'r_0''\right) + z_m\left(r_0'r_0''\right) + \sum_{i,l} r_i\left(b_l'r_0''\right) + b_l\left(r_i'r_0''\right) + \\
&\quad r_0\left(r_i'b_l''\right) + r_i\left(r_0'b_l''\right) + r_0\left(b_l'r_i''\right) + b_l\left(r_0'r_i''\right) + \sum_{j,k} s_j\left(t_k'r_0''\right) + t_k\left(s_j'r_0''\right) + r_0\left(s_j't_k''\right) + s_j\left(r_0't_k''\right) + r_0\left(t_k's_j''\right) + \\
&\quad t_k\left(r_0's_j''\right) + \sum_{i,k} r_i\left(t_k'r_i''\right) + t_k\left(r_i'r_i''\right) + r_i\left(r_i't_k''\right) + \sum_{i,j} s_j\left(s_j'r_i''\right) + r_i\left(s_j's_j''\right) + s_j\left(r_i's_j''\right)\Big]y_m \\
&= \left(r_0 + \sum_{i=1}^{d} r_i u_i + \sum_{j=1}^{e} s_j v_j + \sum_{k=1}^{f} t_k w_k + \sum_{l=1}^{g} b_l x_l + \sum_{m=1}^{h} z_m y_m\right)\Bigg[r_0'r_0'' + \sum_{i=1}^{d}\Big[r_0'\left(r_i''\right) + r_i'\left(r_0''\right)\Big]u_i + \sum_{j=1}^{e}\Big[r_0'\left(s_j''\right) + \\
&\quad s_j'\left(r_0''\right) + \sum_{i} r_i'\left(r_i''\right)\Big]v_j + \sum_{k=1}^{f}\Big[r_0'\left(t_k''\right) + t_k'\left(r_0''\right) + \sum_{i,j} r_i'\left(s_j''\right) + s_j'\left(r_i''\right)\Big]w_k + \sum_{l=1}^{g}\Big[r_0'\left(b_l''\right) + b_l'\left(r_0''\right) + \\
&\quad \sum_{i,k} r_i'\left(t_k''\right) + t_k'\left(r_i''\right) + \sum_{j} s_j'\left(s_j''\right)\Big]x_l + \sum_{m=1}^{h}\Big[r_0'\left(z_m''\right) + z_m'\left(r_0''\right) + \sum_{i,l} r_i'\left(b_l''\right) + b_l'\left(r_i''\right) + \sum_{j,k} s_j'\left(t_k''\right) + t_k'\left(s_j''\right)\Big]y_m\Bigg] \\
&= \left(r_0 + \sum_{i=1}^{d} r_i u_i + \sum_{j=1}^{e} s_j v_j + \sum_{k=1}^{f} t_k w_k + \sum_{l=1}^{g} b_l x_l + \sum_{m=1}^{h} z_m y_m\right)\Bigg[\Bigg(r_0' + \sum_{i=1}^{d} r_i' u_i + \sum_{j=1}^{e} s_j' v_j + \sum_{k=1}^{f} t_k' w_k + \sum_{l=1}^{g} b_l' x_l + \\
&\quad \sum_{m=1}^{h} z_m' y_m\Bigg)\left(r_0'' + \sum_{i=1}^{d} r_i'' u_i + \sum_{j=1}^{e} s_j'' v_j + \sum_{k=1}^{f} t_k'' w_k + \sum_{l=1}^{g} b_l'' x_l + \sum_{m=1}^{h} z_m'' y_m\right)\Bigg]
\end{aligned}
$$

Secondly, we show that this multiplication is distributive over addition

$$
\begin{aligned}
&\left(r_0 + \sum_{i=1}^{d} r_i u_i + \sum_{j=1}^{e} s_j v_j + \sum_{k=1}^{f} t_k w_k + \sum_{l=1}^{g} b_l x_l + \sum_{m=1}^{h} z_m y_m\right)\Bigg[\Bigg(r_0' + \sum_{i=1}^{d} r_i' u_i + \sum_{j=1}^{e} s_j' v_j + \sum_{k=1}^{f} t_k' w_k + \sum_{l=1}^{g} b_l' x_l + \\
&\quad \sum_{m=1}^{h} z_m' y_m\Bigg) + \left(r_0'' + \sum_{i=1}^{d} r_i'' u_i + \sum_{j=1}^{e} s_j'' v_j + \sum_{k=1}^{f} t_k'' w_k + \sum_{l=1}^{g} b_l'' x_l + \sum_{m=1}^{h} z_m'' y_m\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(r_0 + \sum_{i=1}^{d} r_i u_i + \sum_{j=1}^{e} s_j v_j + \sum_{k=1}^{f} t_k w_k + \sum_{l=1}^{g} b_l x_l + \sum_{m=1}^{h} z_m y_m\right)\Bigg[\left(r_0' + r_0''\right) + \sum_{i=1}^{d}\left[r_i' + r_i''\right]u_i + \sum_{j=1}^{e}\left[s_j' + s_j''\right]v_j \\
&\quad \sum_{k=1}^{f}\left[t_k' + t_k''\right]w_k + \sum_{l=1}^{g}\left[b_l' + b_l''\right]w_k + \sum_{m=1}^{h}\left[z_m' + z_m''\right]y_m\Bigg]
\end{aligned}
$$

$$=\Bigg[r_0r_0^{'}+\sum_{i=1}^{d}\left[r_0\left(r_i^{'}\right)+r_i\left(r_0^{'}\right)\right]u_i+\sum_{j=1}^{e}\left[r_0\left(s_j^{'}\right)+s_j\left(r_0^{'}\right)+\sum_{i}r_i\left(r_i^{'}\right)\right]v_j+\sum_{k=1}^{f}\Bigg[r_0\left(t_k^{'}\right)+t_k\left(r_0^{'}\right)+\sum_{i,j}r_i\left(s_j^{'}\right)+$$

$$s_j\left(r_i^{'}\right)\Bigg]w_k+\sum_{l=1}^{g}\left[r_0\left(b_l^{'}\right)+b_l\left(r_0^{'}\right)+\sum_{i,k}r_i\left(t_k^{'}\right)+t_k\left(r_i^{'}\right)+\sum_{j}s_j\left(s_j^{'}\right)\right]x_l+\sum_{m=1}^{h}\Bigg[r_0\left(z_m^{'}\right)+z_m\left(r_0^{'}\right)+$$

$$\sum_{i,l}r_i\left(b_l^{'}\right)+b_l\left(r_i^{'}\right)+\sum_{j,k}s_j\left(t_k^{'}\right)+t_k\left(s_j^{'}\right)\Bigg]y_m\Bigg]$$

$$+$$

$$\Bigg[r_0r_0^{"}+\sum_{i=1}^{d}\left[r_0\left(r_i^{"}\right)+r_i\left(r_0^{"}\right)\right]u_i+\sum_{j=1}^{e}\left[r_0\left(s_j^{"}\right)+s_j\left(r_0^{"}\right)+\sum_{i}r_i\left(r_i^{"}\right)\right]v_j+\sum_{k=1}^{f}\Bigg[r_0\left(t_k^{"}\right)+t_k\left(r_0^{"}\right)+$$

$$\sum_{i,j}r_i\left(s_j^{"}\right)+s_j\left(r_i^{"}\right)\Bigg]w_k+\sum_{l=1}^{g}\left[r_0\left(b_l^{"}\right)+b_l\left(r_0^{"}\right)+\sum_{i,k}r_i\left(t_k^{"}\right)+t_k\left(r_i^{"}\right)+\sum_{j}s_j\left(s_j^{"}\right)\right]x_l+\sum_{m=1}^{h}\Bigg[r_0\left(z_m^{"}\right)+z_m\left(r_0^{"}\right)+$$

$$\sum_{i,l}r_i\left(b_l^{"}\right)+b_l\left(r_i^{"}\right)+\sum_{j,k}s_j\left(t_k^{"}\right)+t_k\left(s_j^{"}\right)\Bigg]y_m\Bigg]$$

$$=\left[\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)\left(r_0^{'}+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m\right)\right]$$

$$+$$

$$\left[\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)\left(r_0^{"}+\sum_{i=1}^{d}r_i^{"}u_i+\sum_{j=1}^{e}s_j^{"}v_j+\sum_{k=1}^{f}t_k^{"}w_k+\sum_{l=1}^{g}b_l^{"}x_l+\sum_{m=1}^{h}z_m^{"}y_m\right)\right]$$

Similarly, it can be shown that

$$\left[\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)+\left(r_0^{'}+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m\right)\right]$$

$$\left(r_0^{"}+\sum_{i=1}^{d}r_i^{"}u_i+\sum_{j=1}^{e}s_j^{"}v_j+\sum_{k=1}^{f}t_k^{"}w_k+\sum_{l=1}^{g}b_l^{"}x_l+\sum_{m=1}^{h}z_m^{"}y_m\right)$$

$$=\left[\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)\left(r_0^{"}+\sum_{i=1}^{d}r_i^{"}u_i+\sum_{j=1}^{e}s_j^{"}v_j+\sum_{k=1}^{f}t_k^{"}w_k+\sum_{l=1}^{g}b_l^{"}x_l+\sum_{m=1}^{h}z_m^{"}y_m\right)\right]$$

$$+$$

$$\left[\left(r_0^{'}+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m\right)\left(r_0^{"}+\sum_{i=1}^{d}r_i^{"}u_i+\sum_{j=1}^{e}s_j^{"}v_j+\sum_{k=1}^{f}t_k^{"}w_k+\sum_{l=1}^{g}b_l^{"}x_l+\sum_{m=1}^{h}z_m^{"}y_m\right)\right]$$

Next, we show that $(1,0,0,0,0,0)$ is the identity under this multiplication on R.

Let $\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)\in R,$ then we need to find $\left(r_0^{'}+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m\right)\in R,$ where $r_0, r_0^{'}\in R_0$ such that

$$\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)\left(r_0^{'}+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m\right)=\left(r_0^{'}+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m\right)\left(r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)=r_0^{'}+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m$$

$$\Rightarrow\left(r_0r_0^{'}+\sum_{i=1}^{d}\left[r_0\left(r_i^{'}\right)+r_i\left(r_0^{'}\right)\right]u_i+\sum_{j=1}^{e}\left[r_0\left(s_j^{'}\right)+s_j\left(r_0^{'}\right)+\sum_i r_i\left(r_i^{'}\right)\right]v_j+\sum_{k=1}^{f}\left[r_0\left(t_k^{'}\right)+t_k\left(r_0^{'}\right)+\sum_{i,j}r_i\left(s_j^{'}\right)+s_j\left(r_i^{'}\right)\right]w_k+\sum_{l=1}^{g}\left[r_0\left(b_l^{'}\right)+b_l\left(r_0^{'}\right)+\sum_{i,k}r_i\left(t_k^{'}\right)+t_k\left(r_i^{'}\right)+\sum_j s_j\left(s_j^{'}\right)\right]x_l+\sum_{m=1}^{h}\left[r_0\left(z_m^{'}\right)+z_m\left(r_0^{'}\right)+\sum_{i,l}r_i\left(b_l^{'}\right)+b_l\left(r_i^{'}\right)+\sum_{j,k}s_j\left(t_k^{'}\right)+t_k\left(s_j^{'}\right)\right]y_m\right)=\left(r_0^{'}+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m\right)$$

This implies that $r_0r_0^{'}=r_0^{'}$; $\sum_{i=1}^{d}\left[r_0\left(r_i^{'}\right)+r_i\left(r_0^{'}\right)\right]u_i=\sum_{i=1}^{d}r_i^{'}u_i$;

$$\sum_{j=1}^{e}\left[r_0\left(s_j^{'}\right)+s_j\left(r_0^{'}\right)+\sum_i r_i\left(r_i^{'}\right)\right]v_j=\sum_{j=1}^{e}s_j^{'}v_j;\quad \sum_{k=1}^{f}\left[r_0\left(t_k^{'}\right)+t_k\left(r_0^{'}\right)+\sum_{i,j}r_i\left(s_j^{'}\right)+s_j\left(r_i^{'}\right)\right]w_k=\sum_{k=1}^{f}t_k^{'}w_k;$$

$$\sum_{l=1}^{g}\left[r_0\left(b_l^{'}\right)+b_l\left(r_0^{'}\right)+\sum_{i,k}r_i\left(t_k^{'}\right)+t_k\left(r_i^{'}\right)+\sum_j s_j\left(s_j^{'}\right)\right]x_l=\sum_{l=1}^{g}b_l^{'}x_l;$$

$$\sum_{m=1}^{h}\left[r_0\left(z_m^{'}\right)+z_m\left(r_0^{'}\right)+\sum_{i,l}r_i\left(b_l^{'}\right)+b_l\left(r_i^{'}\right)+\sum_{j,k}s_j\left(t_k^{'}\right)+t_k\left(s_j^{'}\right)\right]y_m=\sum_{m=1}^{h}z_m^{'}y_m;$$

Upon solving the above set of equations, we find that $r_0=1,\ r_i=0,\ s_j=0,\ t_k=0,\ b_l=0,\ z_m=0$.

Showing that $r_0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m=(1,0,0,0,0,0)$. Therefore R is a unital finite ring.

The ring $R=R_0\oplus\sum_{i=1}^{d}R_0u_i\oplus\sum_{j=1}^{e}R_0v_j\oplus\sum_{k=1}^{f}R_0w_k\oplus\sum_{l=1}^{g}R_0x_l\oplus\sum_{m=1}^{h}R_0y_m$ has the set of zero divisors as

$Z(R)=\sum_{i=1}^{d}R_0u_i\oplus\sum_{j=1}^{e}R_0v_j\oplus\sum_{k=1}^{f}R_0w_k\oplus\sum_{l=1}^{g}R_0x_l\oplus\sum_{m=1}^{h}R_0y_m$. It is worth noting that an element in the constructed ring R is either a zero divisor or a unit. From the definition of multiplication, it follows that $R\big(Z(R)\big)=\big(Z(R)\big)R\subseteq Z(R)$ so that $Z(R)$ is an ideal. Suppose there exists another ideal T in the ring R such that $Z(R)$ is contained in T , then by definition 2.5, T contains an invertible element $x\in R$ such that $xx^{-1}=x^{-1}x=1$ so that $T=R$. Thus $Z(R)$ is the only set in the ring R containing all the zero divisors hence the unique maximal ideal in R since any maximal ideal that is distinct from $Z(R)$ contains a unit. Therefore $Z(R)$ coincides with the Jacobson radical $J(R)$ thereby implying that the ring R is completely primary.

Proposition 3.2: The ring R constructed in section 3.1 is completely primary of characteristic p with Jacobson radical $Z(R)$:

$$Z(R)=\sum_{i=1}^{d}R_0u_i\oplus\sum_{j=1}^{e}R_0v_j\oplus\sum_{k=1}^{f}R_0w_k\oplus\sum_{l=1}^{g}R_0x_l\oplus\sum_{m=1}^{h}R_0y_m$$

$$(Z(R))^2=\sum_{j=1}^{e}R_0v_j\oplus(Z(R))^3$$

$$(Z(R))^3=\sum_{k=1}^{f}R_0w_k\oplus(Z(R))^4$$

$$(Z(R))^4=\sum_{l=1}^{g}R_0x_l\oplus(Z(R))^5$$

$$(Z(R))^5=\sum_{m=1}^{h}R_0y_m$$

$$(Z(R))^6=(0)$$

Proof: In order to show that the set of zero divisors is given by $Z(R)=\sum_{i=1}^{d}R_0u_i\oplus\sum_{j=1}^{e}R_0v_j\oplus\sum_{k=1}^{f}R_0w_k\oplus\sum_{l=1}^{g}R_0x_l\oplus\sum_{m=1}^{h}R_0y_m$, we demonstrate that any element not contained in $Z(R)$ is invertible. Let $1+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\notin Z(R)$ be an element in R .

Suppose $1+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m$ is the inverse of

$r_0'+\sum_{i=1}^{d}r_i'u_i+\sum_{j=1}^{e}s_j'v_j+\sum_{k=1}^{f}t_k'w_k+\sum_{l=1}^{g}b_l'x_l+\sum_{m=1}^{h}z_m'y_m$, then

$$\left(1+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)\left(r_0'+\sum_{i=1}^{d}r_i'u_i+\sum_{j=1}^{e}s_j'v_j+\sum_{k=1}^{f}t_k'w_k+\sum_{l=1}^{g}b_l'x_l+\sum_{m=1}^{h}z_m'y_m\right)$$

$$=(1,0,0,0,0,0)$$

$$\Rightarrow r_0'+\sum_{i=1}^{d}\left[r_i'+r_i(r_0')\right]u_i+\sum_{j=1}^{e}\left[s_j'+s_j(r_0')+\sum_i r_i(r_i')\right]v_j+\sum_{k=1}^{f}\left[t_k'+t_k(r_0')+\sum_{i,j}r_i(s_j')+s_j(r_i')\right]w_k+$$
$$\sum_{l=1}^{g}\left[b_l'+b_l(r_0')+\sum_{i,k}r_i(t_k')+t_k(r_i')+\sum_j s_j(s_j')\right]x_l+\sum_{m=1}^{h}\left[z_m'+z_m(r_0')+\sum_{i,l}r_i(b_l')+b_l(r_i')+\right.$$
$$\left.\sum_{j,k}s_j(t_k')+t_k(s_j')\right]y_m=(1,0,0,0,0,0)$$

Giving $r_0^{'}=1;\ \sum_{i=1}^{d}\left[r_i^{'}+r_i\left(r_0^{'}\right)\right]u_i=0;\ \sum_{j=1}^{e}\left[s_j^{'}+s_j\left(r_0^{'}\right)+\sum_i r_i\left(r_i^{'}\right)\right]v_j=0;$

$$\sum_{k=1}^{f}\left[t_k^{'}+t_k\left(r_0^{'}\right)+\sum_{i,j}r_i\left(s_j^{'}\right)+s_j\left(r_i^{'}\right)\right]w_k=0;\ \sum_{l=1}^{g}\left[b_l^{'}+b_l\left(r_0^{'}\right)+\sum_{i,k}r_i\left(t_k^{'}\right)+t_k\left(r_i^{'}\right)+\sum_j s_j\left(s_j^{'}\right)\right]x_l=0;$$

$$\sum_{m=1}^{h}\left[z_m^{'}+z_m\left(r_0^{'}\right)+\sum_{i,l}r_i\left(b_l^{'}\right)+b_l\left(r_i^{'}\right)+\sum_{j,k}s_j\left(t_k^{'}\right)+t_k\left(s_j^{'}\right)\right]y_m=0;$$

This implies that $r_0^{'}=1,\quad r_i^{'}=-r_i,\quad s_j^{'}=-s_j+\sum_i r_i^2,\quad t_k^{'}=-t_k+\sum_{i,j}\left(2r_is_j-\sum_i r_i^3\right),$

$$b_l^{'}=-b_l+\sum_{i,k}\left(2r_it_k-\sum_{i,j}\left(2r_i^2s_j-\sum_i r_i^4\right)\right)-\sum_j\left(-s_j^2+\sum_i s_jr_i^4\right),$$

$$z_m^{'}=-z_m-\sum_{i,l}\left(-2r_ib_l+\sum_{i,k}\left\{2r_i^2t_k-\sum_{i,j}\left(2r_i^3s_j-\sum_i r_i^5\right)\right\}-\sum_j r_i\left(-s_j^2+s_j\sum_i r_i^2\right)\right)-$$
$$\sum_{j,k}\left(-2s_jt_k+\sum_{i,j}\left\{2r_is_j^2-\sum_i s_jr_i^3\right\}+\sum_{i,k}t_kr_i^2\right)$$

So

$$\left(1+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)^{-1}=\left[1-r_iu_i+\left(-s_j+\sum_i r_i^2\right)v_j+\right.$$
$$\left(-t_k+\sum_{i,j}\left(2r_is_j-\sum_i r_i^3\right)\right)w_k+\left(-b_l+\sum_{i,k}\left(2r_it_k-\sum_{i,j}\left(2r_i^2s_j-\sum_i r_i^4\right)\right)-\sum_j\left(-s_j^2+\sum_i s_jr_i^4\right)\right)x_l+$$
$$\left(-z_m-\sum_{i,l}\left(-2r_ib_l+\sum_{i,k}\left\{2r_i^2t_k-\sum_{i,j}\left(2r_i^3s_j-\sum_i r_i^5\right)\right\}-\sum_j r_i\left(-s_j^2+s_j\sum_i r_i^2\right)\right)-\right.$$
$$\left.\left.\sum_{j,k}\left(-2s_jt_k+\sum_{i,j}\left\{2r_is_j^2-\sum_i s_jr_i^3\right\}+\sum_{i,k}t_kr_i^2\right)\right)y_m\right]$$

Thus, all the elements outside $Z(R)$ are invertible.

The product of any two elements in the set $Z(R)$ is given by

$$\left(0+\sum_{i=1}^{d}r_iu_i+\sum_{j=1}^{e}s_jv_j+\sum_{k=1}^{f}t_kw_k+\sum_{l=1}^{g}b_lx_l+\sum_{m=1}^{h}z_my_m\right)\left(0+\sum_{i=1}^{d}r_i^{'}u_i+\sum_{j=1}^{e}s_j^{'}v_j+\sum_{k=1}^{f}t_k^{'}w_k+\sum_{l=1}^{g}b_l^{'}x_l+\sum_{m=1}^{h}z_m^{'}y_m\right)$$
$$=\left(0+0+\sum_{j=1}^{e}\left(\sum_i r_i\left(r_i^{'}\right)\right)v_j+\sum_{k=1}^{f}\left(\sum_{i,j}r_i\left(s_j^{'}\right)+s_j\left(r_i^{'}\right)\right)w_k+\sum_{l=1}^{g}\left(\sum_{i,k}r_i\left(t_k^{'}\right)+t_k\left(r_i^{'}\right)+\sum_j s_j\left(s_j^{'}\right)\right)x_l+\right.$$
$$\left.\sum_{m=1}^{h}\left(\sum_{i,l}r_i\left(b_l^{'}\right)+b_l\left(r_i^{'}\right)+\sum_{j,k}s_j\left(t_k^{'}\right)+t_k\left(s_j^{'}\right)\right)y_m\right)$$

Thus $\left(Z(R)\right)^2=\left(Z(R)\right)\left(Z(R)\right)=V\oplus W\oplus X\oplus Y$.

The product of any three elements in the set $Z(R)$ is given

$$\left(0+0+\sum_{j=1}^{e}\left(\sum_{i} r_i\left(r_i^{'}\right)\right)v_j+\sum_{k=1}^{f}\left(\sum_{i,j} r_i\left(s_j^{'}\right)+s_j\left(r_i^{'}\right)\right)w_k+\sum_{l=1}^{g}\left(\sum_{i,k} r_i\left(t_k^{'}\right)+t_k\left(r_i^{'}\right)+\sum_{j} s_j\left(s_j^{'}\right)\right)x_l+\right.$$

$$\left.\sum_{m=1}^{h}\left(\sum_{i,l} r_i\left(b_l^{'}\right)+b_l\left(r_i^{'}\right)+\sum_{j,k} s_j\left(t_k^{'}\right)+t_k\left(s_j^{'}\right)\right)y_m\right)\left(0+\sum_{i=1}^{d} r_i^{''}u_i+\sum_{j=1}^{e} s_j^{''}v_j+\sum_{k=1}^{f} t_k^{''}w_k+\sum_{l=1}^{g} b_l^{''}x_l+\sum_{m=1}^{h} z_m^{''}y_m\right)$$

$$=\left(0+0+0+\sum_{k=1}^{f}\left[\sum_{i,j}\left(\sum_{i} r_i^{''}\left(r_i r_i^{'}\right)\right)\right]w_k+\sum_{l=1}^{g}\left[\sum_{i,k}\left(\sum_{i,j}\left(r_i^{''}\left(r_i s_j^{'}\right)+r_i^{''}\left(s_j r_i^{'}\right)\right)\right)+\sum_{j} s_j^{''}\left(\sum_{i}\left(r_i r_i^{'}\right)\right)\right]x_l+\right.$$

$$\left.\sum_{m=1}^{h}\left[\sum_{i,l}\left(\sum_{i,k}\left(r_i^{''}\left(r_i t_k^{'}\right)+r_i^{''}\left(t_k r_i^{'}\right)\right)+r_i^{''}\left(\sum_{j}\left(s_j s_j^{'}\right)\right)\right)+\sum_{j,k}\left(\sum_{i,j}\left(s_j^{''}\left(r_i s_j^{'}\right)+s_j^{''}\left(s_j r_i^{'}\right)\right)+t_k^{''}\left(\sum_{i}\left(r_i r_i^{'}\right)\right)\right)\right]y_m\right)$$

Thus $\left(Z\left(R\right)\right)^3=\left(Z\left(R\right)\right)^2\left(Z\left(R\right)\right)=W\oplus X\oplus Y$

The product of any four elements in the set $Z\left(R\right)$ is given

$$\left(0+0+0+\sum_{k=1}^{f}\left[\sum_{i,j}\left(\sum_{i} r_i^{''}\left(r_i r_i^{'}\right)\right)\right]w_k+\sum_{l=1}^{g}\left[\sum_{i,k}\left(\sum_{i,j}\left(r_i^{''}\left(r_i s_j^{'}\right)+r_i^{''}\left(s_j r_i^{'}\right)\right)\right)+\sum_{j} s_j^{''}\left(\sum_{i}\left(r_i r_i^{'}\right)\right)\right]x_l+\right.$$

$$\left.\sum_{m=1}^{h}\left[\sum_{i,l}\left(\sum_{i,k}\left(r_i^{''}\left(r_i t_k^{'}\right)+r_i^{''}\left(t_k r_i^{'}\right)\right)+r_i^{''}\left(\sum_{j}\left(s_j s_j^{'}\right)\right)\right)+\sum_{j,k}\left(\sum_{i,j}\left(s_j^{''}\left(r_i s_j^{'}\right)+s_j^{''}\left(s_j r_i^{'}\right)\right)+t_k^{''}\left(\sum_{i}\left(r_i r_i^{'}\right)\right)\right)\right]y_m\right)$$

$$\left(0+\sum_{i=1}^{d} r_i^{'''}u_i+\sum_{j=1}^{e} s_j^{'''}v_j+\sum_{k=1}^{f} t_k^{'''}w_k+\sum_{l=1}^{g} b_l^{'''}x_l+\sum_{m=1}^{h} z_m^{'''}y_m\right)$$

$$=\left(0+0+0+0+\sum_{l=1}^{g}\left[\sum_{i,k}\left(\sum_{i,j}\left(\sum_{i} r_i^{'''}\left(r_i^{''}r_i r_i^{'}\right)\right)\right)\right]x_l+\sum_{m=1}^{h}\left[\sum_{i,l}\left(\sum_{i,k}\left[\sum_{i,j}\left(r_i^{'''}\left(r_i^{''}r_i s_j^{'}\right)+r_i^{'''}\left(r_i^{''}s_j r_i^{'}\right)\right)\right]\right)+\right.$$

$$\left.\left(\sum_{j} s_j^{''}\left(\sum_{i}\left(r_i^{''}r_i r_i^{'}\right)\right)\right)+\sum_{j,k}\left(s_j^{'''}\left(\sum_{i,j}\left(\sum_{i}\left(r_i^{''}r_i r_i^{'}\right)\right)\right)\right)\right]y_m\right)$$

Thus $\left(Z\left(R\right)\right)^4=\left(Z\left(R\right)\right)^3\left(Z\left(R\right)\right)=X\oplus Y$

The product of any five elements in the set $Z\left(R\right)$ is given

$$\left(0+0+0+0+\sum_{l=1}^{g}\left[\sum_{i,k}\left(\sum_{i,j}\left(\sum_{i} r_i^{'''}\left(r_i^{''}r_i r_i^{'}\right)\right)\right)\right]x_l+\sum_{m=1}^{h}\left[\sum_{i,l}\left(\sum_{i,k}\left[\sum_{i,j}\left(r_i^{'''}\left(r_i^{''}r_i s_j^{'}\right)+r_i^{'''}\left(r_i^{''}s_j r_i^{'}\right)\right)\right]\right)+\right.$$

$$\left.\left(\sum_{j} s_j^{''}\left(\sum_{i}\left(r_i^{''}r_i r_i^{'}\right)\right)\right)+\sum_{j,k}\left(s_j^{'''}\left(\sum_{i,j}\left(\sum_{i}\left(r_i^{''}r_i r_i^{'}\right)\right)\right)\right)\right]y_m\right)\left(0+\sum_{i=1}^{d} r_i^{''''}u_i+\sum_{j=1}^{e} s_j^{''''}v_j+\sum_{k=1}^{f} t_k^{''''}w_k+\sum_{l=1}^{g} b_l^{''''}x_l+\sum_{m=1}^{h} z_m^{''''}y_m\right)$$

$$=\left(0+0+0+0+0+\sum_{m=1}^{h}\left[\sum_{i,l}\left(\sum_{i,k}\left(\sum_{i,j}\left(\sum_{i} r_i^{''''}\left(r_i^{'''}r_i^{''}r_i r_i^{'}\right)\right)\right)\right)\right]y_m\right)$$

Thus $\left(Z\left(R\right)\right)^5=\left(Z\left(R\right)\right)^4\left(Z\left(R\right)\right)=Y$

The product of any six elements in the set $Z\left(R\right)$ is given

$$\left(0+0+0+0+0+\sum_{m=1}^{h}\left[\sum_{i,l}\left(\sum_{i,k}\left(\sum_{i,j}\left(\sum_{i} r_i^{''''}\left(r_i^{'''}r_i^{''}r_ir_i^{'}\right)\right)\right)\right)\right]y_m\right)\left(0+\sum_{i=1}^{d} r_i^{'''''}u_i+\sum_{j=1}^{e} s_j^{'''''}v_j+\sum_{k=1}^{f} t_k^{'''''}w_k+\right.$$
$$\left.\sum_{l=1}^{g} b_l^{'''''}x_l+\sum_{m=1}^{h} z_m^{'''''}y_m\right)=(0)$$

Since $Z(R)$ is an ideal and $(Z(R))^5 \subseteq ann(Z(R))$, we have
$(Z(R))^6=(Z(R))^5(Z(R))=(Z(R))(Z(R))^5=(0)$

Lemma 3.3: Let R be a ring constructed in section 3.1. Then the unit group

$$U(R)=\,<a>\cdot\left(1+Z(R)\right)\cong\,<a>\times\left(1+Z(R)\right).$$

This is a direct product of the normal subgroup $1+Z(R)$ of the ring R by a cyclic group $<a>$ of order p^r-1.

A complete classification of $U(R)$ requires that the structure of $1+Z(R)$ be determined completely. We will therefore incorporate the ideas of Raghavendran [9], and Chikunji [3], to determine the generators and to classify the group of invertible elements of the ring R for characteristic p. This lemma will be fundamental in outlining the structure.

Lemma 3.4: The multiplicative group $U(R)$ of R is defined by $U(R)=R-Z(R)$, and since $|R|=p^{(1+d+e+f+g+h)r}$, $|Z(R)|=p^{(d+e+f+g+h)r}$, and $|1+Z(R)|=p^{(d+e+f+g+h)r}$, it follows that $|U(R)|=p^{(1+d+e+f+g+h)r}-p^{(d+e+f+g+h)r}=\left(p^r-1\right)p^{(d+e+f+g+h)r}$ which resonates well with the fact that $U(R)\cong\mathbb{Z}_{p^r-1}\times(1+Z(R))$.

Proposition 3.5: Let R be a ring constructed above and of characteristic p with $pu_i=0,\ pv_j=0,\ pw_k=0,$ $px_l=0$ and $py_m=0$. Then the unit group $U(R)$ is a direct product of cyclic groups as follows:

$$U(R)\cong\begin{cases}\mathbb{Z}_{2^r-1}\times\left(\mathbb{Z}_{2^3}^r\right)^d\times(\mathbb{Z}_2^r)^f\times(\mathbb{Z}_2^r)^h, p=2\\ \mathbb{Z}_{3^r-1}\times\left(\mathbb{Z}_{3^2}^r\right)^d\times(\mathbb{Z}_3^r)^e\times(\mathbb{Z}_3^r)^g\times(\mathbb{Z}_3^r)^h, p=3\\ \mathbb{Z}_{5^r-1}\times\left(\mathbb{Z}_{5^2}^r\right)^d\times(\mathbb{Z}_5^r)^e\times(\mathbb{Z}_5^r)^f\times(\mathbb{Z}_5^r)^g, p=5\\ \mathbb{Z}_{p^r-1}\times\left(\mathbb{Z}_p^r\right)^d\times\left(\mathbb{Z}_p^r\right)^e\times\left(\mathbb{Z}_p^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h, p>5\end{cases}$$

Proof: Let $\alpha_1,\cdots,\alpha_r$ be the elements of R_0 with $\alpha_1=1$ so that $\bar{\alpha}_1,\cdots,\bar{\alpha}_r\in R_0/pR_0\cong GR\left(p^r\right)$ form a basis for $GF\left(p^r\right)$ regarded as a vector space over its prime subfield $GF(p)$. We consider the following four cases separately:

$Case\ (i)$: $p=2$. For each $1\le t\le r$, we see that for the elements $1+\alpha_tu_i$, $1+\alpha_tw_k$ and $1+\alpha_ty_m$ in $1+Z(R)$, $(1+\alpha_tu_i)^8=1,\ (1+\alpha_tw_k)^2=1$ and $(1+\alpha_ty_m)^2=1$. For nonnegative integers $\phi_t,\ \theta_t$ and σ_t such that $\phi_t\le 8,$ $\theta_t\le 2$ and $\sigma_t\le 2$, we assert that the equation

$$\prod_{i=1}^{d}\prod_{t=1}^{r}\left\{(1+\alpha_tu_i)^{\phi_t}\right\}\cdot\prod_{k=1}^{f}\prod_{t=1}^{r}\left\{(1+\alpha_tw_k)^{\theta_t}\right\}\cdot\prod_{m=1}^{h}\prod_{t=1}^{r}\left\{(1+\alpha_ty_m)^{\sigma_t}\right\}=\{1\},$$

implies that $\phi_t = 8,\ \theta_t = 2$ and $\sigma_t = 2$ for $1 \le t \le r$.

Suppose

$$A_{ti} = \left\{ \left(1+\alpha_t u_i\right)^{\phi} : \phi = 1,\cdots,8;\ \forall\ 1 \le t \le r \right\},$$

$$B_{tk} = \left\{ \left(1+\alpha_t w_k\right)^{\theta} : \theta = 1,2;\ \forall\ 1 \le t \le r \right\}, \text{ and}$$

$$C_{tm} = \left\{ \left(1+\alpha_t y_m\right)^{\sigma} : \sigma = 1,2;\ \forall\ 1 \le t \le r \right\},$$

we see that A_{ti}, B_{tk}, and C_{tm} are all subgroups of $1+Z(R)$ with their precise orders as indicated in their definition. Since the product of the $(d+f+h)r$ subgroups equal to $p^{(d+f+h)r}$ and the intersection of any pair of the cyclic subgroups gives the identity group, the product by the $(d+f+h)r$ subgroups is direct and exhausts $1+Z(R)$. Thus

$$\begin{aligned} 1+Z(R) &= \prod_{i=1}^{d}\prod_{t=1}^{r} < 1+\alpha_t u_i > \times \prod_{k=1}^{f}\prod_{t=1}^{r} < 1+\alpha_t w_k > \times \prod_{m=1}^{h}\prod_{t=1}^{r} < 1+\alpha_t y_m > \\ &\cong \left(\mathbb{Z}_{2^3}^r\right)^d \times (\mathbb{Z}_2^r)^f \times (\mathbb{Z}_2^r)^h \end{aligned}$$

Case (ii): $p=3$. For each $1 \le t \le r$, we see that for the elements $1+\alpha_t u_i$, $1+\alpha_t v_j$, $1+\alpha_t x_l$ and $1+\alpha_t y_m$ in $1+Z(R)$, $(1+\alpha_t u_i)^9 = 1$, $\left(1+\alpha_t v_j\right)^3 = 1$, $\left(1+\alpha_t x_l\right)^3 = 1$ and $\left(1+\alpha_t y_m\right)^3 = 1$. For nonnegative integers ϕ_t, θ_t, ψ_t and σ_t such that $\phi_t \le 9,\ \theta_t \le 3,\ \psi_t \le 3$ and $\sigma_t \le 3$, we assert that the equation

$$\prod_{i=1}^{d}\prod_{t=1}^{r}\left\{(1+\alpha_t u_i)^{\phi_t}\right\}\cdot\prod_{j=1}^{e}\prod_{t=1}^{r}\left\{\left(1+\alpha_t v_j\right)^{\theta_t}\right\}\cdot\prod_{l=1}^{g}\prod_{t=1}^{r}\left\{(1+\alpha_t x_l)^{\psi_t}\right\}\cdot\prod_{m=1}^{h}\prod_{t=1}^{r}\left\{\left(1+\alpha_t y_m\right)^{\sigma_t}\right\} = \{1\},$$

implies that $\phi_t = 9,\ \theta_t = 3,\ \psi_t = 3$ and $\sigma_t = 3$ for $1 \le t \le r$.

Suppose

$$A_{ti} = \left\{ \left(1+\alpha_t u_i\right)^{\phi} : \phi = 1,\cdots,9;\ \forall\ 1 \le t \le r \right\},$$

$$B_{tj} = \left\{ \left(1+\alpha_t v_j\right)^{\theta} : \theta = 1,2,3;\ \forall\ 1 \le t \le r \right\},$$

$$C_{tl} = \left\{ \left(1+\alpha_t x_l\right)^{\psi} : \psi = 1,2,3;\ \forall\ 1 \le t \le r \right\} \text{and}$$

$$D_{tm} = \left\{ \left(1+\alpha_t y_m\right)^{\sigma} : \sigma = 1,2,3;\ \forall\ 1 \le t \le r \right\},$$

we see that A_{ti}, B_{tj}, C_{tl} and D_{tm} are all subgroups of $1+Z(R)$ with their precise orders as indicated in their definition. Since the product of the $(d+e+g+h)r$ subgroups equal to $p^{(d+e+g+h)r}$ and the intersection of any pair of the cyclic subgroups gives the identity group, the product by the $(d+e+g+h)r$ subgroups is direct and exhausts $1+Z(R)$. Thus

$$1+Z(R)=\prod_{i=1}^{d}\prod_{t=1}^{r}<1+\alpha_t u_i>\times\prod_{j=1}^{e}\prod_{t=1}^{r}<1+\alpha_t v_j>\times\prod_{l=1}^{g}\prod_{t=1}^{r}<1+\alpha_t x_l>\times\prod_{m=1}^{h}\prod_{t=1}^{r}<1+\alpha_t y_m>$$

$$\cong\left(\mathbb{Z}_{3^2}^r\right)^d\times(\mathbb{Z}_3^r)^e\times(\mathbb{Z}_3^r)^g\times(\mathbb{Z}_3^r)^h$$

$Case\ (iii)$: $p=5$. For each $1\le t\le r$, we see that for the elements $1+\alpha_t u_i$, $1+\alpha_t v_j$, $1+\alpha_t w_k$ and $1+\alpha_t x_l$ in $1+Z(R)$, $(1+\alpha_t u_i)^{25}=1$, $\left(1+\alpha_t v_j\right)^5=1$, $\left(1+\alpha_t w_k\right)^5=1$ and $\left(1+\alpha_t x_l\right)^5=1$. For nonnegative integers ϕ_t, θ_t, ψ_t and σ_t such that $\phi_t\le 25$, $\theta_t\le 5$, $\psi_t\le 5$ and $\sigma_t\le 5$, we assert that the equation

$$\prod_{i=1}^{d}\prod_{t=1}^{r}\left\{(1+\alpha_t u_i)^{\phi_t}\right\}\cdot\prod_{j=1}^{e}\prod_{t=1}^{r}\left\{\left(1+\alpha_t v_j\right)^{\theta_t}\right\}\cdot\prod_{k=1}^{f}\prod_{t=1}^{r}\left\{(1+\alpha_t w_k)^{\psi_t}\right\}\cdot\prod_{l=1}^{g}\prod_{t=1}^{r}\left\{(1+\alpha_t x_l)^{\sigma_t}\right\}=\{1\}$$

implies that $\phi_t=25,\ \theta_t=5,\ \psi_t=5$ and $\sigma_t=5$ for $1\le t\le r$.

Suppose

$$A_{ti}=\left\{\left(1+\alpha_t u_i\right)^{\phi}:\phi=1,\cdots,25;\ \forall\ 1\le t\le r\right\},$$

$$B_{tj}=\left\{\left(1+\alpha_t v_j\right)^{\theta}:\theta=1,\ldots,5;\ \forall\ 1\le t\le r\right\},$$

$$C_{tk}=\left\{\left(1+\alpha_t w_k\right)^{\psi}:\psi=1,\ldots,5;\ \forall\ 1\le t\le r\right\}\text{ and}$$

$$D_{tl}=\left\{\left(1+\alpha_t x_l\right)^{\sigma}:\sigma=1,\ldots,5;\ \forall\ 1\le t\le r\right\},$$

we see that A_{ti}, B_{tj}, C_{tk} and D_{tl} are all subgroups of $1+Z(R)$ with their precise orders as indicated in their definition. Since the product of the $(d+e+f+g)r$ subgroups equal to $p^{(d+e+f+g)r}$ and the intersection of any pair of the cyclic subgroups gives the identity group, the product by the $(d+e+f+g)r$ subgroups is direct and exhausts $1+Z(R)$. Thus

$$1+Z(R)=\prod_{i=1}^{d}\prod_{t=1}^{r}<1+\alpha_t u_i>\times\prod_{j=1}^{e}\prod_{t=1}^{r}<1+\alpha_t v_j>\times\prod_{k=1}^{f}\prod_{t=1}^{r}<1+\alpha_t w_k>\times\prod_{l=1}^{g}\prod_{t=1}^{r}<1+\alpha_t x_l>$$

$$\cong\left(\mathbb{Z}_{5^2}^r\right)^d\times(\mathbb{Z}_5^r)^e\times(\mathbb{Z}_5^r)^f\times(\mathbb{Z}_5^r)^g$$

$Case\ (iv)$: $p>5$. For each $1\le t\le r$, we see that for the elements $1+\alpha_t u_i$, $1+\alpha_t v_j$, $1+\alpha_t w_k$, $1+\alpha_t x_l$ and $1+\alpha_t y_m$ in $1+Z(R)$, $(1+\alpha_t u_i)^p=1$, $\left(1+\alpha_t v_j\right)^p=1$, $(1+\alpha_t w_k)^p=1$, $(1+\alpha_t x_l)^p=1$ and $(1+\alpha_t y_m)^p=1$. For nonnegative integers ϕ_t, θ_t, ψ_t, δ_t and σ_t such that $\phi_t\le p$, $\theta_t\le p$, $\psi_t\le p$, $\delta_t\le p$ and $\sigma_t\le p$, we assert that the equation

$$\prod_{i=1}^{d}\prod_{t=1}^{r}\left\{\left(1+\alpha_t u_i\right)^{\phi_t}\right\}\cdot\prod_{j=1}^{e}\prod_{t=1}^{r}\left\{\left(1+\alpha_t v_j\right)^{\theta_t}\right\}\cdot\prod_{k=1}^{f}\prod_{t=1}^{r}\left\{\left(1+\alpha_t w_k\right)^{\psi_t}\right\}\cdot$$
$$\prod_{l=1}^{g}\prod_{t=1}^{r}\left\{\left(1+\alpha_t x_l\right)^{\delta_t}\right\}\cdot\prod_{m=1}^{h}\prod_{t=1}^{r}\left\{\left(1+\alpha_t y_m\right)^{\sigma_t}\right\}=\{1\},$$

implies that $\phi_t = p,\ \theta_t = p,\ \psi_t = p,\ \delta_t = p$ and $\sigma_t = p$ for $1 \le t \le r$.

Suppose

$$A_{ti}=\left\{\left(1+\alpha_t u_i\right)^{\phi}:\phi=1,\cdots,p;\ \forall\ 1\le t\le r\right\},$$

$$B_{tj}=\left\{\left(1+\varepsilon_t v_j\right)^{\theta}:\theta=1,\cdots,p;\ \forall\ 1\le t\le r\right\},$$

$$C_{tk}=\left\{\left(1+\varepsilon_t w_k\right)^{\psi}:\psi=1,\cdots,p;\ \forall\ 1\le t\le r\right\}$$

$$D_{tl}=\left\{\left(1+\varepsilon_t x_l\right)^{\delta}:\delta=1,\cdots,p;\ \forall\ 1\le t\le r\right\}\text{ and}$$

$$E_{tm}=\left\{\left(1+\varepsilon_t y_m\right)^{\sigma}:\sigma=1,\cdots,p;\ \forall\ 1\le t\le r\right\},$$

we see that A_{ti}, B_{tj}, C_{tk}, D_{tl} and E_{tm} are all subgroups of $1+Z(R)$ with their precise orders as indicated in their definition. Since the product of the $(d+e+f+g+h)r$ subgroups equal to $p^{(d+e+f+g+h)r}$ and the intersection of any pair of the cyclic subgroups gives the identity group, the product by the $(d+e+f+g+h)r$ subgroups is direct and exhausts $1+Z(R)$. Thus

$$1+Z(R)=\prod_{i=1}^{d}\prod_{t=1}^{r}<1+\alpha_t u_i>\times\prod_{j=1}^{e}\prod_{t=1}^{r}<1+\alpha_t v_j>\times\prod_{k=1}^{f}\prod_{t=1}^{r}<1+\alpha_t w_k>\times$$
$$\prod_{l=1}^{g}\prod_{t=1}^{r}<1+\alpha_t x_l>\times\prod_{m=1}^{h}\prod_{t=1}^{r}<1+\alpha_t y_m>$$
$$\cong\left(\mathbb{Z}_p^r\right)^d\times\left(\mathbb{Z}_p^r\right)^e\times\left(\mathbb{Z}_p^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h$$

Example: Let $R = GR(2,2)\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ for $p=2,\ r=1,\ d=e=f=g=h=1$ Then $Z(R) = 0\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$. Clearly from lemma 3.3, $U(R)\cong\mathbb{Z}_{p-1}\times\left(1+Z(R)\right)$ where $1+Z(R) = 1\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$. Now the generators of the subgroup $1+Z(R)$ are $\langle 1+u\rangle$ of order 8, $\langle 1+w\rangle$ of order 2 and $\langle 1+y\rangle$ of order 2 as shown below.

$$(1+y)^2=\sum_{\alpha=0}^{2}\binom{2}{\alpha}(y)^{\alpha}$$
$$=\binom{2}{0}(y)^0+\binom{2}{1}(y)^1+\binom{2}{2}(y)^2$$
$$=1+2y+y^2$$

$$=1 \quad (\text{since } \; 2y=0 \quad and \quad y^2=0)$$

$$(1+w)^2=\sum_{\alpha=0}^{2}\binom{2}{\alpha}(w)^{\alpha}$$

$$=\binom{2}{0}(w)^0+\binom{2}{1}(w)^1+\binom{2}{2}(w)^2$$

$$=1+2w+w^2$$

$$=1 \quad (\text{since } \; 2w=0 \quad and \quad w^2=0)$$

$$(1+u)^8=\sum_{\alpha=0}^{8}\binom{8}{\alpha}(u)^{\alpha}$$

$$=\binom{8}{0}(u)^0+\binom{8}{1}(u)^1+\binom{8}{2}(u)^2+\binom{8}{3}(u)^3+\binom{8}{4}(u)^4+\binom{8}{5}(u)^5+\binom{8}{6}(u)^6+\binom{8}{7}(u)^7+\binom{8}{8}(u)^8$$

$$=1+2u+28u^2+56u^3+70u^4+56u^5+28u^6+8u^7+u^8$$

$$=1+4(2u)+14(2v)+28(2w)+35(2x)+28(2y)+0+0+0$$

$$=1 \quad (\text{since } \; 2u=0, \; 2v=0, \; 2w=0, \; 2x=0, \; and \quad 2y=0)$$

Considering nonnegative integers $\phi, \; \theta$ and σ such that $\phi \le 8, \; \theta \le 2$ and $\sigma \le 2$, so that $\left\{(1+u)^{\phi}\right\}\cdot\left\{(1+w)^{\theta}\right\}\cdot\left\{(1+y)^{\sigma}\right\}=\{1\}$, clearly $\phi=8, \; \theta=2$ and $\sigma=2$. Suppose $A=\left\{(1+u)^{\phi}:\phi=1,\cdots,8\right\}, \; B=\left\{(1+w)^{\theta}:\theta=1,2\right\}$, and $C=\left\{(1+y)^{\sigma}:\sigma=1,2\right\}$, we see that $A, B,$ and C are all subgroups of $1+Z(R)$ with their precise orders as indicated in their definition. Since $\langle 1+u\rangle \cap \langle 1+w\rangle \cap \langle 1+y\rangle = 1$, it follows that

$$1+Z(R) = <1+u> \times <1+w> \times <1+y>$$
$$= \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

4. CONCLUSION

This article has constructed a completely primary finite ring whose Jacobson radical is of nilpotent index six of characteristic p and it has lane bare a unique structure of group of invertible elements via characterization. A set of linearly independent generators for $1+Z(R)$ has been established by connecting those generators to the basis $GF(p)$ over its prime field. The relation amongst these generators played essential role in determining the structure of this class of rings. The results obtained in this study contributes significantly to the existing literature on the classification of finite rings. Since for such rings, $Char\, R = p^k, \;\; 1 \le k \le 6$ and this work has only considered a case for $k=1$, future researchers may develop the construction for $2 \le k \le 6$ and investigate as well as determine their structures. Studying the automorphism groups of these rings will as well expose the symmetries of these groups.

REFERENCES

[1] Y. Alkhamees, "Finite rings in which the multiplication of any two zero divisors is zero," Archive Der Mathematik, vol. 37, pp. 144–149, 1981. https://doi.org/10.1007/BF01234337

[2] C. J. Chikunji, "On a class of finite rings," Communications in Algebra, vol. 10, pp. 5049-5081, 1999. https://doi.org/10.1080/00927879908826747

[3] C. J. Chikunji, "Unit groups of cubes radical zero commutative completely primary finite rings," International Journal of Mathematics and Mathematical Sciences, vol. 4, pp. 579-592, 2005. https://doi.org/10.1155/IJMMS.2005.579

[4] C. J. Chikunji, "On unit groups of completely primary finite rings," Mathematical Journal of Okayama University, vol. 50, pp. 149-160, 2008. https://moodle.buan.ac.bw:80/handle/123456789/290

[5] C. J. Chikunji, "Unit groups of finite rings with products of zero divisors in their coefficient subrings," Publications De L'institut Mathematique, vol. 109, pp. 215-220, 2014. https://doi.org/10.2298/PMI1409215C

[6] B. Corbas, "Rings with few zero divisors," Mathematische Annalen, vol. 181, pp. 1-7, 1969. https://doi.org/10.1007/BF01351174

[7] O. M. Oduor, "Automorphisms of a certain class of completely primary finite rings," International Journal of Pure and Applied Mathematics, vol. 74, no. 4, pp. 465-482, 2012.

[8] O. M. Oduor and M. O. Onyango, "Unit groups of some classes of power four radical zero commutative completely primary finite rings," International Journal of Algebra, vol. 8, pp. 357-363, 2014. https://doi.org/10.12988/ija.2014.4431

[9] R. Raghavendran, "Finite associative rings," Compositio Mathematica, vol. 21, pp. 195-469, 1969. http://www.numdam.org/item/cm-1969-21-2-195-01

[10] H. S. Were, M. O. Oduor, and M. N. Gichuki, "Unit groups of classes of five radical zero commutative completely primary finite rings," Journal of Advances in Mathematics and Computer Science, vol. 36, no. 8, pp. 137-154, 2021. Doi:10.9734/JAMCS/2021/v36i830396

[11] H. S. Were, M. O. Oduor, and M. N. Gichuki, "Classification of Units of five radical zero commutative completely primary finite rings with variant orders of second Galois ring module generators," Asian Research Journal of Mathematics, vol. 18, no. 6, pp. 70-78, 2022. Doi:10.9734/ARJOM/2022/v18i630385

[12] H. S. Were and M. O. Oduor, "Classification of unit groups of five radical zero completely primary finite rings whose first and second Galois ring module generators are of order p^k, $k = 2,3,4$," Journal of Mathematics, vol. 2022, pp. 1–11, 2022. https://doi.org/10.1155/2022/7867431

[13] B. R. Wirt, "Finite associative rings," Finite non - commutative local rings [Doctoral Thesis, University of Oklahoma], United States of America. https://hdl.handle.net/11244/3379.

[14] R. S. Wilson, "On the structures of finite rings," Compositio Mathematica, vol. 26, pp. 79-93, 1973. http://www.numdam.org/item?id=CM_1973_26_1_79_0