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A Modified Semi-analytical Method for the Solution of System of Fractional Order Linear and Nonlinear Volterra Integro-differential Equations

ABSTRACT: Background: Finding analytical solution to Volterra integro-differential equations (VIDEs), especially nonlinear types, mostly poses a lot of difficulties and many a time impossible, thus the need to provide semi-analytical solution. **Objective:** This research is concerned with the solutions of system of linear and nonlinear fractional order integro-differential equations with difference kernels. To achieve that, we exploited the advantage of integral transform and one of the existing semi-analytical methods to develop the desired method of solution. **Methods:** One of the recently developed integral transforms, the Shehu transform which generalizes Laplace and Sumudu transforms is systematically integrated into the well-known Adomian decomposition method (ADM) for the purpose of getting a more simplified approach to solution of the class of problems considered. The Shehu transform is first applied to both sides of the given VIDEs with difference kernel, followed by the application of convolution theorem. The ADM is then employed to handle the nonlinearity that are encountered. **Results:** The proposed method; Modified Semi-analytical Method (MSM) is applied to selected problems in the literature, and gives comparatively good results. The method equally produce exact solution whenever the solution is in closed form. The results are given in both tabular and 2D graphical forms for ease of comparison. All the computations are carried out using Mathematica 13.3 with the fractional order derivative interpreted in Caputo sense. **Conclusions:** Since MSM has been successfully used to solve linear and nonlinear VIDEs with difference kernel, the scope of the method can be expanded to cover Volterra-Fredholm integro-differential equations (VFIDEs) in the future studies.

KEYWORDS: Shehu Transform; Integral Equations; Fractional Derivatives; Difference Kernel; Adomian Polynomials

INTRODUCTION

Integro-differential equations (IDEs) are the ones that consists of both integral and differential aspects, these make using them to model real life systems an excellent approach [1]. These equations are mostly used in different disciplines of science and engineering. Despite their usefulness, solving the nonlinear IDEs are most of the time difficult using the known analytical methods. Volterra integro-differential equations (VIDEs) as an example of IDEs, play a crucial role due to their ability to capture memory effects and hereditary properties [2, 3]. The VIDEs are highly relevant in practical applications in viscoelastic materials, population dynamics, heat conduction, and electrochemical processes [4]. Recent developments in research have underscored the relevance of VIDEs in modeling dynamic systems. A typical example is the work of [5] which analyzed the stability of VIDEs in control and biological models, demonstrating their effectiveness in handling impulsive effects and delay dynamics. Likewise in the work reported by [6] where the roles of VIDEs are explored in delay systems, emphasizing their applicability in engineering and theoretical sciences. Furthermore, Liu, Tao, and Zhang [7] proposed a spectral method to solve nonlinear VIDEs with weakly singular kernels, improving computational efficiency and accuracy. All the foregoing studies highlight the

growing significance of VIDEs in both theoretical research and practical problem-solving across various disciplines.
Fractional calculus provides the desired requirements in modeling practical phenomenon where precision is of essence [8]. The application of non-integer-order VIDEs enhances model accuracy in fields such as control theory and signal processing [9]. Expansion of the scope of VIDEs to fractional orders enhances the scope of their applications, since fractional calculus accurately describes real-world situations such as diffusion effects and so on [10]. Fractional order VIDEs provide the desired flexibility in capturing real life phenomena compared to their integer-order counterparts, making them suitable for modeling processes in engineering, biomathematics, mathematical physics, and fluid dynamics [11, 12]. Several studies, including those by Wang [13] and Boulaaras, Jan, and Pham [14], have highlighted the advantages of fractional calculus in describing complex dynamical behaviors that classical integer-order models cannot effectively capture. More recent works by Guo, Yin, and Peng [15], as well as Alshammari, Iqbal, and Ntwiga [16], have demonstrated the increasing importance of fractional-order models in applications such as diffusion processes, viscoelasticity, and bioengineering, reinforcing the need for efficient solution methodologies.
Despite their extensive applications, solving FVIDEs remains a significant challenge due to memory effects and non-local dependencies [17, 18]. Many researchers have attempted to derive analytical or numerical solutions, though exact solutions remain scarce [19–21]. Several analytical methods, including the ADM [22], HPM, VIM, and HAM, have been widely applied, with studies specifically utilizing the VIM [23] and HAM [24, 25]. On the numerical front, researchers have explored methods based on the use operational matrices for nonlinear VIDEs [26] and the multi-wavelet Galerkin method, which employs operational matrices of integration and wavelet transforms [27]. While these methods have shown considerable effectiveness, they often face challenges such as slow convergence, computational instability, and difficulties in handling strong nonlinearities. Moreover, existing approaches frequently require extensive computational resources for higher-order nonlinear problems, underscoring the need for more efficient solution techniques.
The need for more easily accessible methods that are devoid of ambiguities that are inherent in some of the existing methods, such as small parameters to be determined along with solution to original problem. The present work therefore proposed Modified Semi-analytical Methods (MSM) which exploits the advantage in Shehu transform which is the fact that it handles both constant and variable problems, and the Adomian Decomposition Method. The essence of incorporating ADM is to overcome any nonlinear terms that may be encountered in the course of solving the problem. The proposed method has been successfully applied to both linear and nonlinear integro-differential equations with difference kernels has presented in the sequel.

Adomian Polynomial

The Adomian polynomials, denoted by A_q are used to decompose the non-linearity encountered in problems. It is given as

$$A_q = \frac{1}{\Gamma(q+1)}\left[\frac{d^q}{d\xi^q}N\left(\sum_{m=0}^{q}\xi^i u_i\right)\right]_{\xi=0}, \qquad q = 0, 1, 2, \ldots \tag{1}$$

For further reading on ADM and its applications see [28–30].

The Caputo Fractional Derivative

Let $\psi \in \Re_+$ and $\vartheta = \lceil \psi \rceil$. The operator $\mathbb{D}_\zeta^\psi$ is defined by

$$\mathbb{D}_\zeta^\psi f(s) = J_c^{\vartheta-\psi} D^\vartheta f(s) = \frac{1}{(\vartheta-\psi-1)!}\int_c^s (s-\rho)^{\vartheta-\psi-1}\left(\frac{d}{d\rho}\right)^\psi f(\rho)\,d\rho, \tag{2}$$

for $c \le s \le d$, is called the Caputo differential operator of order ψ.
The definition (2) can equally be expressed as

$$D^\psi s^\vartheta = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta-\psi+1)}s^{\vartheta-\psi}. \tag{3}$$

METHODS

Application of Shehu Transform to Differential Coefficients [31]

The Shehu transform of the differential coefficient $g'(s)$ is defined as

$$\mathbb{S}\{g'(s)\} = \left(\frac{\psi}{\vartheta}\right) G(\psi,\vartheta) - f(0). \tag{4}$$

The n^{th} order derivative is given as

$$\mathbb{S}\{g^{(\rho)}(s)\} = \left(\frac{\psi}{\vartheta}\right)^{\rho} G(\psi,\vartheta) - \sum_{i=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(i+1)} g^{(i)}(0), \tag{5}$$

where $G(\psi,\vartheta)$ is the Shehu transform of the function $g(s)$.

Nature of the Problem Considered

In this work, the family of problem considered is

$$\mathbb{D}^{\rho}u(t) = g(t) + \mu u(t) + \int_0^t \left[g(\tau)u(\tau) + h(\tau)F(u(\tau))\right] d\tau, \qquad 0 \le t \le b \tag{6}$$

where ρ is a fractional order derivative, $u(t)$ is the unknown function, $g(\tau)$ and $h(\tau)$ are degenerate kernels which in this study are taken to be difference kernel, $F(u(\tau))$ is the nonlinear term.

Implementation of the Modified Semi-analytical Method (MSM)

We shall apply the Shehu Transform to both sides of (6) as follows:

$$\mathbb{S}\big\{\mathbb{D}^{\rho}u(x)\big\} = \mathbb{S}\left\{ g(t) + \mu u(t) + \int_0^t \left[g(\tau)u(\tau) + h(\tau)F(u(\tau))\right] d\tau \right\}. \tag{7}$$

Applying Shehu transform of derivatives to the term on the left hand side of (7), we have

$$\mathbb{S}\big\{\mathbb{D}^{\rho}u(t)\big\} = \frac{\psi^{\rho}}{\vartheta^{\rho}} U(\psi,\vartheta) - \sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) \tag{8}$$

We then substitute (8) back in (7) to get

$$\left(\frac{\psi}{\vartheta}\right)^{\rho} U(\psi,\vartheta) - \sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) = G(\psi,\vartheta) + \mu\mathbb{S}\big\{u(t)\big\} + \mathbb{S}\Big\{ \int_0^t k_1(t-\tau)u(\tau)d\tau + k_2(t-\tau)u(\tau)d\tau \Big\}, \tag{9}$$

where $k_1(t-\tau)$ and $k_2(t-\tau)$ are difference kernel representing $g(\tau)$ and $h(\tau)$ respectively.

$$\begin{aligned} \left(\frac{\psi}{\vartheta}\right)^{\rho} U(\psi,\vartheta) = \sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} & u^{(j)}(0) + G(\psi,\vartheta) + \mu\mathbb{S}\big\{u(t)\big\} \\ & + \mathbb{S}\big\{k_1(t-\tau)\big\} * \mathbb{S}\big\{u(s)\big\} + \mathbb{S}\big\{k_2(t-\tau)\big\} * \mathbb{S}\big\{F(u(\tau))\big\}. \end{aligned} \tag{10}$$

where $F(u(\tau))$ is the nonlinear term.

$$\begin{aligned} U(\psi,\vartheta) = \frac{\vartheta^{\rho}}{\psi^{\rho}} \Bigg(& \sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) + G(\psi,\vartheta) + \mu\mathbb{S}\big\{u(t)\big\} \\ & + \mathbb{S}\big\{k_1(t-\tau)\big\} * \mathbb{S}\big\{u(s)\big\} + \mathbb{S}\big\{k_2(t-\tau)\big\} * \mathbb{S}\big\{F(u(\tau))\big\} \Bigg). \end{aligned} \tag{11}$$

The initial approximation $u_0(t)$ is obtained from (11) as

$$\mathbb{S}\{u_0(t)\} = \frac{\vartheta^\rho}{\psi^\rho}\left[\sum_{j=0}^{\rho-1}\left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) + G(\psi,\vartheta) + \mu\mathbb{S}\{u(t)\} + \mathbb{S}\{k_1(t-\tau)\}\star\mathbb{S}\{u(s)\}\right], \tag{12}$$

while the recurrence relation is given by the remaining terms in (11) as

$$\mathbb{S}\left[u_{k+1}(t)\right] = \frac{\vartheta^\rho}{\psi^\rho}\left[\mathbb{S}\left\{k_2(t-\tau)\right\} * \mathbb{S}\left\{\sum_{k=0}^{\infty} A_k(s)\right\}\right], \qquad k = 0, 1, 2, ... \tag{13}$$

The Adomian polynomials corresponding to the nonlinear term in (1) derived substituted and used as follows

$$F(u(t)) = \sum_{k=0}^{\infty} A_k(t), \tag{14}$$

where $A_k(t)$ represents the Adomian polynomials.

Numerical Experiment on Linear Problems

Problem 1 [32]
Consider the system of linear VIDEs

$$\begin{aligned} \mathbb{D}_t^\zeta y_1(t) - 2t^2 - \int_0^t \Big((t-\tau)y_1(t) + (t-\tau)y_2(\tau)\Big)\, d\tau = 0, \\ \mathbb{D}_t^\zeta y_2(t) + 3t^2 + \frac{1}{5}t^5 - \int_0^t \Big((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)\Big)\, d\tau = 0. \end{aligned} \tag{15}$$

with the given initial conditions: $y_1(0) = y_2(0) = 1, \qquad 0 < \zeta \leq 1.$

Solution to Problem 1:

$$\begin{aligned} \mathbb{D}_t^\zeta y_1(t) &= 2t^2 + \int_0^t \big((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)\big)\, d\tau, \\ \mathbb{D}_t^\zeta y_2(t) &= -3t^2 - \frac{1}{5}t^5 + \int_0^t \big((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)\big)\, d\tau. \end{aligned} \tag{16}$$

Taking the Shehu transform of both sides, we get

$$\begin{aligned} \mathbb{S}\{\mathbb{D}_t^\zeta y_1(t)\} &= 2\mathbb{S}\{t^2\} + \mathbb{S}\left\{\int_0^t \big((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)\big)\, d\tau\right\}, \\ \mathbb{S}\{\mathbb{D}_t^\zeta y_2(t)\} &= -3\mathbb{S}\{t^2\} - \frac{1}{5}\mathbb{S}\{t^5\} + \mathbb{S}\left\{\int_0^t \big((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)\big)\, d\tau\right\}. \end{aligned} \tag{17}$$

Implementing the Shehu transform of the derivatives and simplifying, we get

$$\begin{aligned} \left(\frac{\psi}{\vartheta}\right)^\zeta Y_1(\psi,\vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} y_1(0) &= 4\left(\frac{\vartheta}{\psi}\right)^3 + \mathbb{S}\{t\} * \mathbb{S}\{y_1(t)\} + \mathbb{S}\{t\} * \mathbb{S}\{y_2(t)\}, \\ \left(\frac{\psi}{\vartheta}\right)^\zeta Y_2(\psi,\vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} y_2(0) &= -6\left(\frac{\vartheta}{\psi}\right)^3 - 24\left(\frac{\vartheta}{\psi}\right)^6 + \mathbb{S}\{t\} * \mathbb{S}\{y_1(t)\} - \mathbb{S}\{t\} * \mathbb{S}\{y_2(t)\}. \end{aligned} \tag{18}$$

Applying initial conditions and simplification gives

$$\begin{aligned} \left(\frac{\psi}{\vartheta}\right)^\zeta Y_1(\psi,\vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} &= 4\left(\frac{\vartheta}{\psi}\right)^3 + \left(\frac{\vartheta}{\psi}\right)^2 Y_1(\psi,\vartheta) + \left(\frac{\vartheta}{\psi}\right)^2 Y_2(\psi,\vartheta), \\ \left(\frac{\psi}{\vartheta}\right)^\zeta Y_2(\psi,\vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} &= -6\left(\frac{\vartheta}{\psi}\right)^3 - 24\left(\frac{\vartheta}{\psi}\right)^6 + \left(\frac{\vartheta}{\psi}\right)^2 Y_1(\psi,\vartheta) - \left(\frac{\vartheta}{\psi}\right)^2 Y_2(\psi,\vartheta). \end{aligned} \tag{19}$$

$$\begin{aligned}
\left(\frac{\psi}{\vartheta}\right)^{\zeta} Y_1(\psi,\vartheta) &= \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} + 4\left(\frac{\vartheta}{\psi}\right)^{3} + \left(\frac{\vartheta}{\psi}\right)^{2} Y_1(\psi,\vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2} Y_2(\psi,\vartheta),\\
\left(\frac{\psi}{\vartheta}\right)^{\zeta} Y_2(\psi,\vartheta) &= \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} - 6\left(\frac{\vartheta}{\psi}\right)^{3} - 24\left(\frac{\vartheta}{\psi}\right)^{6} + \left(\frac{\vartheta}{\psi}\right)^{2} Y_1(\psi,\vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2} Y_2(\psi,\vartheta).
\end{aligned} \tag{20}$$

$$\begin{aligned}
Y_1(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right) + 4\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_1(\psi,\vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_2(\psi,\vartheta),\\
Y_2(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right) - 6\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{6+\zeta} + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_1(\psi,\vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_2(\psi,\vartheta).
\end{aligned} \tag{21}$$

From where we get the initial approximations as:

$$\begin{aligned}
Y_{1,0}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right) + 4\left(\frac{\vartheta}{\psi}\right)^{3+\zeta},\\
Y_{2,0}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right) - 6\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{6+\zeta}.
\end{aligned} \tag{22}$$

Taking the inverse Shehu transform, we obtain

$$\begin{aligned}
y_{1,0}(t) &= 1 + \frac{4t^{2+\zeta}}{\Gamma(3+\zeta)},\\
y_{2,0}(t) &= 1 - \frac{6t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{24t^{5+\zeta}}{\Gamma(6+\zeta)}.
\end{aligned} \tag{23}$$

For the recurrence relation, we have

$$\begin{aligned}
Y_{1,k+1}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,k}(\psi,\vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,k}(\psi,\vartheta),\\
Y_{2,k+1}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,k}(\psi,\vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,k}(\psi,\vartheta).
\end{aligned} \tag{24}$$

When $k = 0$:

$$\begin{aligned}
Y_{1,1}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,0}(\psi,\vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,0}(\psi,\vartheta),\\
Y_{2,1}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,0}(\psi,\vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,0}(\psi,\vartheta).
\end{aligned} \tag{25}$$

$$\begin{aligned}
Y_{1,1}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta}\left[\left(\frac{\vartheta}{\psi}\right) + 4\left(\frac{\vartheta}{\psi}\right)^{3+\zeta}\right] + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta}\left[\left(\frac{\vartheta}{\psi}\right) - 6\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{6+\zeta}\right],\\
Y_{2,1}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta}\left[\left(\frac{\vartheta}{\psi}\right) + 4\left(\frac{\vartheta}{\psi}\right)^{3+\zeta}\right] - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta}\left[\left(\frac{\vartheta}{\psi}\right) - 6\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{6+\zeta}\right].
\end{aligned} \tag{26}$$

$$\begin{aligned}
Y_{1,1}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} + 4\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 6\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{8+2\zeta},\\
Y_{2,1}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} + 4\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} + 6\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 24\left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}.
\end{aligned} \tag{27}$$

$$
\begin{aligned}
Y_{1,1}(\psi,\vartheta) &= 2\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 2\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}, \\
Y_{2,1}(\psi,\vartheta) &= 10\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 24\left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}.
\end{aligned} \tag{28}
$$

Taking the inverse Shehu transform of both sides, we get

$$
\begin{aligned}
y_{1,1}(t) &= \frac{2t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} - \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)}, \\
y_{2,1}(t) &= \frac{10t^{4+2\zeta}}{\Gamma(5+2\zeta)} + \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)}.
\end{aligned} \tag{29}
$$

When $k = 1$:

$$
\begin{aligned}
Y_{1,2}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,1}(\psi,\vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,1}(\psi,\vartheta), \\
Y_{2,2}(\psi,\vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,1}(\psi,\vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,1}(\psi,\vartheta).
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
Y_{1,2}(\psi,\vartheta) = {} & \left(\frac{\vartheta}{\psi}\right)^{2+\zeta}\left[2\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 2\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}\right] \\
& + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta}\left[10\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 24\left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}\right], \\
Y_{2,2}(\psi,\vartheta) = {} & \left(\frac{\vartheta}{\psi}\right)^{2+\zeta}\left[2\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 2\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}\right] \\
& - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta}\left[10\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 24\left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}\right].
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
Y_{1,2}(\psi,\vartheta) &= 2\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 2\left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{10+3\zeta} + 10\left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} + 24\left(\frac{\vartheta}{\psi}\right)^{10+3\zeta}, \\
Y_{2,2}(\psi,\vartheta) &= 2\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 2\left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{10+3\zeta} - 10\left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{10+3\zeta}.
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
Y_{1,2}(\psi,\vartheta) &= 2\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 8\left(\frac{\vartheta}{\psi}\right)^{7+3\zeta}, \\
Y_{2,2}(\psi,\vartheta) &= 2\left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 12\left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} - 48\left(\frac{\vartheta}{\psi}\right)^{10+3\zeta}.
\end{aligned} \tag{33}
$$

Taking the inverse Shehu transform of both sides, we get

$$
\begin{aligned}
y_{1,2}(t) &= \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} + \frac{8t^{6+3\zeta}}{\Gamma(7+3\zeta)}, \\
y_{2,2}(t) &= \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} - \frac{12t^{6+3\zeta}}{\Gamma(7+3\zeta)} - \frac{48t^{9+3\zeta}}{\Gamma(10+3\zeta)}.
\end{aligned} \tag{34}
$$

The general solutions are obtained as:

$$
\begin{aligned}
y_1(t) &= y_{1,0}(t) + y_{1,1}(t) + y_{1,2}(t) + \dots, \quad k = 0,1,2,\dots, \\
y_2(t) &= y_{2,0}(t) + y_{2,1}(t) + y_{2,2}(t) + \dots, \quad k = 0,1,2,\dots.
\end{aligned} \tag{35}
$$

$$\begin{aligned} y_1(t) = 1 + 4\frac{t^{2+\zeta}}{\Gamma(3+\zeta)} + \frac{2t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} \\ - \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)} + \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} + \frac{8t^{6+3\zeta}}{\Gamma(7+3\zeta)} + \dots \end{aligned} \tag{36}$$

$$\begin{aligned} y_2(t) = 1 - \frac{6t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{24t^{5+\zeta}}{\Gamma(6+\zeta)} + \frac{10t^{4+2\zeta}}{\Gamma(5+2\zeta)} + \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)} \\ + \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} - \frac{12t^{6+3\zeta}}{\Gamma(7+3\zeta)} - \frac{48t^{9+3\zeta}}{\Gamma(10+3\zeta)} + \dots \end{aligned} \tag{37}$$

$$y_1(t) = 1 + \frac{6t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)} + \frac{8t^{6+3\zeta}}{\Gamma(7+3\zeta)} + \dots \tag{38}$$

$$\begin{aligned} y_2(t) = 1 - \frac{6t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{24t^{5+\zeta}}{\Gamma(6+\zeta)} + \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)} + \frac{12t^{4+2\zeta}}{\Gamma(5+2\zeta)} \\ - \frac{12t^{6+3\zeta}}{\Gamma(7+3\zeta)} - \frac{48t^{9+3\zeta}}{\Gamma(10+3\zeta)} + \dots \end{aligned} \tag{39}$$

Problem 2 [33]
Consider the system of linear VIDEs

$$\begin{aligned} \mathbb{D}_t^{\zeta} y_1(t) = 1 + t - \frac{t^3}{3} + \int_0^t \big((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)\big)\,d\tau, \\ \mathbb{D}_t^{\zeta} y_2(t) = 1 - t - \frac{t^4}{12} + \int_0^t \big((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)\big)\,d\tau. \end{aligned} \tag{40}$$

With the given initial conditions: $y_1(0) = y_2(0) = 0$, and $0 < \zeta \le 1$.
Solution to Problem 2:
Following the procedures as in problem 1, we have

$$\begin{aligned} y_1(t) = \frac{t^{\zeta}}{\Gamma(1+\zeta)} + \frac{t^{1+\zeta}}{\Gamma(2+\zeta)} - 2\frac{t^{3+\zeta}}{\Gamma(4+\zeta)} + \frac{2t^{2+2\zeta}}{\Gamma(3+2\zeta)} \\ - \frac{2t^{5+2\zeta}}{\Gamma(6+2\zeta)} - \frac{2t^{6+2\zeta}}{\Gamma(7+2\zeta)} + \frac{2t^{4+3\zeta}}{\Gamma(5+3\zeta)} \\ - \frac{4t^{7+3\zeta}}{\Gamma(8+3\zeta)} + \frac{2t^{5+3\zeta}}{\Gamma(6+3\zeta)} + \ldots \end{aligned} \tag{41}$$

$$\begin{aligned} y_2(t) = \frac{t^{\zeta}}{\Gamma(1+\zeta)} - \frac{t^{1+\zeta}}{\Gamma(2+\zeta)} - 2\frac{t^{4+\zeta}}{\Gamma(5+\zeta)} + \frac{2t^{3+2\zeta}}{\Gamma(4+2\zeta)} \\ - \frac{2t^{5+2\zeta}}{\Gamma(6+2\zeta)} + \frac{2t^{6+2\zeta}}{\Gamma(7+2\zeta)} + \frac{2t^{4+3\zeta}}{\Gamma(5+3\zeta)} \\ - \frac{2t^{7+3\zeta}}{\Gamma(8+3\zeta)} - \frac{4t^{8+3\zeta}}{\Gamma(9+3\zeta)} + \ldots \end{aligned} \tag{42}$$

Problem 3 [34]
Consider the system of linear VIDEs

$$\begin{aligned} \mathbb{D}_t^{\zeta} u(t) + \frac{3t^{2\zeta}\zeta\Gamma(3\zeta)}{\Gamma(1+2\zeta)} - \int_0^t (t-\tau)u(\tau)\,d\tau - \int_0^t (t-\tau)v(\tau)\,d\tau = 0, \\ \mathbb{D}_t^{\zeta} v(t) + \frac{2t^{3\zeta+2}}{2+9\zeta+9\zeta^2} + \frac{3t^{2\zeta}\zeta\Gamma(3\zeta)}{\Gamma(1+2\zeta)} + \int_0^t (t-\tau)u(\tau)\,d\tau + \int_0^t (t-\tau)v(\tau)\,d\tau = 0. \end{aligned} \tag{43}$$

With the given initial conditions: $u(0) = v(0) = 0$, $0 \leq t \leq 1$, $0 < \zeta \leq 1$.
Solution to Problem 3:
Implementation of the earlier algorithm, we have

$$\begin{aligned} u(t) &= t^{3\zeta} - \frac{2\Gamma(3\zeta+1)t^{6\zeta+3}}{\Gamma(6\zeta+4)} + \frac{2\Gamma(3\zeta+1)t^{7\zeta+2}}{\Gamma(7\zeta+3)} + \dots \\ v(t) &= -\frac{2\Gamma(3\zeta+1)t^{4\zeta+2}}{\Gamma(4\zeta+3)} - t^{3\zeta} + \frac{2\Gamma(3\zeta+1)t^{5\zeta+1}}{\Gamma(5\zeta+2)} \\ &\quad + \frac{2\Gamma(3\zeta+1)t^{6\zeta+3}}{\Gamma(6\zeta+4)} - \frac{2\Gamma(3\zeta+1)t^{8\zeta+4}}{\Gamma(8\zeta+5)} - \frac{2\Gamma(3\zeta+1)t^{7\zeta+2}}{\Gamma(7\zeta+3)} + \dots \end{aligned} \tag{44}$$

Numerical Experiment on Nonlinear Problems

Problem 4 [33]

Consider the nonlinear VIDEs

$$\mathbb{D}^{\zeta} y(t) - \int_0^t [y(\tau)]^2 d\tau = -1, \qquad 0 \leq x \leq 1, \qquad 0 < \zeta \leq 1. \tag{45}$$

Subject to the initial condition $y(0) = 0$.

Solution to Problem 4:

$$\mathbb{D}^{\zeta} y(t) = \int_0^t [y(\tau)]^2 d\tau - 1. \tag{46}$$

Taking the Shehu transform of both sides, we get

$$\mathbb{S}\{\mathbb{D}^{\zeta} y(t)\} = \mathbb{S}\left\{ \int_0^t [y(\tau)]^2 d\tau - 1 \right\}. \tag{47}$$

$$\left(\frac{\psi}{\vartheta}\right)^{\zeta} Y(\psi, \vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} y(0) = \frac{\vartheta}{\psi} * \mathbb{S}\{y(t)^2\} - \frac{\vartheta}{\psi}. \tag{48}$$

Applying the initial condition, we get

$$\left(\frac{\psi}{\vartheta}\right)^{\zeta} Y(\psi, \vartheta) = \frac{\vartheta}{\psi} * \mathbb{S}\{y(t)^2\} - \frac{\vartheta}{\psi} \tag{49}$$

$$Y(\psi, \vartheta) = -\left(\frac{\vartheta}{\psi}\right)^{1+\zeta} + \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} * \mathbb{S}\{y(t)^2\} \tag{50}$$

The initial approximation and the recurrence relation are obtained from (50) as follows:

$$Y_0(\psi, \vartheta) = -\left(\frac{\vartheta}{\psi}\right)^{1+\zeta}. \tag{51}$$

$$y_0(x) = -\frac{t^{\zeta}}{\Gamma(\zeta+1)}. \tag{52}$$

The recurrence relation becomes

$$Y_{q+1}(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\{A_q(t)\}, \qquad q = 0, 1, 2, ... \tag{53}$$

The Adomian polynomials corresponding to the nonlinear term $y(t)^2$ are derived from

$$A_q = \frac{1}{\Gamma(q+1)} \left[\frac{d^q}{d\xi^q} N\left(\sum_{i=0}^{q} \xi^i y_i \right) \right]_{\xi=0}, \qquad q = 0, 1, 2, ... \tag{54}$$

$N(y) = y^2$ is the nonlinear term.
When $q = 0$, $q = 1$, $q = 2$:

$$A_0 = y_0^2, \quad A_1 = 2y_0y_1, \quad A_2 = y_1^2 + 2y_0y_2, \quad ... \tag{55}$$

When $q = 0$ is implemented in the recurrence relation, we have

$$Y_1(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\big\{A_0(t)\big\}. \tag{56}$$

$$Y_1(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\bigg\{\bigg(-\frac{t^\zeta}{\Gamma(\zeta+1)}\bigg)^2\bigg\}. \tag{57}$$

$$Y_1(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\bigg\{\frac{t^{2\zeta}}{\Gamma(\zeta+1)^2}\bigg\}. \tag{58}$$

$$Y_1(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \bigg[\frac{\Gamma(2\zeta+1)}{\Gamma(\zeta+1)^2}\left(\frac{\vartheta}{\psi}\right)^{2\zeta+1}\bigg]. \tag{59}$$

$$Y_1(\psi, \vartheta) = \frac{\Gamma(2\zeta+1)}{\Gamma(\zeta+1)^2}\left(\frac{\vartheta}{\psi}\right)^{3\zeta+2}. \tag{60}$$

Taking the inverse shehu transform of Y_1, we get

$$y_1(t) = \frac{\Gamma(2\zeta+1)t^{3\zeta+1}}{\Gamma(\zeta+1)^2\Gamma(2\zeta+3)}. \tag{61}$$

When $k = 1$:

$$Y_2(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\big\{A_1(t)\big\}. \tag{62}$$

From Adomian polynomial, $A_1 = 2y_0y_1$

$$Y_2(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\bigg\{2\bigg[-\frac{t^\zeta}{\Gamma(\zeta+1)}\bigg]\bigg[\frac{\Gamma(2\zeta+1)t^{3\zeta+1}}{\Gamma(\zeta+1)^2\Gamma(3\zeta+2}\bigg]\bigg\} \tag{63}$$

$$Y_2(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\bigg\{-2\frac{\Gamma(2\zeta+1)t^{4\zeta+1}}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2}\bigg\} \tag{64}$$

$$Y_2(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \bigg[\frac{-2\Gamma(4\zeta+2)\Gamma(2\zeta+1)}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)}\left(\frac{\vartheta}{\psi}\right)^{4\zeta+2}\bigg]. \tag{65}$$

$$Y_2(\psi, \vartheta) = \frac{-2\Gamma(4\zeta+2)\Gamma(2\zeta+1)}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)}\left(\frac{\vartheta}{\psi}\right)^{5\zeta+3}. \tag{66}$$

Taking the inverse Shehu transform of Y_2, we get

$$y_2(x) = \frac{-2\Gamma(4\zeta+2)\Gamma(2\zeta+1)t^{2+5\zeta}}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)\Gamma(5\zeta+3)}. \tag{67}$$

When $k = 2$:

$$Y_3(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\big\{A_2(t)\big\}. \tag{68}$$

From Adomian polynomial,

$$A_2 = 2y_0y_2 + y_1^2. \tag{69}$$

$$Y_3(\psi,\vartheta)=\left(\frac{\vartheta}{\psi}\right)^{1+\zeta}\mathbb{S}\Big\{2\Big[-\frac{t^{\zeta}}{\Gamma(\zeta+1)}\Big]\Big[\frac{-2\Gamma(4\zeta+2)\Gamma(2\zeta+1)t^{2+5\zeta}}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)\Gamma(5\zeta+3)}\Big] + \Big[\frac{\Gamma(2\zeta+1)t^{3\zeta+1}}{\Gamma(\zeta+1)^2\Gamma(3\zeta+2}\Big]^2\Big\}. \tag{70}$$

$$Y_3(\psi,\vartheta)=\frac{4\Gamma(4\zeta+2)\Gamma(2\zeta+1)\left(\frac{\vartheta}{\psi}\right)^{4+7\zeta}}{\Gamma(\zeta+1)^4\Gamma(3\zeta+2)\Gamma(5\zeta+3)\Gamma(6\zeta+3)}+\frac{\Gamma(2\zeta+1)^2\left(\frac{\vartheta}{\psi}\right)^{4+7\zeta}}{\Gamma(\zeta+1)^4\Gamma(3\zeta+2)^2\Gamma(6\zeta+3)}. \tag{71}$$

Taking the inverse Shehu transform of Y_3, we get

$$y_3(t)=\frac{4\Gamma(4\zeta+2)\Gamma(2\zeta+1)t^{3+7\zeta}}{\Gamma(\zeta+1)^4\Gamma(3\zeta+2)\Gamma(5\zeta+3)\Gamma(6\zeta+3)\Gamma(7\zeta+4)} + \frac{\Gamma(2\zeta+1)^2t^{3+7\zeta}}{\Gamma(\zeta+1)^4\Gamma(3\zeta+2)^2\Gamma(6\zeta+3)\Gamma(7\zeta+4)}. \tag{72}$$

The general Solution is

$$y(t)=y_0(t)+y_1(t)+y_2(t)+y_3(t)+\ldots. \tag{73}$$

$$y(t)=-\frac{t^{\zeta}}{\Gamma(\zeta+1)}+\frac{\Gamma(2\zeta+1)t^{3\zeta+1}}{\Gamma(\zeta+1)^2\Gamma(2\zeta+3)}-\frac{2\Gamma(4\zeta+2)\Gamma(2\zeta+1)t^{2+5\zeta}}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)\Gamma(5\zeta+3)} + \frac{4\Gamma(4\zeta+2)\Gamma(2\zeta+1)t^{3+7\zeta}}{\Gamma(\zeta+1)^4\Gamma(3\zeta+2)\Gamma(5\zeta+3)\Gamma(6\zeta+3)\Gamma(7\zeta+4)} + \frac{\Gamma(2\zeta+1)^2t^{3+7\zeta}}{\Gamma(\zeta+1)^4\Gamma(3\zeta+2)^2\Gamma(6\zeta+3)\Gamma(7\zeta+4)}. \tag{74}$$

Problem 5 [33]

Consider the nonlinear VIDEs

$$\mathbb{D}^{\zeta}y(t)-\int_0^t \exp^{-\tau}[y(\tau)]^2d\tau=1, \qquad 0\le t\le 1, \qquad 3<\zeta\le 4. \tag{75}$$

Subject to the initial condition $y(0)=y'(0)=y''(0)=y'''(0)=0$.
Solution to Problem 5:
Implementing the algorithm as we did in the earlier problems yields

$$\begin{aligned} y(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^{\zeta}}{\Gamma(\zeta+1)}+\frac{t^{\zeta+1}}{\Gamma(\zeta+2)}+\frac{2t^{\zeta+2}}{\Gamma(\zeta+3)}+\frac{4t^{\zeta+3}}{\Gamma(\zeta+4)}+\frac{8t^{\zeta+4}}{\Gamma(\zeta+5)}+\frac{14t^{\zeta+5}}{\Gamma(\zeta+6)}\\ +\frac{20t^{\zeta+6}}{\Gamma(\zeta+7)}+\frac{20t^{\zeta+7}}{\Gamma(\zeta+8)}+\frac{\Gamma(2\zeta+1)t^{3\zeta+1}}{\Gamma(\zeta+1)^2\Gamma(2\zeta+3)}+\frac{2t^{2\zeta+1}}{\Gamma(2\zeta+2)}+\frac{2\Gamma(\zeta+2)t^{2\zeta+2}}{\Gamma(\zeta+1)\Gamma(2\zeta+3)}\\ +\frac{\Gamma(\zeta+3)t^{2\zeta+3}}{\Gamma(\zeta+1)\Gamma(2\zeta+4)}+\frac{\Gamma(2\zeta+4)t^{2\zeta+4}}{3\Gamma(\zeta+1)\Gamma(2\zeta+5)}+\frac{t^{2\zeta+2}}{\Gamma(2\zeta+3)}+\frac{\Gamma(\zeta+3)t^{2\zeta+3}}{\Gamma(\zeta+2)\Gamma(2\zeta+5)}\\ +\frac{\Gamma(\zeta+3)t^{2\zeta+4}}{\Gamma(\zeta+2)\Gamma(2\zeta+5)}+\frac{\Gamma(\zeta+5)t^{2\zeta+5}}{6\Gamma(2\zeta+6)\Gamma(2\zeta+2)}+\frac{\Gamma(2\zeta+2)t^{2+3\zeta}}{6\Gamma(\zeta+1)\Gamma(\zeta+2)\Gamma(3\zeta+3)}\\ +\frac{2\Gamma(2)t^{3+2\zeta}}{\Gamma(\zeta+3)\Gamma(2\zeta+4)}+\frac{2\Gamma(\zeta+4)t^{4+2\zeta}}{\Gamma(\zeta+3)\Gamma(2\zeta+5)}. \end{aligned} \tag{76}$$

RESULTS AND DISCUSSION

In this study, five numerical examples are solved using the proposed Modified Semi-analytical Method (MSM). To validate the effectiveness of our method, these examples were carefully selected from recently published articles that employed different numerical techniques, highlighting the uniqueness and versatility of our approach.
Specifically, Osama & Adyan [32] applied the Homotopy Analysis Method (HAM) and the Variational Iteration Method (VIM) to solve Problems 1 and 2, respectively. Our tabulated results show perfect agreement with theirs, thereby confirming the reliability of our method. Similarly, for Problem 3, we obtained results consistent with those of [34], who utilized the Optimal Homotopy Asymptotic Method. Our results for Problem 4 are precisely the same as those obtained in [33], where the method of solution is the Bernoulli Pseudospectral Method. Additionally, our results for Problem 5 are in perfect agreement with those of [35], who employed the Wavelet Method. Moreover, all results for the problems considered are represented graphically through 2D plots to enhance interpret-ability and provide clearer insights.

Tables

Table 1: Results for Problem 1

t	MSM y_1	MSM (y_2)	HAM (y_1)	HAM (y_2)
0.0	1.000	1.000	1.000	1.000
0.1	1.001	0.999	1.001	0.999
0.2	1.008	0.992	1.008	0.992
0.3	1.027	0.973	1.027	0.973
0.4	1.064	0.936	1.064	0.936
0.5	1.125	0.875	1.125	0.874
0.6	1.216	0.783	1.216	0.782
0.7	1.343	0.655	1.343	0.653
0.8	1.512	0.484	1.512	0.479
0.9	1.729	0.262	1.729	0.253
1.0	2.000	-0.017	2.000	-0.033

Table 2: Results for Problem 2

t	MSM (y_1)	MSM (y_2)	VIM (y_1)	VIM (y_2)
0.0	0.000	0.000	0.000	0.000
0.1	0.105	0.095	0.105	0.095
0.2	0.220	0.180	0.220	0.180
0.3	0.345	0.255	0.345	0.255
0.4	0.480	0.320	0.480	0.320
0.5	0.625	0.375	0.625	0.374
0.6	0.780	0.420	0.780	0.419
0.7	0.945	0.455	0.945	0.452
0.8	1.120	0.480	1.120	0.475
0.9	1.305	0.495	1.305	0.485
1.0	1.500	0.500	1.500	0.483

Table 3: Results for Problem 3 for values of ζ=1

t	MSM $[u(t)]$	MSM $[v(t)]$	OHAM $[u(t)]$	OHAM $[v(t)]$
0.0	0.000	0.000	0.000	0.000
0.1	0.001	-0.001	0.001	-0.001
0.2	0.008	-0.008	0.008	-0.008
0.3	0.027	-0.027	0.027	-0.027
0.4	0.064	-0.064	0.064	-0.064
0.5	0.125	-0.125	0.125	-0.125
0.6	0.216	-0.216	0.216	-0.216
0.7	0.343	-0.343	0.343	-0.343
0.8	0.512	-0.512	0.512	-0.512
0.9	0.729	-0.729	0.729	-0.729
1.0	1.000	-1.000	0.999	-1.000

Table 4: Results for Problem 4

t	MSM ($\zeta = 1$)	MSM ($\zeta = 0.9$)	BPM ($\zeta = 1$)	BPM ($\zeta = 0.9$)
0.0000	0.00000	0.00000	0.00000	0.00000
0.0625	-0.06250	-0.08574	-0.06250	-0.08576
0.1250	-0.12498	-0.15996	-0.12498	-0.15997
0.1875	-0.18740	-0.23024	-0.18740	-0.23025
0.2500	-0.24968	-0.29790	-0.24968	-0.29791
0.3125	-0.31171	-0.36342	-0.31171	-0.36344
0.3750	-0.37336	-0.42699	-0.37336	-0.42702
0.4375	-0.43446	-0.48862	-0.43446	-0.48866
0.5000	-0.49482	-0.54824	-0.49482	-0.54829
0.5625	-0.55423	-0.60572	-0.55423	-0.60576
0.6250	-0.61243	-0.66086	-0.61243	-0.66089
0.6875	-0.66917	-0.71347	-0.66917	-0.71347
0.7500	-0.72416	-0.76330	-0.72115	-0.76327
0.8125	-0.77711	-0.81013	-0.77709	-0.81007
0.8750	-0.82771	-0.85373	-0.82767	-0.85360
0.9375	-0.87565	-0.89386	-0.87557	-0.89363

Table 5: Results for Problem 5

t	MSM ($\zeta = 3.25$)	MSM ($\zeta = 3.5$)	MSM ($\zeta = 3.75$)	CAS ($\zeta = 3.25$)	CAS ($\zeta = 3.5$)	CAS ($\zeta = 3.75$)
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.1052	1.1052	1.1052	1.1053	1.1052	1.1052
0.2	1.2220	1.2217	1.2215	1.2219	1.2216	1.2216
0.3	1.3521	1.3509	1.3502	1.3523	1.3510	1.3510
0.4	1.4975	1.4945	1.4928	1.4968	1.4941	1.4941
0.5	1.6604	1.6545	1.6509	1.6635	1.6565	1.8334
0.7	2.0484	2.0319	2.0210	2.0444	2.0283	2.0293
0.8	2.2793	2.2544	2.2374	2.2776	2.2537	2.2537
0.9	2.5390	2.5032	2.4781	2.5265	2.4949	2.4949

Figures

Graphical Representation of Result

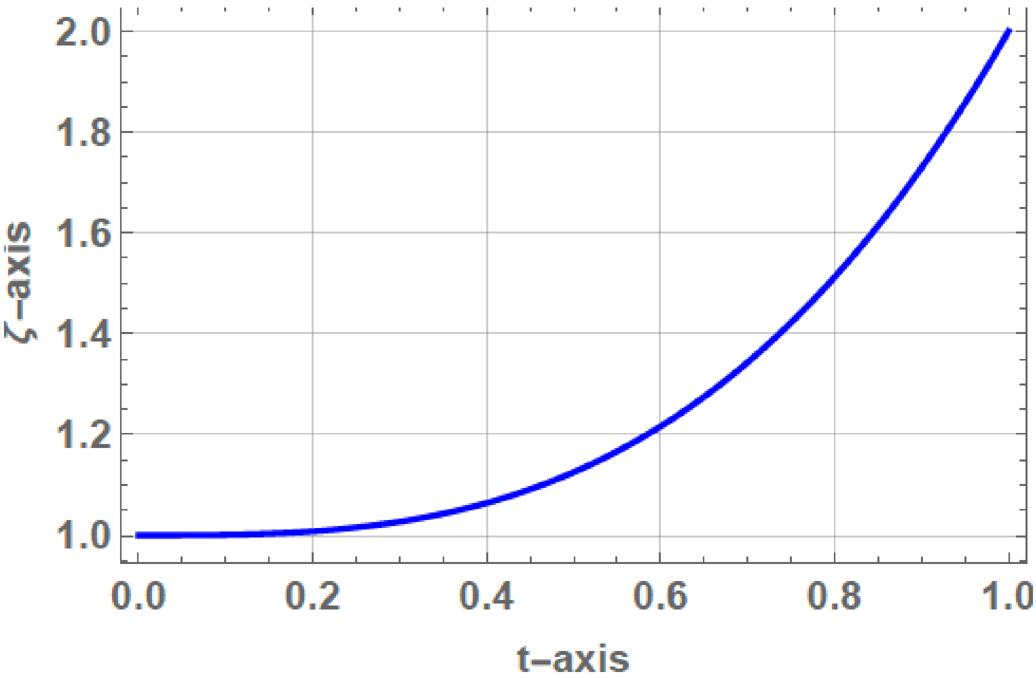

Figure 1: Graph of y_1 for Problem 1

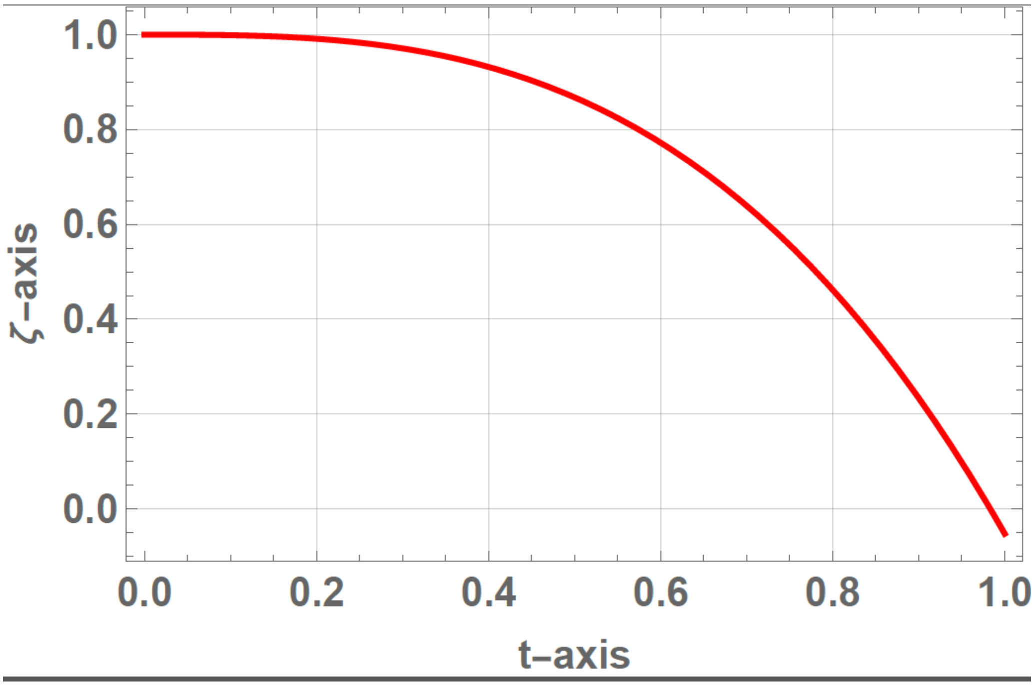

Figure 2: Graph of y_2 for Problem 1

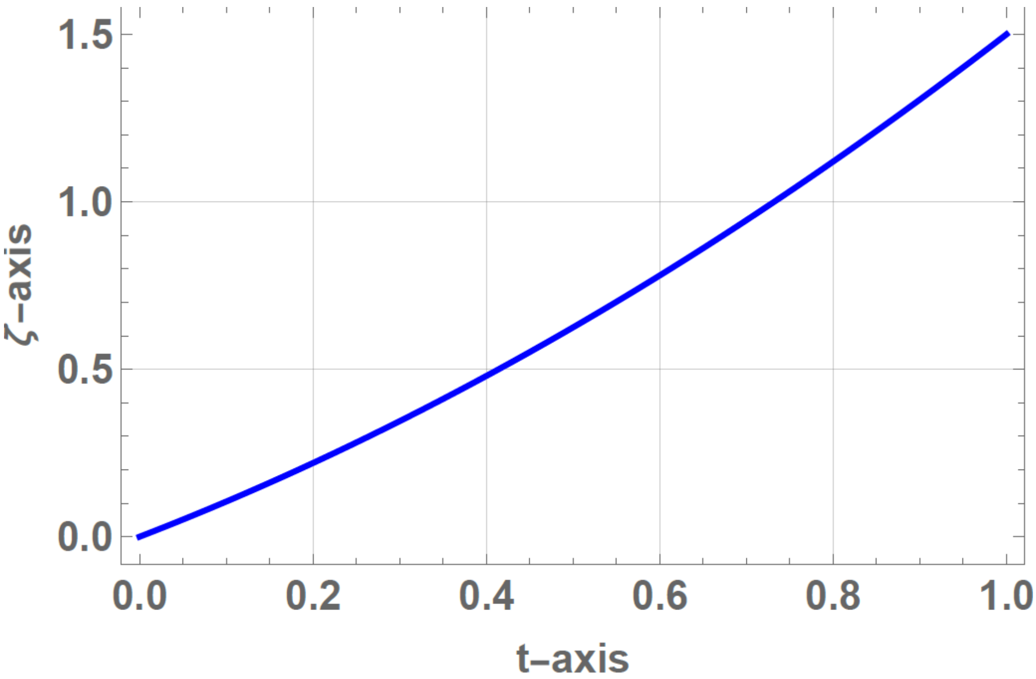

Figure 3: Graph of y_1 for Problem 2

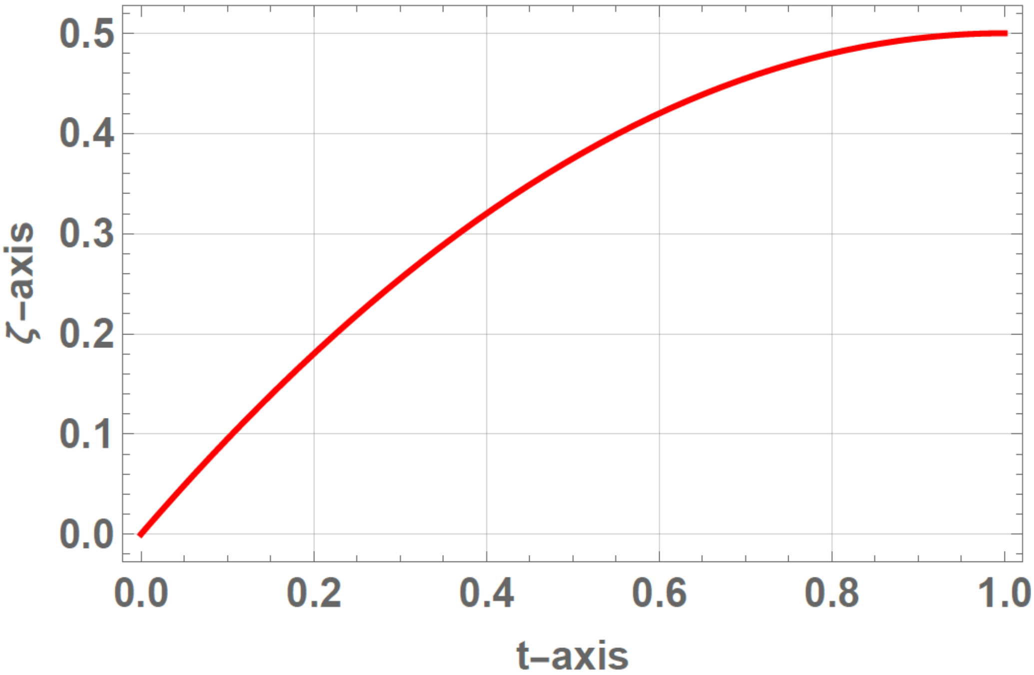

Figure 4: Graph of y_2 for Problem 2

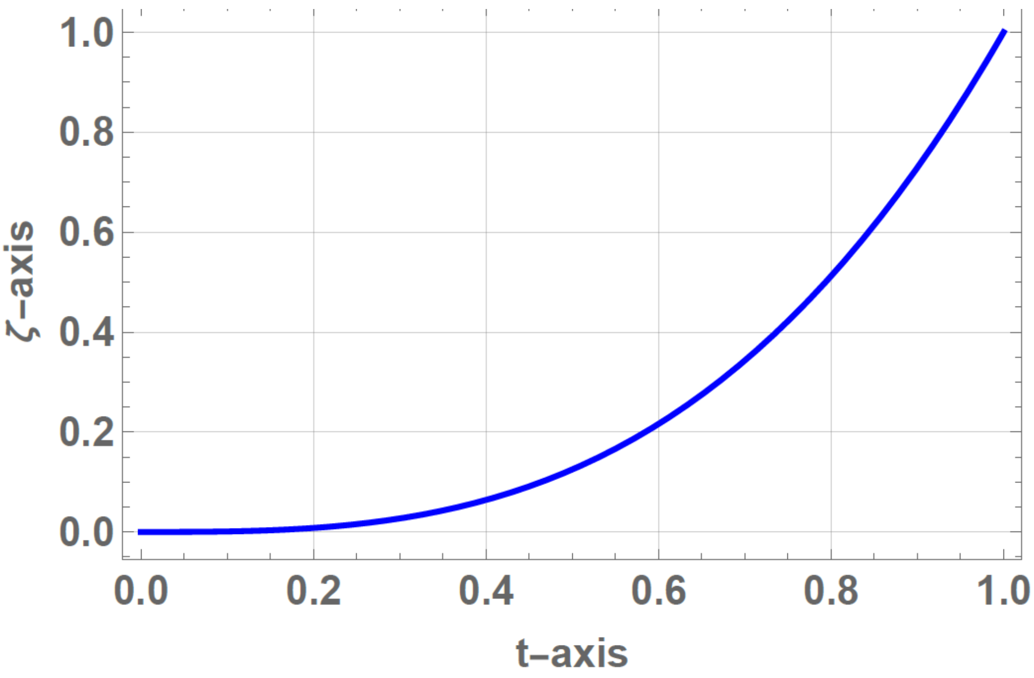

Figure 5: Graph of $u(x)$ for Problem 3

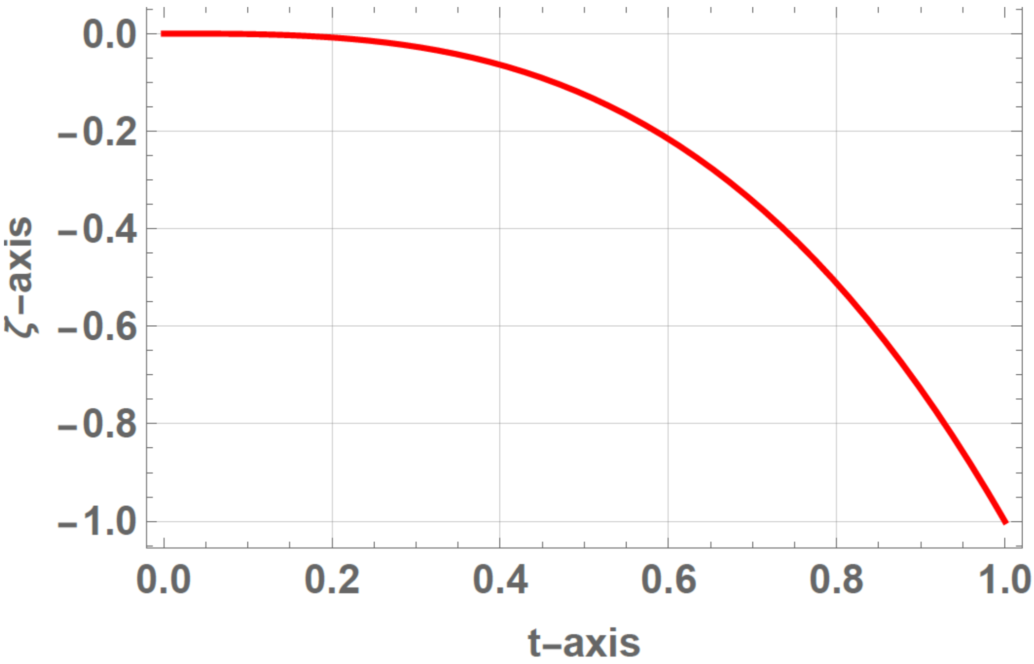

Figure 6: Graph of $v(x)$ for Problem 3

1 2D Graphs for Nonlinear Problems

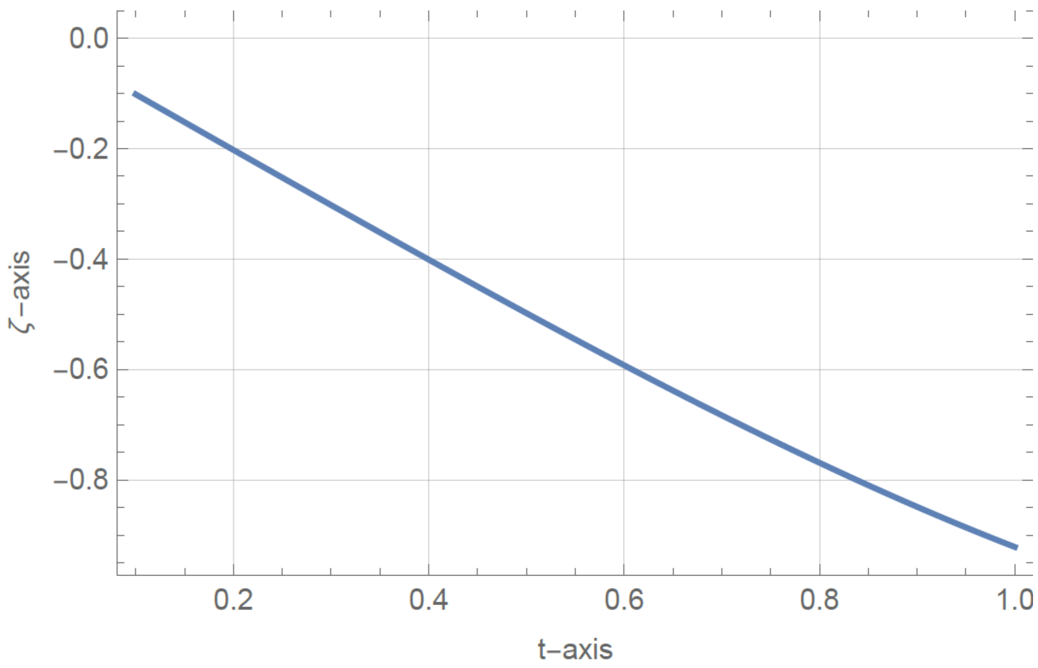

Figure 7: 2D graph for Problem 4

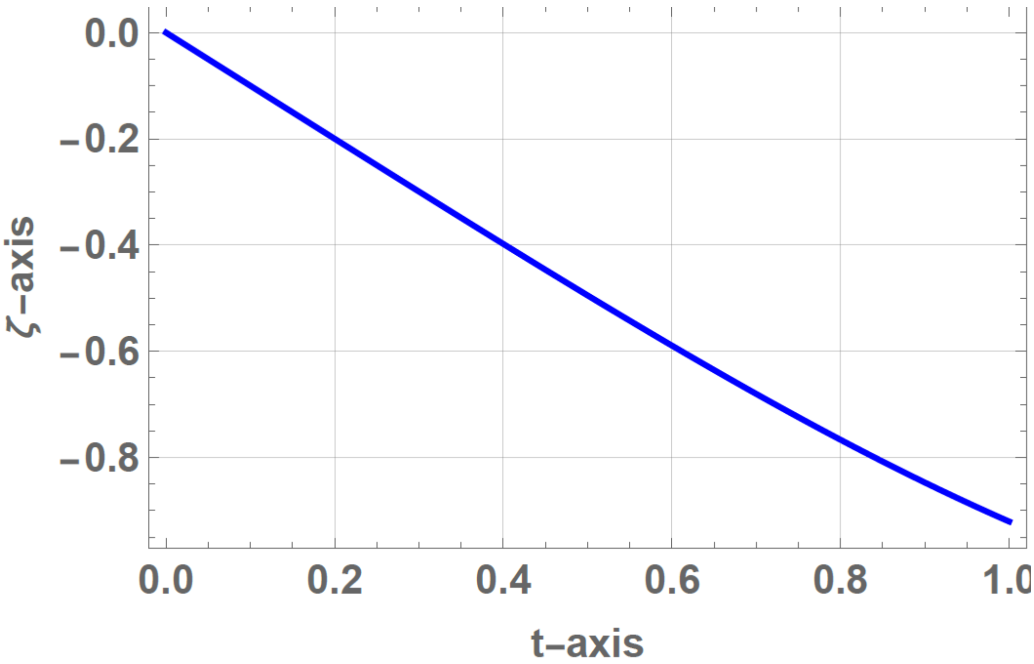

Figure 8: 2D graph for Problem 5

Conclusion

It is evident from the tabulated results in Tables 4, when compared with existing results in the literature that the Modified Table Semi-analytical Method (MSM)is an established mathematical tool for solving both fractional-order nonlinear Volterra integro-differential equations and fractional-order systems of linear Volterra integro-differential e quations. This method has thus proven to be reliable. All computations were performed using Mathematica 13.3.

The method reported in the present research can be investigated with the view of extending it to Volterra-Fredholm Integro-differential Equations (VFIDEs). This thought is premised on the fact that Fredholm integral equation which is an extension of VIDEs, that have been successfully discussed in the present work, in VFIDEs.

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None

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Data availability statements

.None

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Conflicts of interest

The authors declare no conflicts of interest.

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