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Bias Reduction of Maximum Likelihood Estimation In Generalized Ramos-Louzada Distribution

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Abstract

The generalized Ramos-Louzada distribution (GRL) is a potential option for modeling the survival data which considers flexibility while modeling data with decreasing, increasing, reversed-J shaped, and J-shaped hazard rate functions. It should however be noted that the most widely used technique of parameter estimation of GLR is the maximum likelihood technique (MLE). While the MLE is quite efficient in large samples, they are known to be highly biased for small samples. We are therefore forced to come up with essentially nonbiased estimators for GRL parameters. Particularly, we investigate two procedures of bias correction: bootstrap and analytical methods in order to minimize MLE bias up to the second-degree. Two real-world data applications and Monte Carlo simulations are used to countercheck the conclusions of these methods. Simulation results highlight the proposed approaches' performance, which is significantly less biased than the MLE, which is highly Biased. In small sample sizes, the bias is eliminated to about the half of the original amount.

Keywords: Bias correction; survival analysis; generalized Ramos-Louzada; bootstrap.

تقليل التحيز لتقدير الامكان الاعظم في توزيع راموس لوزادا المعمم

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الخلاصة

يوفر توزيع Ramos–Louzada المعمم (GRL) بديلاً لنمذجة بيانات البقاء على قيد الحياة مع مراعاة المرونة في نمذجة البيانات مع وظائف معدل الخطر المتزايدة والمتناقصة على شكل والعكس على شكل . الطريقة الأكثر شيوعًا لتقدير معلمات GLR هي طريقة الامكان الاعظم (MLE). من ناحية أخرى، فإن MLE متحيز بشكل كبير لأحجام العينات الصغيرة. لذلك نحن نقوم لإنشاء مقدرات غير متحيزة تقريبًا لمعلمات GRL. وبشكل أكثر تحديدًا، نحن نركز على طريقتين لتصحيح التحيز ، اسلوب البوستراب والتحليلي، لتقليل تحيزات MLEإلى الدرجة الثانية من التحيز. تتم مقارنة أداء هذه الأساليب من خلال محاكاة مونت كارلو وتطبيقين حقيقيين للبيانات.

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1. Introduction

In statistics and epidemiology, modeling survival data is a crucial activity that is frequently used to assess time-to-event data, such as the amount of time until a certain event occurs, the time until a component fails, or the time until death. The statistical method used to model and examine such data is called survival analysis. A key idea in survival analysis is a survival distribution. Taking into account the existing facts, it describes the likelihood that a given event (such as failure, death, or any other outcome) will not occur before a given time point.

A class of probability distributions known as parametric survival distributions is employed in survival analysis to model time-to-event data. Parametric models assume a certain functional form for the survival distribution, in contrast to non-parametric techniques such as the Kaplan-Meier estimator, which make minimal assumptions about the underlying distribution. Certain aspects and interpretations of these models are essential for comprehending and evaluating survival data.

Ramos and Louzada [1] recently introduced Ramos–Louzada (RL) distribution with survival function as

$$S(x\,|\alpha)=\left(\frac{1}{\alpha-1}\right)\left(\alpha-1+\frac{x}{\alpha}\right)e^{-\frac{x}{\alpha}},\quad x>0,\quad \alpha\geq 2. \tag{1}$$

Generalized Ramos-Louzada (GRL), a novel two-parameter model for the RL distribution that incorporates a power parameter in the basic model, was presented by Al-Mofleh, et al. [2]. Let X be a positive random variable that follows the GRL model, the survival function of random variable X is given by

$$S(x\,|\alpha,\theta)=\left(\frac{1}{\alpha-1}\right)\left(\alpha-1+\frac{x^{\theta}}{\alpha}\right)e^{-\frac{x^{\theta}}{\alpha}}, \tag{2}$$

where $\alpha\geq 2$ and $\theta>0$ are shape parameters. According to Eq. (2), the probability density function (PDF) and CDF of GRL is given, respectively, by

$$f(x;\tau)=\frac{\theta}{\alpha(\alpha-1)}x^{\theta-1}\left(\alpha+\frac{x^{\theta}}{\alpha}-2\right)e^{-\frac{x^{\theta}}{\alpha}},\quad x>0 \tag{3}$$

$$F(x;\tau)=1-\left(\frac{1}{\alpha-1}\right)\left(\alpha-1+\frac{x^{\theta}}{\alpha}\right)e^{-\frac{x^{\theta}}{\alpha}}, \tag{4}$$

where $\tau=(\theta,\alpha)$.

One of the approaches discussed in this paper that should be mentioned is a kind of 'correction,' - adjustment, which is capable to correct the bias up to the second order. The rationale of this 'corrective' approach is that MLE is inconsistent because of the influence of the bias and therefore the appropriate action is to remove this bias in order to get the so called bias corrected MLEs. It is illustrated that the presented bias-corrected MLE of the generalized Ramos-Louzada distribution not only has closed-form expressions in terms of a suitable matrix notation but also minimizes the biases, as well as the root mean square errors, of the parameters. Moreover, the last approach can be discussed in terms of Efron's bootstrap resampling method that can also minimize. It is maintained to the second order by the bias.

Our paper is tidy as follows. Section 2 covers the maximum likelihood estimation of the generalized Ramos-Louzada distribution. In Section 3, bias-corrected maximum likelihood estimation is summarized. The simulation and real application results are listed in Sections 4 and 5 respectively. The conclusion is provided in Section 6.

2. MLE

For sample of size n and $X = (x_1, x_2, ..., x_n)$ from the GRL distribution. The log-likelihood function is given by

$$l(\theta,\alpha) = nln(\theta) - 3nln(\alpha) - 2nln(\alpha - 1) + nln(\alpha - 2) + (3\theta - 2)\sum_{i=1}^{n}\ln(x_i) - \frac{2}{\alpha}\sum_{i=1}^{n}(x_i)^{\theta} \quad (5)$$

Maximize Eq. (5), we have the following equations:

$$\frac{\partial}{\partial\theta} l(\theta,\alpha) = \frac{2n}{\theta} + 3\sum_{i=1}^{n}\ln(x_i) - \frac{2}{\alpha}\sum_{i=1}^{n} x_i^{\theta}\ln(x_i) = 0 \quad (6)$$

$$\frac{\partial}{\partial\alpha} l(\theta,\alpha) = \frac{-3n}{\alpha} - \frac{2n}{(\alpha-1)} + \frac{n}{(\alpha-2)} + 2\frac{2n}{\alpha^2}\sum_{i=1}^{n} x_i^{\theta} = 0 \quad (7)$$

3. Bias-corrected MLEs

For the correction of the bias in the parameter as it is estimated, by the MLE method, the bias corrected maximum likelihood estimation (BCMLE) technique is applied. From the above relationship, when the true parameter value does not equal the average value of the estimates calculated using the large number of samples, then there is Bias in MLE. Since it would be more desirable to have methods that provide improved plug-in estimator, called bias-corrected approaches attempt to solve this systematic mistake or reduce it to a certain extent [3-5].

When evaluating the bias, the common methods are corrective approach and bootstrap which are normally applied [6]. This method is helpful in situations when bias might affect the statistical conclusion of a study. Following the above two guises, numerous authors in the literature addressed the BCMLE problem [7-36].

3.1. The Corrective approach

This approached was developed in part by [37]. Suppose $L(\tau) = l(\theta,\alpha)$, the joint cumulants of the derivatives of Eq. (5) are given by

$$M_{ij} = E\left[\frac{\partial^2 L(\tau)}{\partial\tau_i \partial\tau_j}\right], \quad (8)$$

$$M_{ijl} = E\left[\frac{\partial^3 L(\tau)}{\partial\tau_i \partial\tau_j d\tau_l}\right], \quad (9)$$

$$M_{ij,l} = E\left[\left(\frac{\partial^2 L}{\partial\tau_i \partial\tau_j}\right)\left(\frac{dL}{d\tau_l}\right)\right], \quad (10)$$

with

$$M_{ij}^{(l)} = \frac{\partial M_{ij}}{\partial\tau_l}, i, j, l = 1,2,3,......, p \quad (11)$$

The bias of the s^{th} element of the MLE of τ is, according to Cox and Snell [37],

$$Bias(\hat{\tau}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} M^{si} M^{jl}\left[\frac{1}{2}M_{ijl} + M_{ij,l}\right] + o(n^{-2}) \quad s = 1,2,......., p \quad (12)$$

where M^{ij} is the $(i, j)^{th}$ element of the Fisher information matrix inverted. According to Cordeiro and Cribari-Neto [3], they recommended Eq. (13)

$$Bias\ (\hat{\tau}_s) = \sum_{i=1}^{p} M^{si} \sum_{j=1}^{p}\sum_{l=1}^{p}\left[M_{ij}^{(l)} - \frac{1}{2}M_{ijl}\right]M^{jl} + o(n^{-2}) \quad s = 1,2,....,p \tag{13}$$

Let Fisher's information matrix of τ, $M = \{-M_{ij}\}$, and let $a_{ij}^{(l)} = M_{IJ}^{(l)} - \frac{1}{2}M_{ijl}$ are elements $A^{(l)} = a_{ij}^{(l)}$ matrix for $i, j, l = 1,2,3\ldots,p$. Let $A^{(l)} = \left[a_{ij}^{(l)}\right]$, then $A = \left[A^{(1)}\middle|A^{(2)}\middle|A^{(3)}\middle|.....\middle|A^{(P)}\right]$.

As a result, the bias expression for $\hat{\tau}$ can be expressed as

$$Bias(\hat{\tau}) = M^{-1}A \cdot vec(M^{-1}) + O(n^{-2}). \tag{14}$$

Thus, the BCMLE of τ using CAMLE, $\hat{\tau}^{CA-MLE}$, is given by

$$\hat{\tau}^{CMLE} = \hat{\tau} - M^{-1}A \cdot vec(M^{-1}) \tag{15}$$

where $\hat{\tau}$ is the MLE of τ , $\hat{M} = M\,|_{\tau=\hat{\tau}}$, and $\hat{A} = A\,|_{\tau=\hat{\tau}}$. Whereas the bias of $\hat{\tau}^{CA-MLE}$ is quadratic.

Related to generalized Ramos-Louzada distribution, the derivatives are:

$$\frac{\partial^2 L}{\partial\theta\partial\alpha} = \frac{2\sum_{i=1}^{n} e^{-X_i}\sum_{i=1}^{n} X_i^{\theta}\ln(x_i)}{\alpha^2} \tag{16}$$

$$\frac{\partial^2 L}{\partial\theta^2} = \sum_{i=1}^{n} e^{-X}\left(-\frac{2n}{\theta^2} - \frac{2n}{\alpha}\sum_{i=1}^{n} X_i^{\theta}\ln(x_i)^2\right) \tag{17}$$

$$\frac{\partial^2 L}{\partial\alpha^2} = \sum_{i=1}^{n} e^{-x}\left(\frac{3n}{\alpha^2} + \frac{2n}{(\alpha-1)^2} - \frac{n}{(\alpha-2)^2} - \frac{4}{\alpha^3}\sum_{i=1}^{n} x_i^{\theta}\right) \tag{18}$$

$$\frac{\partial^3 L}{\partial\theta^2\partial\alpha} = \frac{2}{\alpha^2}\sum_{i=1}^{n} e^{-x}\sum_{i=1}^{n} X_i^{\theta}\ln(x_i)^2 \tag{19}$$

$$\frac{\partial^3 L}{\partial\theta\partial\alpha^2} = \frac{4}{\alpha^2}\sum_{i=1}^{n} e^{-x}\sum_{i=1}^{n} X_i^{\theta}\ln(x_i)^2 \tag{20}$$

$$\frac{\partial^3 L}{\partial\theta^3} = \sum_{i=1}^{n} e^{-x}\left(-\frac{4n}{\theta^3} - \frac{2n}{\alpha}\sum_{i=1}^{n} X_i^{\theta}\ln(x_i)^3\right) \tag{21}$$

$$\frac{\partial^3 L}{\partial\alpha^3} = \sum_{i=1}^{n} e^{-x}\left(-\frac{6n}{\alpha^3} + \frac{4n}{(\alpha-1)^3} - \frac{2n}{(\alpha-2)^3} - \frac{12}{\alpha^4}\sum_{i=1}^{n} x_i^{\theta}\right). \tag{22}$$

Then,

$$A = \left[A^{(1)}\middle|A^{(2)}\right] = A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix} \tag{23}$$

with

$$
\begin{aligned}
a_{11}^{(1)} &= M_{11}^{(1)} - \frac{1}{2}M_{111} \\
a_{11}^{(2)} &= M_{11}^{(2)} - \frac{1}{2}M_{112} \\
a_{12}^{(2)} &= M_{12}^{(2)} - \frac{1}{2}M_{122} = a_{21}^{(2)} \\
a_{12}^{(1)} &= M_{12}^{(1)} - \frac{1}{2}M_{112} = a_{21}^{(1)} \\
a_{22}^{(1)} &= M_{22}^{(1)} - \frac{1}{2}M_{122} \\
a_{22}^{(2)} &= M_{22}^{(2)} - \frac{1}{2}M_{222}
\end{aligned} \tag{24}
$$

where Appendix section defines M_{ijl}. The GRL distribution's bias MLE is provided by

$$Bias\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\end{pmatrix} = M^{-1}A\,Vec(M^{-1}) + O(n^{-2}). \tag{25}$$

And then,

$$\begin{pmatrix}\hat{\theta}_{CA-MLE}\\ \hat{\alpha}_{CA-MLE}\end{pmatrix} = \begin{pmatrix}\hat{\theta}_{MLE}\\ \hat{\alpha}_{MLE}\end{pmatrix} - Bias\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\end{pmatrix}. \tag{26}$$

3.2. Bootstrap approach

An alternate approach to generate second-order bias-corrected estimators is presented, which is based on the parametric bootstrap resampling process [38, 39]. Let $X = (x_1, x_2, x_3, \ldots, x_n)$ be a size-n random sample with a distribution function of F from the random variable X. Using the distribution function F, B independent bootstrap samples are created to determine the estimated bias of the MLE of $\hat{\tau}$.

$$Bias(\hat{\tau}_{MLE}) = \frac{1}{B}\sum_{j=1}^{B}(\hat{\tau}^{*}_{j,MLE} - \hat{\tau}_{MLE}) \tag{27}$$

Where $\hat{\tau}^{*}_{j}$ is the MLE of τ from the j^{th} bootstrap sample generated from GRL. Then, the bias-corrected bootstrap approach (BCBoot) is defined as

$$\hat{\tau}_{BC-Boot} = 2\hat{\tau}_{MLE} - \frac{1}{B}\sum_{j=1}^{B}\hat{\tau}^{*}_{j,MLE}. \tag{28}$$

4. Simulation results

The purpose of this simulation study is to evaluate the performance of the MLE, CAMLE, and BCBoot estimators of the parameters of the generalized Ramos-Louzada distribution. The generalized Ramos-Louzada distribution was used to generate samples with three different pair of its parameters: $(\theta = 2, \alpha = 2)$, $(\theta = 0.7, \alpha = 2)$, and $(\theta = 4.5, \alpha = 3.5)$. The sample size is used as 10, 30, 50, and 100. Each case was generating under Monte Carlo samples with 5000 times and 1000 bootstrap samples in each time. The simulation experiments for the proposed approaches have been modeled and executed in R program working on Intel core i7 processor with 2.4 MHz processor speed, and16 GB RAM. The root mean squared error (RMSE) and the bias (Bias) of the estimates, which are defined in Eqs. (29) and (30), respectively, are provided in order to assess the accuracy of the parameter estimates. Tables 1-3 provide a summary of all averaged biases and RMSE results.

$$Bias(\hat{\tau}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\tau}_{i,BC-MLE} - \hat{\tau}_{MLE}) \tag{29}$$

$$RMSE(\tau) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\tau}_{i,BC-MLE} - \hat{\tau}_{MLE})^2} \tag{30}$$

Tables 1-3 allow for the derivation of several conclusions.

- The MLE estimators of α appear to be positively biased for all the simulations that are taken into consideration. This demonstrates how, generally speaking, they exaggerate the parameter α value, especially when the sample size is small. Moreover, the MLE estimators usually show a positive bias, that is, they consistently overestimate the true value of the parameter θ for different sample sizes, when the real value of the parameter θ is equal to or greater than 0.7.
- In every simulation for varying sample sizes, the MLE estimators performed worse in terms of bias and root mean square error (RMSE) than the CAMLE and BCBoot of α and θ. In terms of RMSE, the BCBoot of α and θ also fared better than the CAMLE. Furthermore, BCBoot performed better than CAMLE for θ in terms of bias. On the other hand, for α, CAMLE performed better than BCBoot.
- As sample size n increases, all examined estimators' biases and RMSEs will naturally decrease. This is primarily due to the fact that most estimators in statistical theory perform better as sample size n increases. As previously mentioned, both CAMLE and BCBoot exhibit extremely significant reductions in bias and RMSE for small sample numbers. For example, Table 2 shows that, for n=30, the reduction in RMSE of both CAMLE and BCBoot was approximately 19.43% and 19.87% for θ, and 25.10% and 25.31% for α lower than that of the MLE. Further, the reduction in terms of Bias for the same case of both CAMLE and BCBoot was 76.01% and 51.33% for θ, and 71.51% and 71.79% for α lower than that of the MLE.
- Lastly, even though BCBoot and CAMLE are similarly efficient, BCBoot requires less computing work than CAMLE.

Table 1: Bias and RMSE, on average, when $(\theta = 2, \alpha = 2)$

		θ			α		
n		MLE	CAMLE	BCBoot	MLE	CAMLE	BCBoot
10	RMSE	0.4064	0.3391	0.3376	0.4022	0.3165	0.3158
	Bias	0.2251	0.0165	0.0842	0.2038	0.0228	0.0221
30	RMSE	0.3462	0.2789	0.2774	0.3422	0.2563	0.2556
	Bias	0.2197	0.0111	0.0788	0.1984	0.0174	0.0167
50	RMSE	0.3184	0.2498	0.2483	0.3127	0.2215	0.2235
	Bias	0.2183	0.0097	0.0771	0.1971	0.0161	0.0153
100	RMSE	0.2743	0.2072	0.2055	0.2701	0.1844	0.1837
	Bias	0.2174	0.0088	0.0753	0.1961	0.0151	0.0144

Table 2: Bias and RMSE, on average, when $(\theta = 0.7, \alpha = 2)$

		θ			α		
n		MLE	CAMLE	BCBoot	MLE	CAMLE	BCBoot
10	RMSE	0.4615	0.3938	0.3923	0.4569	0.3712	0.3705
	Bias	0.2798	0.0712	0.1388	0.2585	0.0773	0.0768
30	RMSE	0.4009	0.3336	0.3321	0.3969	0.3112	0.3104
	Bias	0.2743	0.0658	0.1335	0.2531	0.0721	0.0714
50	RMSE	0.3731	0.3045	0.3032	0.3674	0.2762	0.2782
	Bias	0.273	0.0644	0.1318	0.2518	0.0708	0.0703

100	RMSE	0.3291	0.2619	0.2602	0.3247	0.2392	0.2383
	Bias	0.2721	0.0634	0.1303	0.2508	0.0698	0.0692

Table 3: Bias and RMSE, on average, when $(\theta = 4.5, \alpha = 3.5)$

		θ			α		
n		MLE	CAMLE	BCBoot	MLE	CAMLE	BCBoot
10	RMSE	0.5145	0.4472	0.4457	0.5103	0.4246	0.4239
	Bias	0.3332	0.1246	0.1323	0.3119	0.1310	0.1302
30	RMSE	0.4543	0.3872	0.3855	0.4503	0.3644	0.3637
	Bias	0.3278	0.1192	0.1269	0.3063	0.1255	0.1243
50	RMSE	0.4265	0.3579	0.3564	0.4208	0.3296	0.3316
	Bias	0.3264	0.1178	0.1252	0.3053	0.1241	0.1231
100	RMSE	0.3824	0.3153	0.3136	0.3782	0.2925	0.2918
	Bias	0.3255	0.1169	0.1222	0.3041	0.1234	0.1218

5. Real data application

In this section, we illustrate the utility of the proposed bias-corrected estimators for the generalized Ramos-Louzada distribution using two real datasets with small sample sizes. The first dataset shows the 18 electronic devices' lifetime failures [40]. This data was further analyzed by Wang and Wang [33]. The tubes that exhibit leakage at a stress level of 120 psi are represented in the second dataset, with n=30, [41], [42], and by Çetinkaya and Bulut [12]. The GRL distribution estimation is shown in Tables 4 and 5. The BCBoot and CAMLE estimates of θ and α are less than the MLE estimate, as seen in Tables 4 and 5, suggesting that the MLE technique overestimates these parameters.

Figures 1 and 2, respectively, present the study of the generalized Ramos-Louzada distribution pdf for the θ and α values of both datasets in reference to Tables 4 and 5. These Figures show that the density shape based on the MLE approach may be misleading, hence we recommend utilizing CAMLE and BCBoot estimations for both datasets.

Table 4: Generalized Ramos-Louzada distribution parameter estimation for the electronic device data.

	θ	α
MLE	0.7041	17.0851
CAMLE	0.6788	16.4272
BCBoot	0.6592	16.3041

Table 5: Generalized Ramos-Louzada distribution parameter estimation for the show leak data.

	θ	α
MLE	1.841	2.121
CAMLE	1.756	2.108
BCBoot	1.738	2.066

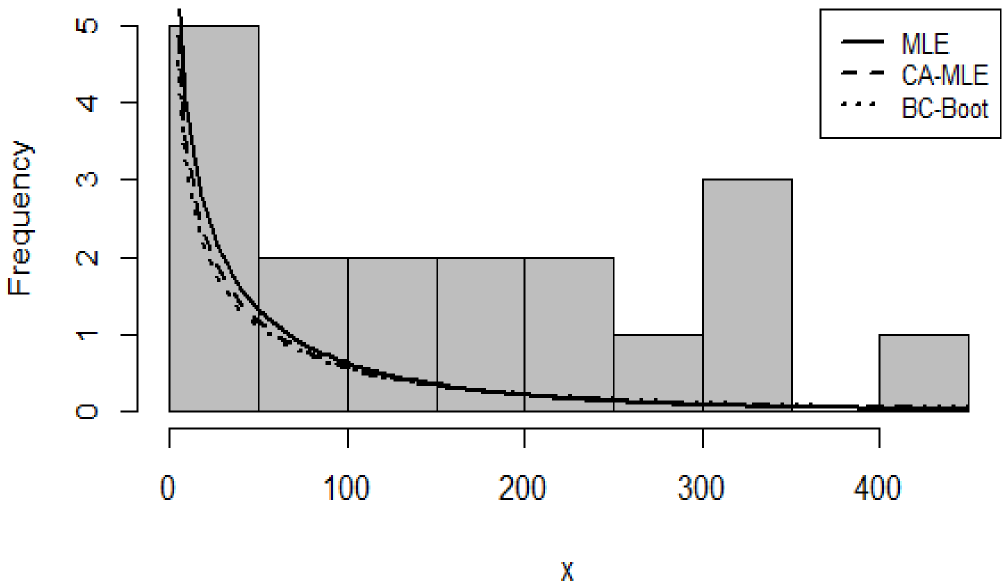

Figure 1: First dataset estimated density function.

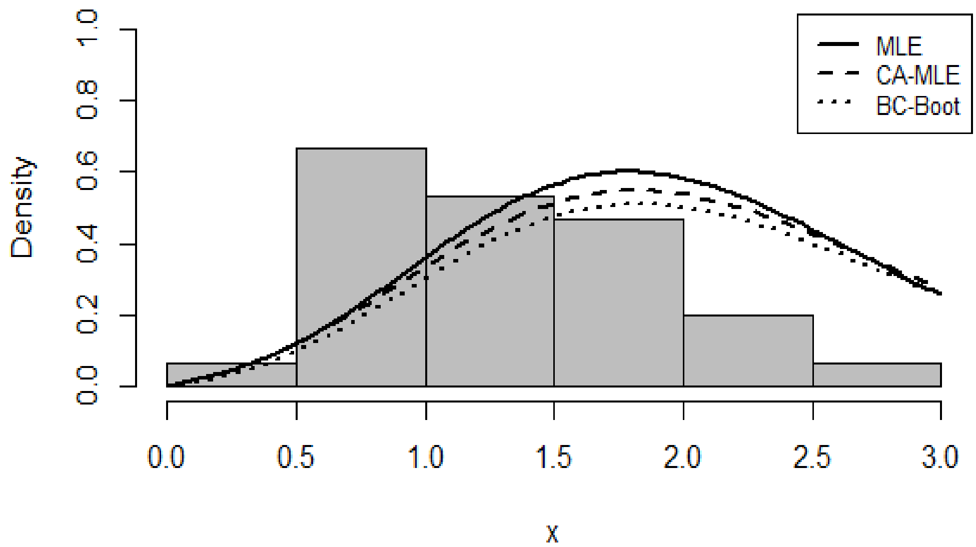

Figure 2: Second dataset estimated density function.

6. Conclusion

This study suggested a corrective technique to get simple closed-form equations for the second-order biases of the MLE of the parameters of the generalized Ramos-Louzada distribution. Specifically: BCBoot and CAMLE. The recently suggested estimators converge to their true value far more quickly than the MLE, as shown by the fact that their biases are of order $O(n^{-2})$ as opposed to $O(n^{-1})$ for the MLE. The numerical data shows that the proposed techniques perform better than the MLE in terms of bias and RMSE, which makes them very attractive. It is strongly recommended to use the bias-corrected estimators that have been given, especially when the sample size is small. We propose to investigate the skewness of the MLE in other distributions in a subsequent paper.

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Appendix

$$M_{11}=E\left[\frac{\partial^2 L}{\partial\theta^2}\right]=-\frac{2n^2(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2))}{\theta^2}-\frac{2n^3(-3\gamma\theta+2\gamma)}{\theta^2}+\frac{4n^3\overline{)\theta+1}}{\alpha\theta^2}-$$

$$\frac{1}{\alpha}(2\overline{)\theta+1}(\psi(\theta+1)^2+\gamma(1,\theta+1))n^2(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2)))-$$

$$\frac{2\overline{)\theta+1}(\psi(\theta+1)^3+3\gamma(1,\theta+1)\psi(\theta+1)+\gamma(2,\theta+1))(3\theta-2)n^3}{\alpha}+\frac{4\overline{)2\theta+1}(\psi(2\theta+1)^2+\gamma(1,2\theta+1))n^3}{\alpha^2}$$

$$M_{22}=E\left[\frac{\partial^2 L}{\partial\alpha^2}\right]=-\frac{1}{\alpha(\alpha+1)^2(\alpha-2)^2}(2n^2(2\alpha^4\overline{)\theta+1}-2\alpha^5-12\alpha^3\overline{)\theta+1}+12\alpha^4+26\alpha^2\overline{)\theta+1}-23\alpha^3-24\alpha\overline{)\theta+1}+$$

$$18\alpha^2+8\overline{)\theta+1}-6\alpha)(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2)))-\frac{1}{\alpha^3(\alpha^4-6\alpha^3+13\alpha^2-\alpha+4)}$$

$$(2n^3(6\theta\alpha^4\psi(\theta+1)\overline{)\theta+1}+6\gamma\theta\alpha^5-36\theta\alpha^3\psi(\theta+1)\overline{)\theta+1}-4\alpha^4\psi(\theta+1)\overline{)\theta+1}-36\gamma\theta\alpha^4-4\gamma\alpha^5+78\theta\alpha^2\psi(\theta+1)\overline{)\theta+1}+$$

$$24\alpha^3\psi(\theta+1)\overline{)\theta+1}+69\gamma\theta\alpha^3+24\gamma\alpha^4-72\alpha\theta\psi(\theta+1)\overline{)\theta+1}-52\alpha^2\psi(\theta+1)\overline{)\theta+1}-54\gamma\theta\alpha^2-46\gamma\alpha^3+24\theta\psi(\theta+1)\overline{)\theta+1}+$$

$$48\alpha\psi(\theta+1)\overline{)\theta+1}+18\gamma\theta\alpha+36\gamma\alpha^2-16\psi(\theta+1)\overline{)\theta+1}-12\lambda\alpha))$$

$$M_{12}=E\left[\frac{\partial^2 L}{\partial\theta\partial\alpha}\right]=\frac{2\psi(\theta+1)\overline{)\theta+1}n^2(2n\ln(\theta-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2))}{\alpha^2}-\frac{4n^3\psi(2\theta+1)\overline{)2\theta+1}}{\alpha^2}+$$

$$\frac{(6\theta n^3\psi(\theta+1)^2\overline{)\theta+1}+6\theta n^3\overline{)\theta+1}\gamma(1,\theta+1)-4n^3\psi(\theta+1)^2\overline{)\theta+1}-4n^3\overline{)\theta+1}\gamma(1,\theta+1))}{\alpha^2}$$

$$M_{12}=M_{21}$$

$$M_{111}=E\left[\frac{\partial^3 L}{\partial\alpha^3}\right]=\frac{1}{\alpha\theta^3}(2n^2(\theta^3\overline{)\theta+1}\psi(\theta+1)^3+3\theta^3\overline{)\theta+1}\psi(\theta+1)\gamma(1,\theta+1)+\theta^3-2\alpha)(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+$$

$$n\ln(\alpha-2)))-\frac{1}{\alpha\theta^3}(2n^3(3\theta^4\overline{)\theta+1}\psi(\theta+1)^4-2\theta^3\overline{)\theta+1}\psi(\theta+1)^4+18\theta^4\overline{)\theta+1}\psi(\theta+1)^2\gamma(1,\theta+1)-12\theta^3\overline{)\theta+1}\psi(\theta+1)^2$$

$$\gamma(1,\theta+1)+12\theta^4\overline{)\theta+1}\psi(\theta+1)\gamma(2,\theta+1)+9\theta^4\overline{)\theta+1}\gamma(1,\theta+1)-8\theta^3\overline{)\theta+1}\psi(\theta+1)\gamma(2,\theta+1)-6\theta^3\overline{)\theta+1}\gamma(1,\theta+1)^2+$$

$$3\theta^4\overline{)\theta+1}\gamma(3,\theta+1)-2\theta^3\overline{)\theta+1}\gamma(3,\theta+1)+6\gamma\alpha\theta-4\gamma\alpha))+\frac{1}{\theta^3\alpha^2}(4n^3(\psi(2\theta+1)^3\overline{)2\theta+1}\theta^3+3\theta^3\psi(2\theta+1)\overline{)2\theta+1}$$

$$\gamma(1,2\theta+1)+\theta^3\overline{)2\theta+1}\gamma(2,2\theta+1)-2\alpha\overline{)\theta+1}))$$

$$M_{222}=E\left[\frac{\partial^3 L}{\partial\alpha^3}\right]=\frac{1}{\alpha^4(\alpha-1)^3(\alpha-2)^3}(4n^2(3\alpha^6\overline{)\theta+1}-2\alpha^7-27\alpha^5\overline{)\theta+1}+18\alpha^6+99\alpha^4\overline{)\theta+1}-60\alpha^5-189\alpha^3\overline{)\theta+1}+$$

$$102\alpha^4+198\alpha^2\overline{)\theta+1}-99\alpha^3-108\alpha\overline{)\theta+1}+54\alpha^2+24\overline{)\theta+1}-12\alpha)(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha)+n\ln(\alpha-2)))+(4n^3$$

$$(-306\gamma\theta\alpha^4+297\gamma\theta\alpha^3+36\gamma\theta\alpha+54\alpha^5\overline{)\theta+1}\psi(\theta+1)+108\gamma\alpha^2-324\theta\alpha\psi(\theta+1)\overline{)\theta+1}+297\theta\alpha^4\psi(\theta+1)\overline{)\theta+1}-567\theta\alpha^4$$

$$\psi(\theta+1)\overline{)\theta+1}+594\theta\alpha^2\psi(\theta+1)\overline{)\theta+1}+204\gamma\alpha^4-198\gamma\alpha^3-24\gamma\alpha-4\gamma\alpha^7+36\gamma\alpha^6-48\psi(\theta+1)\overline{)\theta+1}+6\gamma\theta\alpha^7-54\gamma\theta\alpha^6+$$

$$-81\theta\alpha^5\overline{)\theta+1}\psi(\theta+1)+216\alpha\psi(\theta+1)\overline{)\theta+1}-6\alpha^6\overline{)\theta+1}\psi(\theta+1)-120\gamma\alpha^5+9\theta\alpha^6\overline{)\theta+1}\psi(\theta+1)-162\gamma\theta\alpha^2+378\alpha^3$$

$$\psi(\theta+1)\overline{)\theta+1}-396\alpha^2\psi(\theta+1)\overline{)\theta+1}+180\gamma\theta\alpha^5-198\alpha^4\psi(\theta+1)\overline{)\theta+1}+72\theta\psi(\theta+1)\overline{)\theta+1}))/(\alpha^4(\alpha^6-9\alpha^5+33\alpha^4-$$

$$63\alpha^3+66\alpha^2-36\alpha+8))+\frac{1}{\alpha^5(\alpha-1)^3(\theta-2)^3}(8n^3(2\alpha^7\overline{)\theta+1}-18\alpha^6\overline{)\theta+1}-3\alpha^6\overline{)2\theta+1}+60\alpha^5\overline{)\theta+1}+27\alpha^5\overline{)2\theta+1}-$$

$$102\alpha^4\overline{)2\theta+1}-102\alpha^4\overline{)\theta+1}-99\alpha^4\overline{)2\theta+1}+99\alpha^3\overline{)\theta+1}+189\alpha^3\overline{)2\theta+1}-54\alpha^2\overline{)\theta+1}-198\alpha^2\overline{)2\theta+1}+12\overline{)\theta+1}\alpha+$$

$$108\alpha\overline{)2\theta+1}-14\overline{)2\theta+1}))$$

$$M_{112}=E\left[\frac{\partial^3 L}{\partial\theta^2\partial\alpha}\right]=\frac{1}{\alpha^2}((2n^2\overline{)\theta+1}\psi(\theta+1)^2+2n^2\overline{)\theta+1}\gamma(1,\theta+1))(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2))+$$

$$6n^3\theta\psi(\theta+1)^3\overline{)\theta+1}\theta+18\theta n^3\psi(\theta+1)\overline{)\theta+1}\gamma(1,\theta+1)+6\theta n^3\overline{)\theta+1}\gamma(2,\theta+1)-4n^3\psi(\theta+1)^3\overline{)\theta+1}-12n^3\psi(\theta+1)\overline{)\theta+1}$$

$$\gamma(1,\theta+1)-4n^3\overline{)\theta+1}\gamma(2,\theta+1)-\frac{4\overline{)2\theta+1}(\psi(2\theta+1)^2+\gamma(1,2\theta+1))n^3}{\alpha})$$

$M_{221}=E\left[\frac{\partial^3 L}{\partial\alpha^2\partial\theta}\right]=\frac{1}{\alpha^3}(-4\psi(\theta+1)\overline{)\theta+1}n^2(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2))-12\psi(\theta+1)^2\overline{)\theta+1}\theta n^3-12\overline{)\theta+1}$ where

$\gamma(1,\theta+1)\theta n^3+8\psi(\theta+1)^2\overline{)\theta+1}n^3+8n^3\overline{)\theta+1}\gamma(1,\theta+1)n^3+\frac{8\psi(2\theta+1)\overline{)2\theta+1}n^3}{\alpha})$

$\psi(.)$ denotes digamma and $\gamma(.)$ denotes Riemann's zeta function, We have too

$M_{11}^{(1)}=\frac{\partial M_{11}}{\partial\theta}=\frac{6n^3\gamma}{\theta^2}+\frac{4n^3(-3\gamma\theta+2\gamma)}{\theta^3}+\frac{4n^3\psi(\theta+1)\overline{)\theta+1}}{\alpha\theta^2}-\frac{8n^3\overline{)\theta+1}}{\alpha\theta^3}-\frac{1}{\alpha}(2\psi(\theta+1)\overline{)\theta+1}(\psi(\theta+1)^2+$

$\gamma(1,\theta+1))n^2(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2)))-\frac{1}{\alpha}(2\overline{)\theta+1}(2\gamma(1,\theta+1)\psi(\theta+1)+\gamma(2,\theta+1))n^2$

$(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2)))-\frac{4\overline{)\theta+1}(\psi(\theta+1)^2+\gamma(1,\theta+1))n^3}{\theta\alpha}-\frac{1}{\alpha}(2\psi(\theta+1)\overline{)\theta+1}(\psi(\theta+1)^3+$

$3\gamma(1,\theta+1)\psi(\theta+1)+\gamma(2,\theta+1))(3\theta-2)n^3)-\frac{1}{\alpha}(2\overline{)\theta+1}(3\psi(\theta+1)^2\gamma(1,\theta+1)+3\gamma(2,\theta+1)\psi(\theta+1)+3\gamma(1,\theta+1)+$

$\gamma(3,\theta+1))(3\theta-2)n^3)-\frac{6\overline{)\theta+1}(\psi(\theta+1)^3+3\gamma(1,\theta+1)\psi(\theta+1)+\gamma(2,\theta+1))\theta n^3}{\alpha}+$

$\frac{8\psi(2\theta+1)\overline{)2\theta+1}(\psi(2\theta+1)^2+\gamma(1,2\theta+1))n^3}{\alpha^2}+\frac{4\overline{)2\theta+1}(4\psi(2\theta+1)\gamma(1,2\theta+1)+2\gamma(2,2\theta+1))n^3}{\alpha^2}$

$M_{11}^{(2)}=\frac{\partial M_{11}}{\partial\alpha}=-\frac{4n^3\overline{)\theta+1}}{\theta^2\alpha^2}+\frac{1}{\alpha}\left(2\overline{)\theta+1}(\psi(\theta+1)^2+\gamma(1,\theta+1)^2+\gamma(1,\theta+1))n^2(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2))\right)$

$-\frac{2\overline{)\theta+1}(\psi(\theta+1)^2+\gamma(1,\theta+1))n^2\left(-\frac{3n}{\alpha}-\frac{2n}{\alpha-1}+\frac{n}{\alpha-2}\right)}{\alpha}+\frac{2\overline{)\theta+1}(\psi(\theta+1)^3+3\gamma(1,\theta+1)\psi(\theta+1)+\gamma(2,\theta+1))(3\theta-2)n^3}{\alpha^2}$

$-\frac{8\overline{)2\theta+1}(\psi(2\theta+1)^2+\gamma(1,2\theta+1))n^3}{\alpha^3}$

$M_{12}^{(1)}=\frac{\partial M_{12}}{\partial\theta}=\frac{2\gamma(1,\theta+1)\overline{)\theta+1}n^2(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2))}{\alpha^2}-\frac{8\overline{)2\theta+1}n^3\gamma(1,2\theta+1)}{\alpha^3}+$

$\frac{4\psi(\theta+1)\overline{)\theta+1}n^3}{\theta\alpha^2}-\frac{8\overline{)2\theta+1}n^3\gamma(1,2\theta+1)^2}{\alpha^3}+\frac{2\psi(\theta+1)^2\overline{)\theta+1}n^2(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2))}{\alpha^2}+$

$\frac{1}{\alpha^2}(18\psi(\theta+1)\overline{)\theta+1}\theta n^3\gamma(1,\theta+1)+6\psi(\theta+1)^3\overline{)\theta+1}\theta n^3+6\psi(\theta+1)^2\overline{)\theta+1}n^3+6\overline{)\theta+1}\gamma(2,\theta+1)\theta n^3+6\overline{)\theta+1}$

$\gamma(1,\theta+1)n^3-12\psi(\theta+1)\overline{)\theta+1}n^3\gamma(1,\theta+1)-4\gamma(\theta+1)^3\overline{)\theta+1}n^3-4\overline{)\theta+1}\gamma(2,\theta+1)n^3)$

$$M_{22}^{(1)}=\frac{\partial M_{22}}{\partial\theta}=-\frac{1}{\alpha^{3}(\alpha-1)^{2}(\alpha-2)^{2}}(2n^{2}(2\psi(\alpha+1))\overline{\theta+1}\alpha^{4}-12\psi(\theta+1))\overline{\theta+1}\alpha^{3}+26\psi(\theta+1))\overline{\theta+1}\alpha^{2}-24\psi(\theta+1)$$

$$)\overline{\theta+1}\alpha+8\psi(\theta+1))\overline{\theta+1})(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2)))-\frac{1}{\alpha^{3}(\alpha-1)^{2}(\alpha-2)^{2}\theta}(4n^{3}(2)\overline{\theta+1}\alpha^{4}-$$

$$2\alpha^{5}-12)\overline{\theta+1}\alpha^{3}+12\alpha^{4}+26)\overline{\theta+1}\alpha^{2}-23\alpha^{3}-24)\overline{\theta+1}\alpha+18\alpha^{2}+8)\overline{\theta+1}-6\alpha))-\frac{1}{\alpha^{3}(\alpha^{4}-6\alpha^{3}+13\alpha^{2}-12\alpha+4)}$$

$$(2n^{3}(-54\gamma\alpha^{2}-36\psi(\theta+1)^{2})\overline{\theta+1}\theta\alpha^{3}+78\psi(\theta+1)^{2})\overline{\theta+1}\theta\alpha^{2}-36)\overline{\theta+1}\gamma(1,\theta+1)\theta\alpha^{3}+78)\overline{\theta+1}\gamma(1,\theta+1)\theta\alpha^{2}-72$$

$$\psi(\theta+1)^{2})\overline{\theta+1}\theta\alpha-72)\overline{\theta+1}\gamma(1,\theta+1)\theta\alpha+6\gamma(1,\theta+1))\overline{\theta+1}\theta\alpha^{4}-4\psi(\theta+1)^{2})\overline{\theta+1}\alpha^{4}+24\psi(\theta+1)^{2})\overline{\theta+1}\theta-36\gamma\alpha+$$

$$69\gamma\alpha^{3}+18\gamma\alpha-4\gamma(1,\theta+1))\overline{\theta+1}\alpha^{4}+24\gamma(1,\theta+1))\overline{\theta+1}\theta-16\psi(\theta+1)^{2})\overline{\theta+1}+24\psi(\theta+1))\overline{\theta+1}-16\gamma(1,\theta+1))\overline{\theta+1}-$$

$$72\psi(\theta+1))\overline{\theta+1}\alpha+48\psi(\theta+1)^{2})\overline{\theta+1}\alpha+48)\overline{\theta+1}\gamma(1,\theta+1)\alpha+6\gamma\alpha^{5}+24\psi(\theta+1)^{2})\overline{\theta+1}\alpha^{3}-52\psi(\theta+1)^{2})\overline{\theta+1}\alpha^{2}-$$

$$36\psi(\theta+1))\overline{\theta+1}\alpha^{3}+24)\overline{\theta+1}\gamma(1,\theta+1)\alpha^{3}-78\psi(\theta+1))\overline{\theta+1}\alpha^{2}-52)\overline{\theta+1}\gamma(1,\theta+1)\alpha^{2}+6\psi(\theta+1)^{2})\overline{\theta+1}\theta\alpha^{4}+6\psi(\theta+1)$$

$$)\overline{\theta+1}\alpha^{4})-\frac{1}{(\alpha-1)^{2}\alpha^{4}(\alpha-2)^{2}}(4n^{3}(2)\overline{\theta+1}\psi(\theta+1)\alpha^{5}-12\psi(\theta+1))\overline{\theta+1}\alpha^{4}-4\psi(2\theta+1))\overline{2\theta+1}\alpha^{4}+23\psi(\theta+1))\overline{\theta+1}\alpha^{3}+$$

$$24\psi(2\theta+1))\overline{2\theta+1}\alpha^{3}-18\psi(\theta+1))\overline{\theta+1}\alpha^{2}-52\psi(2\theta+1))\overline{2\theta+1}\alpha^{2}+6\psi(\theta+1))\overline{\theta+1}\alpha+48\psi(2\theta+1))\overline{2\theta+1}\alpha-16\psi(2\theta+1))\overline{2\theta+1}))$$

$$M_{12}^{(2)}=\frac{\partial M_{12}}{\partial\alpha}=\frac{4\psi(\theta+1))\overline{\theta+1}n^{2}(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2))}{\alpha^{3}}+\frac{12\psi(2\theta+1))\overline{2\theta+1}n^{3}}{\alpha^{4}}$$

$$\frac{2(6\psi(\theta+1)^{2})\overline{\theta+1}\theta n^{3}+6)\overline{\theta+1}\gamma(1,\theta+1)\theta n^{3}-4\psi(\theta+1)^{2})\overline{\theta+1}n^{3}-4)\overline{\theta+1}\gamma(1,\theta+1)n^{3})}{\alpha^{3}}+$$

$$\frac{2\psi(\theta+1))\overline{\theta+1}n^{2}\left(-\frac{3n}{\alpha}-\frac{2n}{\alpha-1}+\frac{n}{\alpha-2}\right)}{\alpha^{2}}$$

$$
\begin{aligned}
&M_{22}^{(2)}=\frac{\partial M_{22}}{\partial\alpha}=-\frac{1}{\alpha^{3}(\alpha-1)^{2}(\alpha-2)^{2}}(2n^{2}(8)\overline{\theta+1}\alpha^{3}-10\alpha^{4}-36)\overline{\theta+1}\alpha^{2}+48\alpha^{3}+52)\overline{\theta+1}\alpha-69\alpha^{2}-24)\overline{\theta+1}+36\alpha-6)\\
&(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2)))+\frac{1}{\alpha^{4}(\alpha-1)^{2}(\alpha-2)^{2}}(6n^{2}(2)\overline{\theta+1}\alpha^{4}-2\alpha^{5}-12)\overline{\theta+1}\alpha^{3}+12\alpha^{4}+\\
&26)\overline{\theta+1}\alpha^{2}-23\alpha^{3}-24)\overline{\theta+1}\alpha+18\alpha^{2}+8)\overline{\theta+1}-6\alpha)(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2)))+\frac{1}{\alpha^{3}(\alpha-1)^{3}(\alpha-2)^{2}}\\
&(4n^{2}(2)\overline{\theta+1}\alpha^{4}-2\alpha^{5}-12)\overline{\theta+1}\alpha^{3}+12\alpha^{4}+26)\overline{\theta+1}\alpha^{2}-23\alpha^{3}-24)\overline{\theta+1}\alpha+18\alpha^{2}+8)\overline{\theta+1}-6\alpha)(2n\ln(\theta)-3n\ln(\alpha)\\
&-2n\ln(\alpha-1)+n\ln(\alpha-2)))+\frac{1}{\alpha^{3}(\alpha-1)^{2}(\alpha-2)^{3}}(4n^{2}(2)\overline{\theta+1}\alpha^{4}-2\alpha^{5}-12)\overline{\theta+1}\alpha^{3}+12\alpha^{4}+26)\overline{\theta+1}\alpha^{2}-23\alpha^{3}-\\
&24)\overline{\theta+1}\alpha+18\alpha^{2}+8)\overline{\theta+1}-6\alpha)(2n\ln(\theta)-3n\ln(\alpha)-2n\ln(\alpha-1)+n\ln(\alpha-2)))-\frac{1}{\alpha^{3}(\alpha-1)^{2}(\alpha-2)^{2}}\\
&\left(2n^{2}(2)\overline{\theta+1}\alpha^{4}-2\alpha^{5}-12)\overline{\theta+1}\alpha^{3}+12\alpha^{4}+26)\overline{\theta+1}\alpha^{2}-23\alpha^{3}-24)\overline{\theta+1}\alpha+18\alpha^{2}+8)\overline{\theta+1}-6\alpha)\left(-\frac{3n}{\alpha}-\frac{2n}{\alpha-1}+\frac{n}{\alpha-1}\right)\right)\\
&-\frac{1}{\alpha^{3}(\alpha^{4}-6\alpha^{3}+13\alpha^{2}-12\alpha+4)}(2n^{3}(\psi(\theta+1))\overline{\theta+1}\theta\alpha^{3}+30\gamma\theta\alpha^{4}-108\psi(\theta+1))\overline{\theta+1}\theta\alpha^{2}-16\psi(\theta+1))\overline{\theta+1}\alpha^{3}-144\gamma\theta\alpha^{3}-\\
&20\gamma\alpha^{4}+156\psi(\theta+1))\overline{\theta+1}\theta\alpha+72\psi(\theta+1))\overline{\theta+1}\alpha^{2}+207\gamma\alpha^{2}\theta+96\gamma\alpha^{3}-72\psi(\theta+1))\overline{\theta+1}\theta-104\psi(\theta+1))\overline{\theta+1}\alpha-\\
&108\gamma\theta\alpha-138\gamma\alpha^{2}+48\psi(\theta+1))\overline{\theta+1}+18\gamma\theta+72\gamma\alpha-12\gamma))+\frac{1}{\alpha^{4}(\alpha^{4}-6\alpha^{3}+13\alpha^{2}-12\alpha+4)}(6n^{3}(6\psi(\theta+1)\\
&)\overline{\theta+1}\theta\alpha^{4}+6\gamma\theta\alpha^{5}-36\psi(\theta+1))\overline{\theta+1}\theta\alpha^{3}-4\psi(\theta+1))\overline{\theta+1}\alpha^{4}-36\gamma\theta\alpha^{4}-\\
&4\gamma\alpha^{5}-78\psi(\theta+1))\overline{\theta+1}\theta\alpha^{2}+24\psi(\theta+1))\overline{\theta+1}\alpha^{3}+69\gamma\alpha^{3}\theta+24\gamma\alpha^{4}-72\psi(\theta+1))\overline{\theta+1}\theta\alpha-52\psi(\theta+1))\overline{\theta+1}\alpha^{2}-\\
&54\gamma\theta\alpha^{2}-46\gamma\alpha^{3}+24\psi(\theta+1))\overline{\theta+1}\theta+48\psi(\theta+1))\overline{\theta+1}\alpha+18\gamma\theta\alpha+36\gamma\alpha^{2}-16\psi(\theta+1))\overline{\theta+1}-12\gamma\alpha))+\\
&\frac{1}{\alpha^{3}(\alpha^{4}-6\alpha^{3}+13\alpha^{2}-12\alpha+4)^{2}}(2n^{3}(6\psi(\theta+1))\overline{\theta+1}\theta\alpha^{4}+6\gamma\theta\alpha^{5}-36\psi(\theta+1))\overline{\theta+1}\theta\alpha^{3}-4\psi(\theta+1))\overline{\theta+1}\alpha^{4}-36\gamma\theta\alpha^{4}-\\
&4\gamma\alpha^{5}-78\psi(\theta+1))\overline{\theta+1}\theta\alpha^{2}+24\psi(\theta+1))\overline{\theta+1}\alpha^{3}+69\gamma\alpha^{3}\theta+24\gamma\alpha^{4}-72\psi(\theta+1))\overline{\theta+1}\theta\alpha-52\psi(\theta+1))\overline{\theta+1}\alpha^{2}-\\
&54\gamma\theta\alpha^{2}-46\gamma\alpha^{3}+24\psi(\theta+1))\overline{\theta+1}\theta+48\psi(\theta+1))\overline{\theta+1}\alpha+18\gamma\theta\alpha+36\gamma\alpha^{2}-16\psi(\theta+1))\overline{\theta+1}-12\gamma\alpha))(4\alpha^{3}-18\alpha^{2}+\\
&26\alpha-12))-\frac{1}{\alpha^{4}(\alpha-1)^{2}(\alpha-2)^{2}}(4n^{3}(10)\overline{\theta+1}\alpha^{4}-48)\overline{\theta+1}\alpha^{3}-8)\overline{2\theta+1}\alpha^{3}+69)\overline{\theta+1}\alpha^{2}+36)\overline{2\theta+1}\alpha^{2}-36)\overline{\theta+1}\alpha-\\
&52)\overline{2\theta+1}\alpha+6)\overline{\theta+1}+24)\overline{2\theta+1}))+\frac{1}{\alpha^{4}(\alpha-1)^{3}(\alpha-2)^{2}}(8n^{3}(2)\overline{\theta+1}\alpha^{5}-12)\overline{\theta+1}\alpha^{4}-2)\overline{2\theta+1}\alpha^{4}+23)\overline{\theta+1}\alpha^{3}+\\
&12)\overline{2\theta+1}\alpha^{3}-18)\overline{\theta+1}\alpha^{2}-26)\overline{2\theta+1}\alpha^{2}+6)\overline{\theta+1}\alpha+24)\overline{2\theta+1}\alpha-8)\overline{2\theta+1}))+\frac{1}{\alpha^{5}(\alpha-1)^{2}(\alpha-2)^{2}}(16n^{3}(2)\overline{\theta+1}\alpha^{5}-\\
&12)\overline{\theta+1}\alpha^{4}-2)\overline{2\theta+1}\alpha^{4}+23)\overline{\theta+1}\alpha^{3}+12)\overline{2\theta+1}\alpha^{3}-18)\overline{\theta+1}\alpha^{2}-26)\overline{2\theta+1}\alpha^{2}+6)\overline{\theta+1}\alpha+24)\overline{2\theta+1}\alpha-8)\overline{2\theta+1}))+\\
&\frac{1}{\alpha^{4}(\alpha-1)^{2}(\alpha-2)^{3}}(8n^{3}(2)\overline{\theta+1}\alpha^{5}-12)\overline{\theta+1}\alpha^{4}-2)\overline{2\theta+1}\alpha^{4}+23)\overline{\theta+1}\alpha^{3}+\\
&12)\overline{2\theta+1}\alpha^{3}-18)\overline{\theta+1}\alpha^{2}-26)\overline{2\theta+1}\alpha^{2}+6)\overline{\theta+1}\alpha+24)\overline{2\theta+1}\alpha-8)\overline{2\theta+1}))
\end{aligned}
$$