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Third-Order Sandwich Results for Analytic Univalent Functions Involving a New Hadamard Product Operator

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Abstract

The objective of this paper is to examine the outcomes of third-order differential subordination and superordination for analytic functions within the region $D = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$, specifically focusing on the utilization of the novel Hadamard operator $K_{\eta,\lambda}^{\delta} f(z)$. The results are derived by analyzing relevant categories of allowable functions. New findings have been found about differential subordination and superordination, along with the discovery of several sandwich theorems. Furthermore, other specific instances are also seen. The qualities and outcomes of differential subordination exhibit symmetry with the properties of differential superordination, leading to the formulation of the sandwich theorems.

Keywords: Differential subordination, Hadamard product, Superordination, Analytic function, Sandwich theorem, Third-order.

نتائج ساندويتش من الدرجة الثالثة للدوال التحليلية أحادية التكافؤ باستخدام ضرب مؤثر جديد

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الخلاصة

الهدف من هذه الورقة هو دراسة نتائج التبعية التفاضلية من الدرجة الثالثة والتفوق للوظائف التحليلية داخل المنطقة $\{z: z \in \mathbb{C} \text{ and } |z| < 1\}$، مع التركيز بشكل خاص على استخدام مؤثر الجديد $K_{\eta,\lambda}^{\delta} f(z)$. يتم استخلاص النتائج من خلال تحليل الفئات ذات الصلة من الوظائف المسموح بها. تم العثور على نتائج جديدة حول التبعية التفاضلية والتنسيق الفائق، إلى جانب اكتشاف العديد من نظريات الساندويتش. علاوة على ذلك، يتم أيضًا رؤية حالات محددة أخرى. تظهر صفات ونتائج التبعية التفاضلية تماثلًا مع خصائص التبعية التفاضلية، مما يؤدي إلى صياغة نظريات الساندويتش.

1. Introduction

Consider $H(D)$ as a set of analytic functions within the open unit disk D, defined as $D = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$. For $n \in N = \{1,2,3,\cdots\}$ such that $a \in \mathbb{C}$, and let $H[a,n] = \{f: f \in H(D) \text{ and } f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$ and suppose that $H_0 = H[0,1]$. Consider $A \subset H(D)$, which denotes the subset of functions in D that are both analytic and have been normalized. The Taylor-Maclaurin series is a mathematical series that takes the form:

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in D). \tag{1.1}$$

Assume that both f and g belong to $H(D)$. We assert that f is a subordinate of g, (or g is a superordinate of f), denoted

$$f \prec g \text{ in } D \text{ or } f(z) \prec g(z), (z \in \mathbb{C}),$$

when there is a Schwarz function $h \in H$, defined in the unit disk D, that is analytic and satisfies $h(0) = 0$ with $|h(z)| < 1$ $(z \in \mathbb{C})$, it means $f(z) = g(h(z)), (z \in \mathbb{C})$. Furthermore, if the function g is injective within the domain D. Subsequently, we can construct the following equivalency based on the references provided in $[1 - 3]$.

$f(z) \prec g(z) \leftrightarrow f(0) = g(0)$ and $f(D) \subset g(D)$.

Let $f \in A$ be a function defined by Equation (1.1). Consider $g \in A$ defined as follows:

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

The Hadamard product, often known as convolution, of functions f and g is written as:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), (z \in D).$$

Lemma 1.1. The operator Q_λ^η is defined as the new operator of $f \in A$ for $\eta \in \mathbb{R}\backslash\{-1,-2,\dots\}, |z| = 1,\ 0 \le \lambda < 1$. It is defined as follows:

$$Q_\lambda^\eta f(z) = \frac{1}{(\eta+2)^{\lambda+2}\Gamma(\lambda+2)} \int_0^\infty t^{\lambda-1} e^{-\left[\frac{t}{\eta+2}\right]} f(zt^2)dt$$
$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+2n)(\eta+2)^{2n-2}}{\Gamma(\lambda+2)} a_n z^n. \tag{1.2}$$

Proof.

$$Q_\lambda^\eta f(z) = \frac{1}{(\eta+2)^{\lambda+2}\Gamma(\lambda+2)} \int_0^\infty t^{\lambda-1} e^{-\left[\frac{t}{\eta+2}\right]} f(zt^2)dt$$
$$= \frac{1}{(\eta+2)^{\lambda+2}\Gamma(\lambda+2)} \int_0^\infty t^{\lambda-1} e^{-\left[\frac{t}{\eta+2}\right]} \left[zt^2 + \sum_{n=2}^{\infty} a_n z^n t^{2n}\right] dt.$$

Let $x = \frac{t}{\eta+2}$, then if $t = 0$, we get $x = 0$, $t = \infty$, we get $x = \infty$ and $t = (\eta+2)x$, then $dt = (\eta+2)dx$. Thus

$$Q_\lambda^\eta f(z) = \frac{1}{(\eta+2)^{\lambda+2}\Gamma(\lambda+2)} \left\{ z \int_0^\infty (\eta+2)^{\lambda+2} x^{\lambda+1} e^{-x} dx \right.$$
$$\left. + \sum_{n=2}^{\infty} a_n z^n \int_0^\infty (\eta+2)^{\lambda+2n} x^{\lambda-1+2n} e^{-x} dx \right\}$$
$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+2n)(\eta+2)^{2n-2}}{\Gamma(\lambda+2)} a_n z^n.$$

The proof is complete.

Definition 1.2. The operator $K_{\eta,\lambda}^\delta f(z): A \to A$, where $f \in A$, is defined by convolving the new operator $Q_\lambda^\eta f(z)$ with the general Hurwitz- lerch zeta function $\emptyset(z, \delta, \eta)$ [4].

$$K_{\eta,\lambda}^\delta f(z) = Q_\lambda^\eta f(z) * \left((\eta+2)^{\delta+2}\left[\emptyset(z, \delta+2, \eta+1) - (\eta+1)^{-\delta-2}\right]\right)$$

$$= \mathfrak{z} + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda + 2n)(\eta + 2)^{2n+\delta}}{\Gamma(\lambda + 2)(n + \eta + 1)^{\delta+2}} a_n \mathfrak{z}^n, \tag{1.3}$$

where ($\eta \in \mathbb{R}\backslash\{-1, -2, \dots\}$, $\delta \in \mathbb{C}$, $Re(\delta) > 1$, $|\mathfrak{z}| = 1$, $0 \le \lambda < 1$).
According to Equation (1.3), it is simple to prove the following identity:

$$\mathfrak{z}\left(\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})\right)' = (\eta + 2)\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z}) - (\eta + 1)\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}). \tag{1.4}$$

Antonino and Miller [1] have extended the idea of second-order differential subordination and superordination in the domain D, as initially established by Miller and Mocanu [2,3,5], to the third-order scenario. This extension is also referenced in [6,7]. The features of functions p that meet the third-order differential subordination were determined:

$$\{\Theta(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2 G''(\mathfrak{z}), \mathfrak{z}^3 G'''(\mathfrak{z}); \mathfrak{z}) : \mathfrak{z} \in D\} \subset \Omega.$$

Additionally, this applies to third-order differential superordination:

$$\Omega \subset \{\Theta(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2 G''(\mathfrak{z}), \mathfrak{z}^3 G'''(\mathfrak{z}); \mathfrak{z}) : \mathfrak{z} \in D\},$$

where Ω be a set within $\mathbb{C}$, G be an analytic function with $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$.

In recent studies, multiple writers have examined various implementations of the second-order differential subordination and superordination idea. They have also produced sandwich outcomes, as evidenced by references [8]. Additionally, third-order outcomes have been explored for diverse classes, as indicated by references [6,7,9]. To explore intriguing applications of differential subordination and superordination in other mathematical disciplines, we can consult references $[10 - 12]$.
Ponnusamy and Juneja's work [13] built the concept of third-order differential subordination. Tang et al. introduced a recent study that is a good example of this (see [6,7]).

The second and third-order terms are used interchangeably. Uneven subordination piqued the interest of many academics in this field. (see [8,9], $[14 - 25]$).

In this study, we analyze a collection of appropriate admissible functions related to the integral operator with established precise requirements on the normalized analytic function, referred to as the sandwich condition.

2- Preliminaries

The concepts and lemmas listed below are necessary for the demonstration of our findings.

Definition 2.1. [**1**]. Consider the function $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$ with assume that a function $h(\mathfrak{z})$ is univalent within D. Given that the function $G(\mathfrak{z})$ is analytic in D with fulfilling the given third-order differential subordination:

$$\Theta(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2 G''(\mathfrak{z}), \mathfrak{z}^3 G'''(\mathfrak{z}); \mathfrak{z}) \prec h(\mathfrak{z}). \tag{2.1}$$

thus $G(\mathfrak{z})$ is denoted as a solution of the differential subordination (2.1). Moreover, a given univalent function $\mathcal{T}(\mathfrak{z})$ is referred to as a dominant of the solutions of Equation (2.1). Alternatively, a dominant is defined as $G(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z})$ for any $G(\mathfrak{z})$ that meets Equation (2.1). The best dominant is defined as a dominant $\tilde{\mathcal{T}}(\mathfrak{z})$ which fulfills the condition $\tilde{\mathcal{T}}(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z})$ for every dominant $\mathcal{T}(\mathfrak{z})$ of (2.1).

Definition 2.2. [**1**]. Consider the set $\mathbb{Q}$, which consists of every function $\mathcal{T}$ that is both univalent and analytic on $\overline{D}\backslash E(\mathcal{T})$, where
$E(\mathcal{T}) = \{\xi : \xi \in \partial D : \lim_{\mathfrak{z}\to\xi} \mathcal{T}(\mathfrak{z}) = \infty\}$,
with $min|\mathcal{T}'(\xi)| = \rho > 0$ for $\xi \in \partial D\backslash E(\mathcal{T})$. Additionally, we can represent the subclass of $\mathbb{Q}$, where $\mathcal{T}(0) = a$, as $Q(a)$, where $\mathbb{Q}(0) = \mathbb{Q}_0$ and $\mathbb{Q}(1) = \mathbb{Q}_1$.

The method of subordination is used for an appropriate class of admissible functions.
The following class of admissible functions was given by Antonino and Miller [1].

Definition 2.3. [**1**]. Consider Ω, a set of complex numbers, and let $\mathcal{T} \in \mathbb{Q}$ and $n \in N\backslash\{1\}$. The class of admissible functions $\Psi_n[\Omega, \mathcal{T}]$ has functions $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$, which fulfills the following admissibility conditions:

$$\mathcal{T}(\mathrm{r}, s, \mathrm{t}, \mathrm{u}; ʒ) \notin \Omega,$$

whenever

$$\mathrm{r} = \mathcal{T}(\xi), \qquad s = k\xi\mathcal{T}'(\xi), \qquad Re\left(\frac{\mathrm{t}}{s} + 1\right) \geq kRe\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)} + 1\right),$$

and

$$Re\left(\frac{\mathrm{u}}{s}\right) \geq k^2 Re\left(\frac{\xi^2\mathcal{T}'''(\xi)}{\mathcal{T}'(\xi)}\right),$$

where $ʒ \in D, \xi \in \partial D\backslash E(\mathcal{T})$, with $k \geq n$.

Lemma 2.4. [**1**]. Suppose that $G \in H[a, n]$ such that $n \geq 2$, and $\mathcal{T} \in \mathbb{Q}(a)$ satisfying the following conditions:

$$Re\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)}\right) \geq 0, \left|\frac{ʒG'(ʒ)}{\mathcal{T}'(\xi)}\right| \leq k,$$

where $ʒ \in D, \xi \in \partial D \setminus E(\mathcal{T}), k \geq n$. If Ω is a set within $\mathbb{C}$, $\Theta \in \Psi_n[\Omega, \mathcal{T}]$, with

$$\Theta(G(ʒ), ʒG'(ʒ), ʒ^2G''(ʒ), ʒ^3G'''(ʒ); ʒ) \in \Omega,$$

then

$$G(ʒ) \prec \mathcal{T}(ʒ), \quad (ʒ \in D).$$

Definition 2.5. [**7**]. Consider the function $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$. Let $h(ʒ)$ be an analytic in D. Given the function $G(ʒ)$ with

$$\Theta(G(ʒ), ʒG'(ʒ), ʒ^2G''(ʒ), ʒ^3G'''(ʒ); ʒ),$$

are univalent in D and fulfill the given third-order differential superordination:

$$h(ʒ) \prec \Theta(G(ʒ), ʒG'(ʒ), ʒ^2G''(ʒ), ʒ^3G'''(ʒ); ʒ), \tag{2.2}$$

if $G(ʒ)$ satisfies differential superordination, it is considered a solution. Further, an analytic function $\mathcal{T}$ is referred to as a subordinant of the solutions of the differential superordination, or it's just a subordinant, if $\mathcal{T}(ʒ) \prec G(ʒ)$ in each $G(ʒ)$ fulfilling Equation (2.2). A univalent subordnant $\tilde{\mathcal{T}}(ʒ)$ which fulfill $\mathcal{T}(ʒ) \prec \tilde{\mathcal{T}}(ʒ)$ for every subordinants $\mathcal{T}(ʒ)$ of (2.2) is known to be the best subordinant.

Definition 2.6. [**7**]. Consider Ω, a set of the complex numbers, and let $\mathcal{T} \in H[a, n]$ such that $\mathcal{T}'(ʒ) \neq 0$. The function class $\Psi'_n[\Omega, \mathcal{T}]$ is defined as the set of functions $\Theta : \mathbb{C}^4 \times \overline{D} \to \mathbb{C}$ which fulfills the next admissibility conditions:

$$\Theta(\mathrm{r}, s, \mathrm{t}, \mathrm{u};\ \xi) \in \Omega,$$

whenever

$$\mathrm{r} = \mathcal{T}(ʒ), s = \frac{ʒ\mathcal{T}'(ʒ)}{m}, Re\left(\frac{\mathrm{t}}{s} + 1\right) \leq \frac{1}{m} Re\left(\frac{ʒ\mathcal{T}''(ʒ)}{\mathcal{T}'(ʒ)} + 1\right),$$

and

$$Re\left(\frac{\mathrm{u}}{s}\right) \leq \frac{1}{m^2} Re\left(\frac{ʒ^2\mathcal{T}'''(ʒ)}{\mathcal{T}'(ʒ)}\right),$$

where $ʒ \in D, \xi \in \partial D$, with $m \geq n \geq 2$.

Lemma 2.7. [7]. Consider $\mathcal{T} \in H[a, n]$, and $\Theta \in \Psi'_n[\Omega, \mathcal{T}]$. Assuming that

$$\Theta(G(ʒ), ʒG'(ʒ), ʒ^2G''(ʒ), ʒ^3G'''(ʒ); ʒ)$$

is a univalent within D with $G \in \mathbb{Q}(a)$ that fulfills the following conditions:

$$Re\left(\frac{\xi\mathcal{T}''(z)}{\mathcal{T}'(z)}\right) \geq 0, \left|\frac{zG'(z)}{\mathcal{T}'(z)}\right| \leq m,$$

where $z \in D, \xi \in \partial D$, with $m \geq n \geq 2$, then

$$\Omega \subset \{\Theta(G(z), zG'(z), z^2G''(z), z^3G'''(z); z): z \in D\},$$

indicates that

$$\mathcal{T}(z) \prec G(z), \quad (z \in D).$$

The present study applies the methods described in the works of Antonino and Miller [1], Jeyaraman and Suresh [23], and Tang et al. [6,25] to examine the third-order differential subordination and superordination outcomes. Various situations are considered, as documented in references [18,24]. This paper examines specific categories of permissible functions and presents novel findings about third-order differential subordination and superordination for analytic functions within the domain D, concerning the operator $K_{\eta,\lambda}^{\delta}f(z)$.

3- Results on third-order differential subordination

Here, we give differential subordination results using the operator $K_{\eta,\lambda}^{\delta}f(z)$.

Definition 3.1. Consider Ω as a set $\mathbb{C}$, and let $\mathcal{T} \in \mathbb{Q}_0 \cap H_0$. The function class $\Im_j[\Omega, \mathcal{T}]$ is defined as the set of functions $\Theta : \mathbb{C}^4 \times D \to \mathbb{C}$ which fulfills the admissibility condition:

$$\Theta(a, b, c, d; z) \notin \Omega,$$

whenever

$$a = \mathcal{T}(\xi), b = \frac{\xi k\mathcal{T}'(\xi) + (\eta+1)\mathcal{T}(\xi)}{\eta+2},$$

$$Re\left(\frac{(\eta+2)[(\eta+2)c-2(\eta+1)b]+(\eta+1)^2a}{(\eta+2)b-(\eta+1)a}\right) \geq kRe\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)}+1\right),$$

and

$$Re\left(\frac{(\eta+2)^3[d-3(c-b)]-[3(\eta+1)(\eta+2)a+(\eta+2)b]+[(\eta+1)a(1-(\eta+1)^2)]}{(\eta+2)b-(\eta+1)a}\right) \geq k^2Re\left(\frac{\xi^2\mathcal{T}'''(\xi)}{\mathcal{T}'(\xi)}\right),$$

where $z \in D, \xi \in \partial D \setminus E(\mathcal{T})$ with $k \in \mathbb{N} \setminus \{1\}$.

Theorem 3.2. Suppose that $\Theta \in \Im_j[\Omega, \mathcal{T}]$. Given that the functions $f \in A$ and $\mathcal{T} \in \mathbb{Q}_0 \cap H_0$, they both fulfill the next condition:

$$Re\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)}\right) \geq 0, \qquad \left|\frac{K_{\eta,\lambda}^{\delta-1}f(z)}{\mathcal{T}'(\xi)}\right| \leq k, \tag{3.1}$$

and

$$\{\Theta\left(K_{\eta,\lambda}^{\delta}f(z), K_{\eta,\lambda}^{\delta-1}f(z), K_{\eta,\lambda}^{\delta-2}f(z), K_{\eta,\lambda}^{\delta-3}f(z); z\right): z \in D\} \subset \Omega, \tag{3.2}$$

then

$$K_{\eta,\lambda}^{\delta}f(z) \prec \mathcal{T}(z), \qquad (z \in D).$$

Proof. Consider the function $G(z)$, which is analytic in D by

$$G(z) = K_{\eta,\lambda}^{\delta}f(z). \tag{3.3}$$

From Equation (1.4) with differentiating (3.3) concerning z, we obtain

$$K_{\eta,\lambda}^{\delta-1}f(z) = \frac{zG'(z) + (\eta+1)G(z)}{\eta+2}. \tag{3.4}$$

By a similar argument, yields

$$K_{\eta,\lambda}^{\delta-2}f(z) = \frac{z^2G''(z) + [2(\eta+1)+1]zG'(z) + (\eta+1)^2G(z)}{(\eta+2)^2}, \tag{3.5}$$

and

$$\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z}) = \frac{\mathfrak{z}^3 G'''(\mathfrak{z}) + 3(\eta+2)\mathfrak{z}^2 G''(\mathfrak{z}) + [3(\eta+1)(\eta+2)+1]\mathfrak{z} G'(\mathfrak{z}) + (\eta+1)^3 G(\mathfrak{z})}{(\eta+2)^3}. \tag{3.6}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(\mathfrak{r}, s, \mathfrak{t}, \mathfrak{u}) = \mathfrak{r}, \qquad b(\mathfrak{r}, s, \mathfrak{t}, \mathfrak{u}) = \frac{s + (\eta+1)\mathfrak{r}}{\eta+2},$$

$$c(\mathfrak{r}, s, \mathfrak{t}, \mathfrak{u}) = \frac{\mathfrak{t} + (2\eta+3)s + (\eta+1)^2 \mathfrak{r}}{(\eta+2)^2}, \tag{3.7}$$

and

$$d(\mathfrak{r}, s, \mathfrak{t}, \mathfrak{u}) = \frac{\mathfrak{u} + 3(\eta+2)\mathfrak{t} + [3(\eta+1)(\eta+2)+1]s + (\eta+1)^3 \mathfrak{r}}{(\eta+2)^3}. \tag{3.8}$$

Let $\mathfrak{X}(\mathfrak{r}, s, \mathfrak{t}, \mathfrak{u}) = \Theta(a, b, c, d) =$

$$\Theta\left(\begin{array}{c} \mathfrak{r}, \dfrac{s + (\eta+1)\mathfrak{r}}{\eta+2}, \dfrac{\mathfrak{t} + (2\eta+3)s + (\eta+1)^2 \mathfrak{r}}{(\eta+2)^2}, \\ \dfrac{\mathfrak{u} + 3(\eta+2)\mathfrak{t} + [3(\eta+1)(\eta+2)+1]s + (\eta+1)^3 \mathfrak{r}}{(\eta+2)^3}; \mathfrak{z} \end{array}\right). \tag{3.9}$$

The proof will be utilized in the following Lemma 2.4. By utilizing Equations (3.3) through (3.6), and also Equation (3.9), we obtain

$$\mathfrak{X}(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2 G''(\mathfrak{z}), \mathfrak{z}^3 G'''(\mathfrak{z}); \mathfrak{z}) = \Theta\big(\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z}); \mathfrak{z}\big). \tag{3.10}$$

Hence, Equation (3.2) leads to

$$\mathfrak{X}(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2 G''(\mathfrak{z}), \mathfrak{z}^3 G'''(\mathfrak{z}); \mathfrak{z}) \in \Omega,$$

note that

$$\frac{\mathfrak{t}}{s} + 1 = \frac{(\eta+2)[(\eta+2)c - 2(\eta+1)b] + (\eta+1)^2 a}{(\eta+2)b - (\eta+1)a}$$

and

$$\frac{\mathfrak{u}}{s} = \frac{(\eta+2)^3[d - 3(c-b)] - (\eta+2)[3(\eta+1)a + b] + [(\eta+1)a(1-(\eta+1)^2)]}{(\eta+2)b - (\eta+1)a}.$$

As a result, the admissibility condition in Definition 3.1, for $\Theta \in \mathfrak{J}_j[\Omega, \mathcal{T}]$ is equivalent to the condition $\mathfrak{X} \in \Psi_2[\Omega, \mathcal{T}]$ as stated in Definition 2.3, such that $n = 2$. As a result, using Equation (3.1) and Lemma 2.4, we get

$$\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z}).$$

This completes the proof.

The next result is a generalization of Theorem 3.2 to the case when the effects of $\mathcal{T}(\mathfrak{z})$ on ∂D is unknown.

Corollary 3.3. Consider $\Omega \subset \mathbb{C}$ and the function $\mathcal{T}$ be univalent in D with $\mathcal{T}(0) = 1$. Assume that $\Theta \in \mathfrak{J}_j[\Omega, \mathcal{T}_\rho]$ where $\rho \in (0,1)$, and $\mathcal{T}_\rho(\mathfrak{z}) = \mathcal{T}(\rho\mathfrak{z})$. Given the function $f \in A$ as well as $\mathcal{T}_\rho \in \mathbb{Q}_0$, and if they satisfy the following conditions:

$$Re\left(\frac{\xi \mathcal{T}_\rho''(\xi)}{\mathcal{T}_\rho'(\xi)}\right) \geq 0, \left|\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathcal{T}_\rho'(\xi)}\right| \leq k, \big(\mathfrak{z} \in D, \xi \in \partial D \backslash E(\mathcal{T}_\rho) \text{with } k \geq 2.\big),$$

and

$$\Theta\big(\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z}); \mathfrak{z}\big) \in \Omega,$$

then

$$\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}) \prec \mathcal{T}_\rho(\mathfrak{z}), (\mathfrak{z} \in D).$$

Proof. Using the Theorem 3.2, we obtain

$$K_{\eta,\lambda}^{\delta} f(z) \prec \mathcal{T}_\rho(z), (z \in D).$$

Corollary asserts the following conclusion that Equation (3.1) is now deduced from the subordination characteristic that follows: $\mathcal{T}_\rho(z) \prec \mathcal{T}(z)$, $(z \in D)$.

If $\Omega \neq \mathbb{C}$ is a domain with only one connection, then $\Omega = h(D)$ for the purpose of conformal mapping $h(z)$ of D onto Ω. The class in this situation is $\Im_j[h(D), \mathcal{T}]$ is written as $\Im_j\ [h, \mathcal{T}]$. This is a direct result of the Theorem 3.2 and Corollary 3.3.

Theorem 3.4. Suppose that $\Theta \in \Im_j[h, \mathcal{T}]$. Let $f \in A$, and let $\mathcal{T} \in \mathbb{Q}_0 \cap H_0$, and they fulfill the following conditions:

$$Re\left(\frac{\xi \mathcal{T}_\rho''(\xi)}{\mathcal{T}_\rho'(\xi)}\right) \geq 0\,, \left|\frac{K_{\eta,\lambda}^{\delta-1} f(z)}{\mathcal{T}_\rho'(\xi)}\right| \leq k, \tag{3.11}$$

and

$$\Theta\left(K_{\eta,\lambda}^{\delta} f(z), K_{\eta,\lambda}^{\delta-1} f(z), K_{\eta,\lambda}^{\delta-2} f(z), K_{\eta,\lambda}^{\delta-3} f(z); z\right) \prec h(z). \tag{3.12}$$

Then

$$K_{\eta,\lambda}^{\delta-1} f(z) \prec \mathcal{T}(z)\,, (z \in D).$$

The subsequent outcome is a direct consequence of Corollary 3.3.

Corollary 3.5. Consider $\Omega \subset \mathbb{C}$ and the function $\mathcal{T}$ be univalent in D with $\mathcal{T}(0) = 1$. Assume that $\Theta \in \Im_j[\Omega, \mathcal{T}]$ where $\rho \in (0,1)$, and $\mathcal{T}_\rho(z) = \mathcal{T}(\rho z)$. Given the function $f \in A$ and $\mathcal{T}_\rho$, and if they satisfy the following conditions:

$$Re\left(\frac{\xi \mathcal{T}_\rho''(\xi)}{\mathcal{T}_\rho'(\xi)}\right) \geq 0\,, \qquad \left|\frac{K_{\eta,\lambda}^{\delta-1} f(z)}{\mathcal{T}_\rho'(\xi)}\right| \leq k\,, (z \in D, \xi \in \partial D \backslash E(\mathcal{T}_\rho) \text{ and } k \geq 2)$$

and

$$\Theta\left(K_{\eta,\lambda}^{\delta} f(z), K_{\eta,\lambda}^{\delta-1} f(z), K_{\eta,\lambda}^{\delta-2} f(z), K_{\eta,\lambda}^{\delta-3} f(z); z\right) \prec h(z),$$

thus

$$K_{\eta,\lambda}^{\delta-1} f(z) \prec \mathcal{T}(z), \qquad (z \in D).$$

The following theorem makes the differential subordination (3.12) of the best dominant.

Theorem 3.6. Consider the function h in D. Assume that h is univalent. Let $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$ and $\mathfrak{X}$ be given by Equation (3.10). Consider the equation of differentiation:

$$\mathfrak{X}(G(z), zG'(z), z^2 G''(z), z^3 G'''(z); z) = h(z). \tag{3.13}$$

Then there exists a solution $\mathcal{T}(z)$ such that $\mathcal{T}(0) = 1$, that fulfills Equation (3.1). If $f \in A$ and fulfills condition (3.12), such that

$$\Theta\left(K_{\eta,\lambda}^{\delta} f(z), K_{\eta,\lambda}^{\delta-1} f(z), K_{\eta,\lambda}^{\delta-2} f(z), K_{\eta,\lambda}^{\delta-3} f(z); z\right).$$

If a function is analytic in the region D, then

$$K_{\eta,\lambda}^{\delta} f(z) \prec \mathcal{T}(z), \qquad (z \in D)$$

and $\mathcal{T}(z)$ is the best dominant.

Proof. According to Theorem 3.2, $\mathcal{T}$ is a dominant of Equation (3.12). Since $\mathcal{T}$ fulfills Equation (3.13), it this is additionally a solution to Equation (3.12). Thus, $\mathcal{T}$ will be dominated by any dominants. Therefore, $\mathcal{T}$ is the best dominant.

The proof of the theorem is completed.

Utilizing Definition 3.1, and the special case $\mathcal{T}(z) = Mz$ ($M > 0$), the class of admissible functions $\Im_j[\Omega, \mathcal{T}]$, given as $\Im_j[\Omega, M]$, expresses itself as follows.

Definition 3.7. Consider Ω as a set $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Im_j[\Omega, M]$ consists of functions $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$ that fulfills the condition

$$\Theta\left(\begin{array}{c} Me^{i\theta}, \dfrac{\big(k+(\eta+1)\big)Me^{i\theta}}{\eta+2}, \dfrac{L+[[2(\eta+1)+1]k+(\eta+1)^2]Me^{i\theta}}{(\eta+2)^2}, \\ \dfrac{N+3(\eta+2)L+[(3(\eta+1)(\eta+2)+1)k+(\eta+1)^3]Me^{i\theta}}{(\eta+2)^3}; ʒ \end{array}\right) \notin \Omega, \qquad (3.14)$$

whenever $ʒ \in D$,

$$Re\big(Le^{-i\theta}\big) \geq kM(k-1),$$

and

$$Re\big(Ne^{-i\theta}\big) \geq 0, \qquad \forall \theta \in \mathbb{R}, k \geq 2.$$

Corollary 3.8. Consider $\Theta \in \Im_j[\Omega, M]$. Given a function $f \in A$ satisfies:

$$\left|K_{\eta,\lambda}^{\delta-1} f(ʒ)\right| \leq kM, \qquad (ʒ \in D, k \geq 2; M > 0)$$

and

$$\Theta\big(K_{\eta,\lambda}^{\delta} f(ʒ), K_{\eta,\lambda}^{\delta-1} f(ʒ), K_{\eta,\lambda}^{\delta-2} f(ʒ), K_{\eta,\lambda}^{\delta-3} f(ʒ); ʒ\big) \in \Omega,$$

then

$$\left|K_{\eta,\lambda}^{\delta} f(ʒ)\right| < M.$$

In the particular case where $\Omega = \mathcal{T}(D) = \{w: |w| < M\}$, the class $\Im_j[\Omega, \mathcal{T}]$ is simply referred to as $\Im_j[M]$. Corollary 3.8 can now be used as follows:

Corollary 3.9. Assume that $\Theta \in \Im_j[M]$. If the function $f \in A$ and fulfills the following conditions:

$$\left|K_{\eta,\lambda}^{\delta-1} f(ʒ)\right| \leq kM, \qquad (ʒ \in D, k \geq 2; M > 0)$$

and

$$\left|\big(K_{\eta,\lambda}^{\delta} f(ʒ), K_{\eta,\lambda}^{\delta-1} f(ʒ), K_{\eta,\lambda}^{\delta-2} f(ʒ), K_{\eta,\lambda}^{\delta-3} f(ʒ); ʒ\big)\right| < M,$$

then

$$\left|K_{\eta,\lambda}^{\delta} f(ʒ)\right| < M.$$

Corollary 3.10. Given that $k \geq 2, and\ M > 0$. If the function $f \in A$ and satisfies its conditions:

$$\left|K_{\eta,\lambda}^{\delta-1} f(ʒ)\right| \leq kM,$$

and

$$\left|K_{\eta,\lambda}^{\delta-1} f(ʒ) - K_{\eta,\lambda}^{\delta} f(ʒ)\right| \leq \frac{M}{\eta+2},$$

then

$$\left|K_{\eta,\lambda}^{\delta} f(ʒ)\right| \leq M.$$

Proof. Consider $\Theta(a, b, c, d; ʒ) = b - a,\ \Omega = h(D)$, with $h(ʒ) = \frac{Mʒ}{\eta+2}$, $ʒ \in D, M > 0$.

Make use of the Corollary 3.8. We must prove it $\Theta \in \Im_j[\Omega, M]$, in other words, the admissibility condition of Equation (3.14) is fulfilled. This follows readily, since it is seen that

$$|\Theta(a, b, c, d; ʒ)| = \left|\frac{(k-1)}{\eta+2} Me^{i\theta}\right| = \frac{k-1}{\eta+2} M \geq \frac{M}{\eta+2},$$

whenever $ʒ \in D, \theta \in \mathbb{R}$ and $k \geq 2$.

Definition 3.11. Consider Ω as a set in $\mathbb{C}$, and let $\mathcal{T} \in \mathbb{Q}_1 \cap H_1$. The class $\Im_{j,1}[\Omega, \mathcal{T}]$ of functions that are admissible consists of those functions $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$, which fulfills the given admissibility conditions:

$$\Theta(a, b, c, d; z) \notin \Omega,$$

whenever

$$a = \mathcal{T}(\xi), \qquad b = \frac{k\xi\mathcal{T}'(\xi) + (\eta + 2)\mathcal{T}(\xi)}{\eta + 2},$$

$$Re\left(\frac{(\eta + 2)[a + c - 2b]}{b - a}\right) \geq kRe\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)} + 1\right),$$

and

$$Re\left(\frac{(\eta+2)^2(d-a)-3(\eta+2)(\eta+3)(c-a)+(b-a)[3(\eta+3)^2-1]}{b-a}\right) \geq k^2 Re\left(\frac{\xi^2\mathcal{T}'''(\xi)}{\mathcal{T}'(\xi)}\right),$$

where $z \in D, \xi \in \partial D \backslash E(\mathcal{T})$ and $k \geq 2$.

Theorem 3.12. Consider a function $\Theta \in \Im_{j,1}[\Omega, \mathcal{T}]$. If $f \in A$ be a function and $\mathcal{T} \in \mathbb{Q}_1 \cap H_1$, fulfilling the given conditions:

$$Re\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)}\right) \geq 0, \qquad \left|\frac{K_{\eta,\lambda}^{\delta-1}f(z)}{z\mathcal{T}(z)}\right| \leq k, \tag{3.15}$$

and

$$\left\{\Theta\left(\frac{K_{\eta,\lambda}^{\delta}f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-1}f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-2}f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-3}f(z)}{z}; z\right) : z \in D\right\} \subset \Omega, \tag{3.16}$$

thus

$$\frac{K_{\eta,\lambda}^{\delta}f(z)}{z} \prec \mathcal{T}(z), \qquad (z \in D).$$

Proof. The analytic function should be defined as $G(z)$ within D by

$$G(z) = \frac{K_{\eta,\lambda}^{\delta}f(z)}{z}. \tag{3.17}$$

Using the Equations (1.4) and (3.17), we have

$$\frac{K_{\eta,\lambda}^{\delta-1}f(z)}{z} = \frac{zG'(z) + (\eta + 2)G(z)}{\eta + 2}. \tag{3.18}$$

Using a comparable line of reasoning, we obtain

$$\frac{K_{\eta,\lambda}^{\delta-2}f(z)}{z} = \frac{z^2G''(z) + [2(\eta + 2) + 1]zG'(z) + (\eta + 2)^2G(z)}{(\eta + 2)^2}, \tag{3.19}$$

and

$$\frac{K_{\eta,\lambda}^{\delta-3}f(z)}{z} = \frac{z^3G'''(z) + 3(\eta + 3)z^2G''(z) + [3(\eta + 2)(\eta + 3) + 1]zG'(z) + (\eta + 2)^3G(z)}{(\eta + 2)^3}. \tag{3.20}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r, s, t, u) = r, \qquad b(r, s, t, u) = \frac{s + (\eta + 2)r}{\eta + 2},$$

$$c(r, s, t, u) = \frac{t + (2\eta + 5)s + (\eta + 2)^2r}{(\eta + 2)^2}, \tag{3.21}$$

and

$$d(r, s, t, u) = \frac{u + 3(\eta + 2)t + [3(\eta + 2)(\eta + 3) + 1]s + (\eta + 2)^3r}{(\eta + 2)^3}. \tag{3.22}$$

Let

$$\mathfrak{X}(\mathrm{r}, s, \mathrm{t}, \mathrm{u}) = \Theta(a, b, c, d; z) =$$

$$\Theta\left(\begin{array}{c} \mathrm{r}, \dfrac{s + (\eta+2)\mathrm{r}}{\eta+2}, \dfrac{\mathrm{t} + [2(\eta+2)+1]s + (\eta+2)^2\mathrm{r}}{(\eta+2)^2}, \\ \dfrac{\mathrm{u} + 3(\eta+2)\mathrm{t} + [3(\eta+2)(\eta+3)+1]s + (\eta+2)^3\mathrm{r}}{(\eta+2)^3}; z \end{array}\right). \tag{3.23}$$

The proof will utilize Lemma 2.4. Equations are used from the Equation (3.17) to (3.20), and from the Equation (3.23), we have

$$\mathfrak{X}(G(z), zG'(z), z^2G''(z), z^3G'''(z); z) =$$

$$\Theta\left(\frac{K_{\eta,\lambda}^{\delta} f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-1} f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-2} f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-3} f(z)}{z}; z\right). \tag{3.24}$$

Hence, Equation (3.16) becomes

$$\mathfrak{X}(G(z), zG'(z), z^2G''(z), z^3G'''(z); z) \in \Omega,$$

Note that,

$$\frac{\mathrm{t}}{s} + 1 = \frac{(\eta+2)[a + c - 2b]}{b - a},$$

and

$$\frac{\mathrm{u}}{s} = \frac{(\eta+2)^2(d-a) - 3(\eta+2)(\eta+3)(c-a) + (b-a)[3(\eta+3)^2 - 1]}{b - a}.$$

As a result, the admissibility condition for $\Theta \in \mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$ in Definition 3.11 is the same as the admissibility criterion for $\mathfrak{X} \in \Psi_2[\Omega, \mathcal{T}]$ as stated in the Definition 2.3 with $n = 2$. As a result, using Equation (3.13) and Lemma 2.4, we get

$$\frac{K_{\eta,\lambda}^{\delta} f(z)}{z} \prec \mathcal{T}(z).$$

The proof is complete.

Assuming $\Omega \neq \mathbb{C}$, and is a simply connected domain, then $\Omega = h(D)$ for some conformal mapping $h(z)$ of D onto Ω. In this situation, the class $\mathfrak{I}_{j,1}[h(D), \mathcal{T}]$ can be written as $\mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$. This follows the immediate consequence of Theorem 3.12, as follows:

Theorem 3.13. Consider a function $\Theta \in \mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$. If the functions $f \in A$ and $\mathcal{T} \in \mathbb{Q}_1 \cap H_1$ fulfills the following conditions:

$$Re\left(\frac{\xi \mathcal{T}_\rho''(\xi)}{\mathcal{T}_\rho'(\xi)}\right) \geq 0, \qquad \left|\frac{K_{\eta,\lambda}^{\delta-1} f(z)}{z\mathcal{T}_\rho'(\xi)}\right| \leq k, \tag{3.25}$$

and

$$\Theta\left(\frac{K_{\eta,\lambda}^{\delta} f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-1} f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-2} f(z)}{z}, \frac{K_{\eta,\lambda}^{\delta-3} f(z)}{z}; z\right) \prec h(z), \tag{3.26}$$

then

$$\frac{K_{\eta,\lambda}^{\delta} f(z)}{z} \prec \mathcal{T}(z), \qquad (z \in D).$$

Given the Definition 3.11 and the specific case when $\mathcal{T}(z) = Mz$, $M > 0$, the class functions that are admissible $\mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$, written as $\mathfrak{I}_{j,1}[\Omega, M]$ is expressed as follows.

Definition 3.14. Consider Ω to be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$ consists functions $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$ that fulfill the following conditions:

$$\Theta\left(\begin{array}{c} Me^{i\theta}, \dfrac{k+(\eta+2)Me^{i\theta}}{\eta+2}, \dfrac{L+([(2\eta+5)]k+(\eta+2)^2)Me^{i\theta}}{(\eta+2)^2}, \\ \dfrac{N+3(\eta+3)L+([3(\eta+2)(\eta+3)+1]k+(\eta+2)^3)Me^{i\theta}}{(\eta+2)^3}; \mathfrak{z} \end{array}\right) \notin \Omega, \quad (3.27)$$

whenever

$$\mathfrak{z} \in D, \qquad Re\left(Le^{-i\theta}\right) \geq kM(k-1),$$

and

$$Re\left(Ne^{-i\theta}\right) \geq 0, \qquad \forall \theta \in \mathbb{R}; k \geq 2.$$

Corollary 3.15. Consider a function $\Theta \in \mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$. Let $f \in A$ that fulfills the following conditions:

$$\left|\frac{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}{\mathfrak{z}}\right| \leq kM, \qquad (\mathfrak{z} \in D, k \geq 2; M > 0),$$

and

$$\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}{\mathfrak{z}}; \mathfrak{z}\right) \in \Omega,$$

then

$$\left|\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathfrak{z}}\right| < M.$$

In the particular case where $\Omega = \mathcal{T}(D) = \{w: |w| < M\}$, the class $\mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$ is simply referred to as $\mathfrak{I}_{j,1}[M]$. Corollary 3.15 can now be used as follows:

Corollary 3.16. Consider a function $\Theta \in \mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$. If the function $f \in A$ and fulfills the following conditions:

$$\left|\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathfrak{z}}\right| \leq kM, \qquad (\mathfrak{z} \in D, k \geq 2; M > 0),$$

and

$$\left|\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}{\mathfrak{z}}; \mathfrak{z}\right)\right| < M,$$

thus

$$\left|\frac{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}{\mathfrak{z}}\right| < M.$$

Definition 3.17. Assume that $\mathcal{T} \in \mathbb{Q}_1 \cap H_1$, and consider Ω to be a set within $\mathbb{C}$. The class $\mathfrak{I}_{j,2}[\Omega, \mathcal{T}]$ of admissible functions contains functions $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$, which fulfills each of the following admissibility requirements:

$$\Theta(a, b, c, d; \mathfrak{z}) \notin \Omega,$$

whenever

$$a = \mathcal{T}(\xi), \qquad b = \frac{1}{\eta+2}\left[\frac{k\xi\mathcal{T}'(\xi) + (\eta+2)\big(\mathcal{T}(\xi)\big)^2}{\mathcal{T}(\xi)}\right],$$

$$Re\left(\frac{(\eta+2)[2a^2+cb-3ab]}{b-a}\right) \geq k\, Re\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)}+1\right),$$

and

$Re\Big(\big[bc(d-c)(\eta+2)^2 - b(\eta+2)^2(c-b)(1-b-c+3a) - 3(\eta+2)(c-b)b + 2(b-a) + 3a(\eta+2)(b-a) + (b-a)^2(\eta+2)\big((b-c)(\eta+2) - 3 - 4a(\eta+2)\big) + a^2(\eta+2)^2(b-a)\big](b-a)^{-1}\Big) \geq k^2 Re\left(\frac{\xi^2\mathcal{T}'''(\xi)}{\mathcal{T}'(\xi)}\right)$,

where $\mathfrak{z} \in D,\ \xi \in \partial D \backslash E(\mathcal{T})$ and $k \geq 2$.

Theorem 3.18. Consider the function $\Theta \in \Im_{j,2}[\Omega, \mathcal{T}]$. Given that the functions $f \in A$ and $\mathcal{T} \in \mathbb{Q}_1 \cap H_1$ fulfill the following conditions:

$$Re\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)}\right) \geq 0, \qquad \left|\frac{K_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}\right| \leq k, \tag{3.28}$$

and

$$\left\{\Theta\left(\frac{K_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta} f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-4} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}; \mathfrak{z}\right) : \mathfrak{z} \in D\right\} \subset \Omega, \tag{3.29}$$

then

$$\frac{K_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta} f(\mathfrak{z})} \prec \mathcal{T}(\mathfrak{z}), \qquad (\mathfrak{z} \in D).$$

Proof. The analytic function should be defined $G(\mathfrak{z})$ in D by

$$G(\mathfrak{z}) = \frac{K_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta} f(\mathfrak{z})}. \tag{3.30}$$

From Equations (1.4) and (3.30), we have

$$\frac{K_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})} = \frac{1}{\eta+2}\left[\frac{\mathfrak{z}G'(\mathfrak{z}) + (\eta+2)G^2(\mathfrak{z})}{G(\mathfrak{z})}\right] = \frac{A}{\eta+2}. \tag{3.31}$$

By a similar argument, we have

$$\frac{K_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})} = \frac{B}{\eta+2}, \tag{3.32}$$

and

$$\frac{K_{\eta,\lambda}^{\delta-4} f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})} = \frac{1}{\eta+2}[B + B^{-1}(C + A^{-1}D - A^{-2}C^2)], \tag{3.33}$$

where

$$B = \frac{\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})} + (\eta+2)G(\mathfrak{z}) + \frac{\frac{\mathfrak{z}^2G''(\mathfrak{z}) + \mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})} - \left(\frac{\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})}\right)^2 + (\eta+2)\mathfrak{z}G'(\mathfrak{z})}{\frac{\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})} + (\eta+2)G(\mathfrak{z})},$$

$$C = \frac{\mathfrak{z}^2G''(\mathfrak{z}) + \mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})} - \left(\frac{\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})}\right)^2 + (\eta+2)\mathfrak{z}G'(\mathfrak{z}),$$

and

$$D = \frac{\mathfrak{z}^3G'''(\mathfrak{z}) + 3\mathfrak{z}^2G''(\mathfrak{z}) + \mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})} - \frac{3\mathfrak{z}^2\left(G'(\mathfrak{z})\right)^2 + 3\mathfrak{z}^3G''(\mathfrak{z})G'(\mathfrak{z})}{\left(G(\mathfrak{z})\right)^2} + 2\left(\frac{\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})}\right)^3 + (\eta+2)\mathfrak{z}^2G''(\mathfrak{z}) + (\eta+2)\mathfrak{z}G'(\mathfrak{z}).$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(\mathfrak{r}, s, \mathfrak{t}, \mathfrak{u}) = \mathfrak{r}, \qquad b(\mathfrak{r}, s, \mathfrak{t}, \mathfrak{u}) = \frac{1}{\eta+2}\left[\frac{s + (\eta+2)\mathfrak{r}^2}{\alpha}\right] = \frac{E}{\eta+2},$$

$$c(\mathfrak{r},s,\mathfrak{t},\mathfrak{u}) = \frac{1}{\eta+2}\left[\frac{s+(\eta+2)\mathfrak{r}^2}{\mathfrak{r}} + \frac{\frac{\mathfrak{t}+s}{\mathfrak{r}} - \left(\frac{s}{\mathfrak{r}}\right)^2 + (\eta+2)s}{\frac{s}{\mathfrak{r}} + (\eta+2)\mathfrak{r}}\right]$$

$$= \frac{F}{\eta+2}, \qquad (3.34)$$

and

$$d(\mathfrak{r},s,\mathfrak{t},\mathfrak{u}) = \frac{1}{\eta+2}[F + F^{-1}(L + HE^{-1} - E^{-2}L^2)], \qquad (3.35)$$

where

$$L = \frac{\mathfrak{t}+s}{\mathfrak{r}} - \left(\frac{s}{\mathfrak{r}}\right)^2 + (\eta+2)s,$$

and

$$H = \frac{\mathfrak{u}+3\mathfrak{t}+s}{\mathfrak{r}} - 3\left(\frac{s}{\mathfrak{r}}\right)^2 - 3\frac{s\mathfrak{t}}{\mathfrak{r}^2} + 2\left(\frac{s}{\mathfrak{r}}\right)^3 + (\eta+2)(s+\mathfrak{t}).$$

Let

$$\mathfrak{X}(\mathfrak{r},s,\mathfrak{t},\mathfrak{u}) = \Theta(a,b,c,d) =$$
$$\Theta\left(\mathfrak{r}, \frac{E}{\eta+2}, \frac{F}{\eta+2}, \frac{1}{\eta+2}[F + F^{-1}(L + HE^{-1} - E^{-2}L^2)]\right). \qquad (3.36)$$

The proof will make use of Lemma 2.4. By utilizing Equations (3.30) through Equation (3.33), and by using Equation (3.36), we get

$$\mathfrak{X}(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2G''(\mathfrak{z}), \mathfrak{z}^3G'''(\mathfrak{z}); \mathfrak{z}) =$$
$$\Theta\left(\frac{K_{\eta,\lambda}^{\delta-1}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta}f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-2}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-1}f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-3}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-2}f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-4}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-3}f(\mathfrak{z})}; \mathfrak{z}\right). \qquad (3.37)$$

Hence, Equation (3.29) leads to

$$\mathfrak{X}(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2G''(\mathfrak{z}), \mathfrak{z}^3G'''(\mathfrak{z}); \mathfrak{z}) \in \Omega.$$

We note that

$$\frac{\mathfrak{t}}{s} + 1 = \frac{(\eta+2)[2a^2 + cb - 3ab]}{b-a},$$

and

$\frac{\mathfrak{u}}{s} = \big[bc(d-c)(\eta+2)^2 - b(\eta+2)^2(c-b)(1-b-c+3a) - 3(\eta+2)(c-b)b + 2(b-a) + 3a(\eta+2)(b-a) + (b-a)^2(\eta+2)\big((b-c)(\eta+2) - 3 - 4a(\eta+2)\big) + a^2(\eta+2)^2(b-a)\big](b-a)^{-1}.$

Thus, the admissibility condition for $\Theta \in \mathfrak{I}_{j,2}[\Omega, \mathcal{T}]$ in Definition 3.17 is the same as the criteria of admissibility for $\Theta \in \Psi_2[\Omega, \mathrm{q}]$ as stated within the Definition 2.3 with $n = 2$. As a result, by utilizing Equation (3.30) and Lemma 2.4, we obtain

$$\frac{K_{\eta,\lambda}^{\delta-1}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta}f(\mathfrak{z})} \prec \mathcal{T}(\mathfrak{z}).$$

This completes the proof.

Assuming $\Omega \neq \mathbb{C}$, and is a simply connected domain, then $\Omega = h(D)$ for some conformal mapping $h(\mathfrak{z})$ of D onto Ω. In this situation, the class $\mathfrak{I}_{j,2}[h(D), \mathcal{T}]$ can be written as $\mathfrak{I}_{j,2}[\Omega, \mathcal{T}]$. This follows the immediate consequence of Theorem 3.18 which is stated below without proof.

Theorem 3.19. Consider a function $\Theta \in \mathfrak{I}_{j,2}[\Omega, \mathcal{T}]$. Given that the functions $f \in A$ with $\mathcal{T} \in \mathbb{Q}_1$, and they fulfill the following conditions (3.29) and

$$\Theta\left(\frac{K_{\eta,\lambda}^{\delta-1}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta}f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-2}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-1}f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-3}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-2}f(\mathfrak{z})}, \frac{K_{\eta,\lambda}^{\delta-4}f(\mathfrak{z})}{K_{\eta,\lambda}^{\delta-3}f(\mathfrak{z})}; \mathfrak{z}\right) \prec h(\mathfrak{z}),$$

therefore,

$$\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1}f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)} \prec \mathcal{T}(z), \qquad (z \in D).$$

4- Results on third-order differential superordination

Definition 4.1. Consider Ω as a set in the complex plane $\mathbb{C}$, and let $\mathcal{T} \in \mathbb{Q}_0 \cap H_0$ such that $\mathcal{T}'(z) \neq 0$. The class of admissible functions $\mathfrak{J}_j'[\Omega, \mathcal{T}]$ includes functions $\Theta: \mathbb{C}^4 \times D \to \mathbb{C}$, which fulfills the particular admissibility conditions:

$$\Theta(a, b, c, d; \xi) \in \Omega,$$

whenever

$$a = \mathcal{T}(z), \qquad b = \frac{z\mathcal{T}'(z) + m\,(\eta+1)\mathcal{T}(z)}{m\,(\eta+2)},$$

$$Re\left(\frac{(\eta+2)[(\eta+2)c-2(\eta+1)b]+(\eta+1)^2a]}{(\eta+2)b-(\eta+1)a}\right) \leq \frac{1}{m} Re\left(\frac{z\mathcal{T}''(z)}{\mathcal{T}'(z)}+1\right),$$

and

$$Re\left(\frac{(\eta+2)^3[d-3(c-b)]-(\eta+2)[3(\eta+1)a+b]+[(\eta+1)a(1-(\eta+1)^2)]}{(\eta+2)b-(\eta+1)a}\right) \leq \frac{1}{m^2} Re\left(\frac{z^2\mathcal{T}'''(z)}{\mathcal{T}'(z)}\right),$$

where $z \in D, \xi \in \partial D$ and $m \geq 2$.

Theorem 4.2. Let $\Theta \in \mathfrak{J}_j'[\Omega, \mathcal{T}]$. If the functions $f \in A$, with $\mathrm{K}_{\eta,\lambda}^{\delta}f(z) \in \mathbb{Q}_0$ and if $\mathcal{T} \in H_0$ and $\mathcal{T}'(z) \neq 0$, fulfilling the following conditions:

$$Re\left(\frac{\xi\mathcal{T}''(z)}{\mathcal{T}'(z)}\right) \geq 0, \qquad \left|\frac{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)}{\mathcal{T}'(z)}\right| \leq m, \tag{4.1}$$

and the function

$$\Theta\left(\mathrm{K}_{\eta,\lambda}^{\delta}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-1}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-2}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-3}f(z); z\right),$$

is univalent within D, thus

$$\Omega \subset \left\{\Theta(\mathrm{K}_{\eta,\lambda}^{\delta}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-1}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-2}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-3}f(z); z \in D\right\}, \tag{4.2}$$

implies that

$$\mathcal{T}(z) \prec \mathrm{K}_{\eta,\lambda}^{\delta}f(z), \qquad (z \in D).$$

Proof. The function $G(z)$ is given by Equation (3.3), while $\mathfrak{X}$ is defined by Equation (3.8). Since $\Theta \in \mathfrak{J}_j'[\Omega, \mathcal{T}]$. Using Equations (3.10) and (4.2), we get

$$\Omega \subset \{\Theta(G(z), zG'(z), z^2G''(z), z^3G'''(z); z): z \in D\}.$$

Using equations (3.7) and (3.8), We can observe that the admissibility condition for $\Theta \in \mathfrak{J}_j'[\Omega, \mathcal{T}]$ in Definition 4.1 is the same as the admissibility criterion for $\mathfrak{X} \in \Psi_n'[\Omega, \mathcal{T}]$ as stated in the Definition 2.6 such that $n = 2$. Hence $\mathfrak{X} \in \Psi_2'[\Omega, \mathcal{T}]$ as well as by Equation (4.2) and Lemma 2.7, we obtain

$$\mathcal{T}(z) \prec \mathrm{K}_{\eta,\lambda}^{\delta}f(z), \qquad (z \in D).$$

This completes the proof.

Assuming $\Omega \neq \mathbb{C}$, and is a simply connected domain, then $\Omega = h(D)$ for some conformal mapping $h(z)$ of D onto Ω. In this situation, the class $\mathfrak{J}_j{}'[h(D), \mathcal{T}]$ can be written as $\mathfrak{J}_j{}'[\mathrm{h}, \mathcal{T}]$. This follows the immediate consequence of Theorem 4.2 which is stated below.

Theorem 4.3. Suppose that $\Theta \in \mathfrak{J}_j{}'[\mathrm{h}, \mathcal{T}]$ and h be analytic in D. If the function $f \in A$, with $\mathrm{K}_{\eta,\lambda}^{\delta}f(z) \in \mathbb{Q}_0$ and $\mathcal{T} \in H_0$ and $\mathcal{T}'(z) \neq 0$, fulfilling the following conditions (4.1) and the function

$$\Theta\left(\mathrm{K}_{\eta,\lambda}^{\delta}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-1}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-2}f(z), \mathrm{K}_{\eta,\lambda}^{\delta-3}f(z); z\right),$$

is univalent within D, thus

$$h(\mathfrak{z}) \prec \Theta\left(\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z}); \mathfrak{z}\right), \tag{4.3}$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}), \qquad (\mathfrak{z} \in D).$$

Theorem 4.2 and 4.3 may only be utilized to get third-order differential superordination of the form subordination of Equation (4.2) or (4.3).

The next theorem shows the existence of the best subordinant of Equation (4.3) with suitable Θ.

Theorem 4.4. Consider a function h that is univalent function within D and $\Theta: \mathbb{C}^4 \times \overline{D} \to \mathbb{C}$ with $\mathfrak{X}$ be given by Equation (3.9). Consider the given differential equation:

$$\mathfrak{X}(\mathcal{T}(\mathfrak{z}), \mathfrak{z}\mathcal{T}'(\mathfrak{z}), \mathfrak{z}^2\mathcal{T}''(\mathfrak{z}), \mathfrak{z}^3\mathcal{T}'''(\mathfrak{z}); \mathfrak{z}) = h(\mathfrak{z}), \tag{4.4}$$

has a solution $\mathcal{T}(\mathfrak{z}) \in \mathbb{Q}_0$. If the functions $f \in A$, and $\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}) \in \mathbb{Q}_0$, and if $\mathcal{T} \in H_0$ with $\mathcal{T}'(\mathfrak{z}) \neq 0$, , and these functions fulfill the criteria Equation (4.1) and the function

$$\Theta\left(\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z}); \mathfrak{z}\right),$$

is analytic within D, thus

$$h(\mathfrak{z}) \prec \Theta\left(\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z}), \mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z}); \mathfrak{z}\right),$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z}), \qquad (\mathfrak{z} \in D)$$

with $\mathcal{T}(\mathfrak{z})$ is the best subordinant.

Proof. According to Theorem 4.2 and 4.3, it is clear that $\mathcal{T}$ is a subordinant of Equation (4.3). As $\mathcal{T}$ fulfills of Equation (4.4), it is also a solution of Equation (4.3). Thus, $\mathcal{T}$ will be subordinant by all subordinants. Hence, $\mathcal{T}$ is the best subordinant. Then proof of Theorem 4.4 well be completed.

Definition 4.5. Consider Ω as a set in the complex plane $\mathbb{C}$, and let $\mathcal{T} \in H_1$ such that $\mathcal{T}'(\mathfrak{z}) \neq 0$. The class of admissible functions $\mathfrak{I}'_{j,1}[\Omega, \mathcal{T}]$ includes functions $\Theta: \mathbb{C}^4 \times \overline{D} \to \mathbb{C}$, which fulfills each of the following admissibility requirements:

$$\Theta(a, b, c, d; \xi) \in \Omega,$$

whenever

$$a = \mathcal{T}(\mathfrak{z}), \qquad b = \frac{\mathfrak{z}\mathcal{T}'(\mathfrak{z}) + m\,(\eta+2)\mathcal{T}(\mathfrak{z})}{m\,(\eta+2)},$$

$$Re\left(\frac{(\eta+2)[c+a-2b]}{b-a}\right) \leq \frac{1}{m} Re\left(\frac{\mathfrak{z}\mathcal{T}''(\mathfrak{z})}{\mathcal{T}'(\mathfrak{z})} + 1\right),$$

and

$$Re\left(\frac{(\eta+2)^2(d-a) - 3(\eta+2)(\eta+3)(c-a) + (b-a)\left[3(\eta+3)^2 - 1\right]}{b-a}\right) \leq \frac{1}{m^2} Re\left(\frac{\mathfrak{z}^2\mathcal{T}'''(\mathfrak{z})}{\mathcal{T}'(\mathfrak{z})}\right),$$

where $\mathfrak{z} \in D, \xi \in \partial D$ with $m \geq 2$.

Theorem 4.6. Consider a function $\Theta \in \mathfrak{I}'_{j,1}[\Omega, \mathcal{T}]$. If the function $f \in A$ and $\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathfrak{z}} \in \mathbb{Q}_1$, and if $\mathcal{T} \in \mathbb{Q}_1 \cap H_1$ with $\mathcal{T}'(\mathfrak{z}) \neq 0$, it satisfies the following conditions:

$$Re\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathfrak{z}\mathcal{T}'(\mathfrak{z})}\right| \leq m, \tag{4.5}$$

and the function

$$\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-2}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-3}f(z)}{z};z\right),$$

is univalent within D, thus

$$\Omega\subset\left\{\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-2}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-3}f(z)}{z};z\right):z\in D\right\},\qquad(4.6)$$

implies that

$$\mathcal{T}(z)\prec\frac{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)}{z},\qquad(z\in D).$$

Proof. The function $G(z)$ is defined by Equation (3.17) and Θ is defined by Equation (3.23). Given that $\Theta\in\mathfrak{J}'_{j,1}[\Omega,\mathcal{T}]$, from Equations (3.24) and (4.6) we have the following:

$$\Omega\subset\{\mathfrak{X}(G(z),zG'(z),z^2G''(z),z^3G'''(z);z):z\in D\}.$$

Equations (3.21) and (3.22) demonstrate that admissibility is a necessary condition for $\Theta\in\mathfrak{J}'_{j,1}[\Omega,\mathcal{T}]$ by Definition 4.1 is equivalent to the admissibility condition for $\mathfrak{X}$ as defined in Definition 2.3 and $n=2$. Therefore, it follows that $\mathfrak{X}\in\Psi'_2[\Omega,\mathcal{T}]$, by utilizing (4.6) and Lemma 2.7, we obtain

$$\mathcal{T}(z)\prec\frac{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)}{z},\qquad(z\in D).$$

This completes the proof.

Assuming $\Omega\neq\mathbb{C}$, and is a simply connected domain, then $\Omega=h(D)$ for some conformal mapping $h(z)$ of D onto Ω. In this situation, the class $\mathfrak{J}'_{j,1}[h(D),\mathcal{T}]$ can be written as $\mathfrak{J}'_{j,1}[h,\mathcal{T}]$. This follows the immediate consequence of Theorem 4.6.

Theorem 4.7. Suppose that $\Theta\in\mathfrak{J}'_{j,1}[h,\mathcal{T}]$ and h is an analytic function in D. If the functions $f\in A$, with $\mathcal{T}\in H_1$ and $\mathcal{T}'(z)\neq 0$, and they satisfy criteria of Equation (4.5) with the function

$$\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-2}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-3}f(z)}{z};z\right),$$

is univalent within D, thus

$$h(z)\prec\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-2}f(z)}{z},\frac{\mathrm{K}_{\eta,\lambda}^{\delta-3}f(z)}{z};z\right),$$

implies that

$$\mathcal{T}(z)\prec\frac{\mathrm{K}_{\eta,\lambda}^{\delta}f(z)}{z},\qquad(z\in D).$$

Definition 4.8. Consider Ω as a set in $\mathbb{C}$, and let $\mathcal{T}\in H_1$ with $\mathcal{T}'(z)\neq 0$. The class $\mathfrak{J}'_{j,2}[\Omega,\mathcal{T}]$ of admissible functions $\mathfrak{J}'_{j,2}[\Omega,\mathcal{T}]$ includes functions $\Theta:\mathbb{C}^4\times\overline{D}\to\mathbb{C}$, which fulfill the given admissibility conditions:

$$\Theta(a,b,c,d;\xi)\in\Omega,$$

whenever

$$a=\mathcal{T}(z),\qquad b=\frac{1}{\eta+2}\left[\frac{z\mathcal{T}'(z)+m\,(\eta+2)\big(\mathcal{T}(z)\big)^2}{m\,\mathcal{T}(z)}\right],$$

$$Re\left(\frac{(\eta+2)[cb+2a^2-3ab]}{b-a}\right)\leq\frac{1}{m}\,Re\left(\frac{z\mathcal{T}''(z)}{\mathcal{T}'(z)}+1\right),$$

and

$Re\Big([bc(d-c)(\eta+2)^2 - b(\eta+2)^2(c-b)(1-b-c+3a) - 3(\eta+2)(c-b)b + 2(b-a) + 3a(\eta+2)(b-a) + (b-a)^2(\eta+2)\big((b-c)(\eta+2) - 3 - 4a(\eta+2)\big) + a^2(\eta+2)^2(b-a)](b-a)^{-1}\Big) \le \frac{1}{m^2} Re\left(\frac{\mathfrak{z}^2\mathcal{T}'''(\mathfrak{z})}{\mathcal{T}'(\mathfrak{z})}\right)$,
where $\mathfrak{z} \in D,\ \xi \in \partial D \backslash E(\mathcal{T})$ with $m \ge 2$.

Theorem 4.9. Consider a function $\Theta \in \Im'_{j,2}[\Omega, \mathcal{T}]$. If the function $f \in A$ and $\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})} \in \mathbb{Q}_1$ and if $\mathcal{T} \in H_1$ with $\mathcal{T}'(\mathfrak{z}) \neq 0$, given these conditions:

$$Re\left(\frac{\xi\mathcal{T}''(\xi)}{\mathcal{T}'(\xi)}\right) \ge 0, \qquad \left|\frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}\right| \le m, \tag{4.7}$$

and the function

$$\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-4} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}; \mathfrak{z}\right),$$

is univalent within D, thus

$$\Omega \subset \left\{\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-4} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}; \mathfrak{z}\right) : \mathfrak{z} \in D\right\}, \tag{4.8}$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}, \qquad (\mathfrak{z} \in D).$$

Proof. The function $G(\mathfrak{z})$ is defined by Equation (3.30), while Θ is defined by Equation (3.36). Given that $\Theta \in \Im'_{j,2}[\Omega, \mathcal{T}]$, we can deduce from Equations (3.37) and (4.8) that

$$\Omega \subset \{\mathfrak{X}(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2G''(\mathfrak{z}), \mathfrak{z}^3G'''(\mathfrak{z}); \mathfrak{z}) : \mathfrak{z} \in D\}.$$

From Equations (3.34) and (3.35), it is clear that admissibility is a requirement $\Theta \in \Im'_{j,2}[\Omega, \mathcal{T}]$ by Definition 4.8 is identical to the admissibility condition for $\mathfrak{X}$ as stated by Definition 2.6, and $n = 2$. Therefore, it follows that $\mathfrak{X} \in \Psi'_2[\Omega, \mathcal{T}]$, by utilizing (4.6) and Lemma 2.7, we obtain

$$\mathcal{T}(\mathfrak{z}) \prec \frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}, \qquad (\mathfrak{z} \in D).$$

This completes the proof.

Theorem 4.10. Consider a function $\Theta \in \Im'_{j,2}[\Omega, \mathcal{T}]$. If the function $f \in A$ and $\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})} \in \mathbb{Q}_1$, and if $\mathcal{T} \in \mathbb{Q}_1 \cap H_1$ such that $\mathcal{T}'(\mathfrak{z}) \neq 0$, fulfilling the given conditions (4.7), and the function

$$\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-4} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}; \mathfrak{z}\right),$$

is univalent within D, thus

$$h(\mathfrak{z}) \prec \Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(\mathfrak{z})}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-4} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(\mathfrak{z})}; \mathfrak{z}\right),$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(\mathfrak{z})}{\mathrm{K}_{\eta,\lambda}^{\delta} f(\mathfrak{z})}, \qquad (\mathfrak{z} \in D).$$

5- Sandwich results

By combining Theorems 3.4 and 4.3, we get the next sandwich-type theorem.

Theorem 5.1. Consider two analytic functions, h_1 and $\mathcal{T}_1$, defined in the unit disk D. Let h_2 be univalent function in D and $\mathcal{T}_2 \in \mathbb{Q}_0$ such that $\mathcal{T}_1(0) = \mathcal{T}_2(0) = 1$ and $\Theta \in \Im_j[h_2, \mathcal{T}_2] \cap \Im'_j[h_1, \mathcal{T}_1]$. If the function $f \in A$ with $\mathrm{K}_{\eta,\lambda}^{\delta} f(z) \in \mathbb{Q}_0 \cap H_0$ and the function

$$\Theta\left(\mathrm{K}_{\eta,\lambda}^{\delta} f(z), \mathrm{K}_{\eta,\lambda}^{\delta-1} f(z), \mathrm{K}_{\eta,\lambda}^{\delta-2} f(z), \mathrm{K}_{\eta,\lambda}^{\delta-3} f(z); z\right),$$

is univalent within D, and if the Equations (3.1) and (4.1) are fulfilled, thus

$$h_1(z) \prec \Theta\left(\mathrm{K}_{\eta,\lambda}^{\delta} f(z), \mathrm{K}_{\eta,\lambda}^{\delta-1} f(z), \mathrm{K}_{\eta,\lambda}^{\delta-2} f(z), \mathrm{K}_{\eta,\lambda}^{\delta-3} f(z); z\right) \prec h_2(z)$$

implies that

$$\mathcal{T}_1(z) \prec \mathrm{K}_{\eta,\lambda}^{\delta} f(z) \prec \mathcal{T}_2(z), \qquad (z \in D). \tag{5.1}$$

Combining Theorems 3.13 and 4.7, The next sandwich-type theorem is obtained.

Theorem 5.2. Consider two analytic functions, h_1 and $\mathcal{T}_1$, defined in the unit disk D. Let h_2 be univalent function in D and $\mathcal{T}_2 \in \mathbb{Q}_1$ such that $\mathcal{T}_1(0) = \mathcal{T}_2(0) = 1$ with $\Theta \in \Im_{j,1}[h_2, \mathcal{T}_2] \cap \Im'_{j,1}[h_1, \mathcal{T}_1]$. If the function $f \in A$ and $\frac{\mathrm{K}_{\eta,\lambda}^{\delta} f(z)}{z} \in \mathbb{Q}_1 \cap H_1$, and the function

$$\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta} f(z)}{z}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(z)}{z}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(z)}{z}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(z)}{z}; z\right),$$

is univalent in D, and the Equations (3.15) and (4.5) are fulfilled, we can conclude that

$$h_1(z) \prec \Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta} f(z)}{z}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(z)}{z}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(z)}{z}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(z)}{z}; z\right) \prec h_2(z),$$

implies that

$$\mathcal{T}_1(z) \prec \frac{\mathrm{K}_{\eta,\lambda}^{\delta} f(z)}{z} \prec \mathcal{T}_2(z), \qquad (z \in D). \tag{5.2}$$

Theorem 5.3. Consider two analytic functions, h_1 and $\mathcal{T}_1$, defined in the unit disk D. Let h_2 be univalent function in D and $\mathcal{T}_2 \in \mathbb{Q}_1$ such that $\mathcal{T}_1(0) = \mathcal{T}_2(0) = 1$ and $\Theta \in \Im_{j,2}[h_2, \mathcal{T}_2] \cap \Im'_{j,2}[h_1, \mathcal{T}_1]$. If the function $f \in A$ and $\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta} f(z)} \in \mathbb{Q}_1 \cap H_1$, and the function

$$\Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta} f(z)}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(z)}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(z)}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-4} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(z)}; z\right),$$

is univalent within D, and the Equations (3.28) and (4.7) are fulfilled, we can conclude that

$$h_1(z) \prec \Theta\left(\frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta} f(z)}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(z)}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta-2} f(z)}, \frac{\mathrm{K}_{\eta,\lambda}^{\delta-4} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta-3} f(z)}; z\right) \prec h_2(z),$$

implies that

$$\mathcal{T}_1(z) \prec \frac{\mathrm{K}_{\eta,\lambda}^{\delta-1} f(z)}{\mathrm{K}_{\eta,\lambda}^{\delta} f(z)} \prec \mathcal{T}_2(z), \qquad (z \in D). \tag{5.3}$$

6-Conclusions:

The study successfully extends the concepts of differential subordination and superordination to third-order scenarios for analytic functions within the open unit disk. By employing the new operator $\mathrm{K}_{\eta,\lambda}^{\delta} f(z)$ novel results and properties were discovered that contribute significantly to the existing body of knowledge in complex analysis. Future research could explore the application of the third-order differential subordination and superordination concepts in other branches of mathematics, such as geometric function theory and approximation theory. This could uncover new relationships and further extend the utility of the operator $\mathrm{K}_{\eta,\lambda}^{\delta} f(z)$. Investigating higher-order differential operators beyond the third order could reveal deeper insights and more generalized results. This progression could potentially lead to new mathematical models and theories that encapsulate a broader range of differential subordinations and superordinations.

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