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Some Properties of Convex Fuzzy Compact Spaces

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Abstract
The first goal in this paper is using the definition of convex fuzzy normed space to present the definition of convex fuzzy compact space. After that an example is introduced to improve the existence of such space. The second aim is to introduce important concepts and prove properties of convex fuzzy compact space. Moreover, we proved if $\mathcal{Z}$ is convex fuzzy compact then $\mathcal{Z}$ is totally convex fuzzy bounded, as well as we prove if $\mathcal{Z}$ is convex fuzzy compact then $\mathcal{Z}$ is convex fuzzy complete.

Keywords: Convex fuzzy compact space, Convex fuzzy normed space, Totally convex fuzzy bounded space, Relatively convex fuzzy compact space, Convex fuzzy Complete space.

بعض خواص الفضاءات المتراصة الضبابية المحدبة

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الخلاصه
الهدف الأول في هذه البحث هو استخدام تعريف الفضاء القياسي الضبابي المحدب مع خواصه الأساسية لتقديم مفهوم الفضاء المتراص الضبابي المحدب . بعد ذلك تم تقديم مثال لإثبات الوجود لمثل هذا الفضاء. الهدف الثاني تقديم مفاهيم مهمة وبرهان خواص للفضاء المتراص الضبابي المحدب. بالإضافة الى ذلك تم برهان اذا كان $\mathcal{Z}$ فضاء متراص ضبابي محدب عندئذ يكون مقيد ضبابي محدب كلي، وكذلك برهنا انه اذا كان $\mathcal{Z}$ فضاء متراص ضبابي محدب عندئذ يكون فضاء كامل ضبابي محدب.

1. Introduction

Katsaras's type fuzzy norm is the first definition that introduced in 1984 on linear space then its properties, topological structures, fixed point theorems were studied by many authors. J. Z. Xiao and X.H. Zhu in 2002 [1] studied linearly topological structures and property of fuzzy normed linear space. T. Bag and S. Samanta in 2003 [2] studied finite dimensional fuzzy normed linear spaces. A. Amini and R. Saadati in 2004 [3] presented some properties of continuous t-norm and s-norms. J. Xiao and X. Zhu in 2004 [4] introduce fuzzy normed spaces of operators and its completeness.

T. Bag and S.K. Samanta in 2005 [5] studied fuzzy bounded linear operators. T. Bag and S. Samanta in 2006 [6] proved fixed point theorems on fuzzy normed spaces. Again T. Bag and

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S.K. Samanta in 2007 [7] proved some fixed point theorems in fuzzy normed linear spaces. T. Bag, S.K. Samanta in 2008 [8] presented a comparative study among several types of fuzzy norms on a linear space defined by various authors has been made. I.Sadeqi and I. F. Kia. in 2009 [9] were first show that the two notations of fuzzy continuity and topological continuity are equivalent, also they proved that fuzzy normed spaces are topological vector spaces.

I.Golet in 2010 [10] generalized fuzzy norms which is defined on a set of objects endowed with a structure of linear space. Then he analyzed the relationships between these fuzzy norms and the fuzzy metrics between these fuzzy norms and the topological structures on the base linear space. M. Goudarzi and S. Vaezpour in 2010 [11] presentd best simultaneous approximation in fuzzy normed spaces. D. Oregan, D. and R. Saadati in 2010 [12] present L-Random and fuzzy normed spaces and classical theory.

A. Hasankhani. A. Nazari and M. Saheli in 2010 [13] introduced some properties of fuzzy Hilbert spaces and norm of operators. C. Alegre, S. Romaguera in 2010 [14] presented characterizations of fuzzy metrizable topological vector spaces and their asymmetric generalization in terms of fuzzy (quasi) norms. M. Janfada, H. Baghani and O. Baghani in 2011[15] presented some properties of the space of all weakly fuzzy bounded linear operators. After that Nădăban and Dzitac 2014 [16] studied the atomic decompositions of fuzzy normed spaces. Again, Nădăban in 2015 [17] proved basic properties of fuzzy continuous mapping in fuzzy normed spaces. Moreover, Nădăban introduced in [18] the notion fuzzy of Euclidean normed spaces for data mining applications. After that Mohammed, Ahmed and Fadhel [19] proved the fixed point theorem in fuzzy normed space. In this direction Nădăban with other authors proved other properties of fuzzy normed space, see [20, 21, 22].

Sabri in 2021[23] studied the fuzzy convergence of sequence and fuzzy compact operators on standard fuzzy normed spaces. Through 2023 [24] Sabri and Ahmed proved best proximity point theorem for $\tilde{\alpha}- \tilde{\psi}$ −contractive type mapping in fuzzy normed space. Daher and Kider [25] published a paper on line contains a study of convex fuzzy normed space. Finally, some properties of fuzzy set appear in [26]. A Lot of research work has been done on fuzzy topological spaces and their properties, see [27-30].

The main goal of this research is to prove properties of convex fuzzy compact space. This research consists of three sections: in section two the convex fuzzy normed space and its basic properties are recalled. In section three we proved the main results in this paper.

2. Properties of convex fuzzy normed spaces

In this section we recall some basic concepts, definitions and theorems for convex fuzzy normed spaces that proved in [26], that will be needed in the section of results and discussion.

Definition 2.1 :

If $\mathcal{A}_{\mathbb{R}}:\mathbb{R} \to [0, 1]$ satisfying

(i) $0< \mathcal{A}_{\mathbb{R}}(\delta) \leq 1$, $\delta \neq 0$.

(ii) (ii) $\mathcal{A}_{\mathbb{R}}(\gamma) \,.\, \mathcal{A}_{\mathbb{R}}(\delta) \geq \mathcal{A}_{\mathbb{R}}(\gamma\delta)$.

(iii) $\mathcal{A}_{\mathbb{R}}(\gamma)=0 \Leftrightarrow \gamma=0$.

(iv) $\sigma\mathcal{A}_{\mathbb{R}}(\gamma) + \mu\, \mathcal{A}_{\mathbb{R}}(\delta) \geq \mathcal{A}_{\mathbb{R}}(\gamma+\delta)$;

for all $\sigma, \mu \in [0, 1]$ with $\sigma + \mu = 1$, and for all $\gamma, \delta \in \mathbb{R}$.

Thus $(\mathbb{R}, \mathcal{A}_{\mathbb{R}})$ is **convex fuzzy absolute value space** (or, c-FAVS)

Definition 2.2 :

Suppose that $\mathcal{Z}$ is a vector space over $\mathbb{R}$. If $\mathcal{N}: \mathcal{U} \to [0, 1]$ satisfying

(i) $0< \mathcal{N}(y) \leq 1$, $y \neq 0$.
(ii) $\mathcal{N}(\alpha y) \leq \mathcal{A}_{\mathbb{R}}(\alpha)\, \mathcal{N}(y)$ for all $0\neq \alpha \in \mathbb{R}$.
(iii) $\mathcal{N}(y) = 0$ if and only if $y=0$.
(iv) $\mathcal{N}(y + d) \leq \gamma\mathcal{N}(y) + \delta\mathcal{N}(d)$, where $\gamma + \delta=1$,
for all $y, d \in \mathcal{Z}$. Hence, $(\mathcal{Z}, \mathcal{N})$ is **convex fuzzy normed space** (or, c-FNS).

Example 2.3:
Define $\mathcal{N}(k) = max_{\alpha\in[d,b]}\, \mathcal{A}_{\mathbb{R}}[k(\alpha)]$ for all $k \in \mathcal{W}$ where $\mathcal{W}$=C[d, b]. Then, $(\mathcal{W}, \mathcal{N})$ is a c-FNS.

Definition 2.4:
If $(\mathcal{Z}, \mathcal{N})$ is c-FNS and let (u_k) be a sequence in $\mathcal{Z}$ then (u_k) is **fuzzy converges** to y if for all $0< \alpha < 1$ there exists N satisfies $\mathcal{N}(u_k - y) < \alpha$, for all k≥N. In this case we write $\lim_{n\to\infty} \mathfrak{N}(u_k - y)$,or $\lim_{k\to\infty} u_k = y$ or $u_k \to y$.

Theorem 2.5:
If $(\mathcal{Z}_1, \mathcal{N}_1)$ and $(\mathcal{Z}_2, \mathcal{N}_2)$ are two c-FNS then $(\mathcal{Z}, \mathcal{N})$ is c-FNS where $\mathcal{Z}= \mathcal{Z}_1 \times \mathcal{Z}_2$ and $\mathcal{N}[(y_1, y_2)]=\gamma\mathcal{N}_1(y_1) + \delta\mathcal{N}_2(y_2)$ for all $(y_1, y_2) \in \mathcal{Z}$ where γ , $\delta \in[0, 1]$ with $\gamma + \delta=1$.

Corollary 2.6:
If $(\mathcal{Z}, \mathcal{N})$ is c-FNS, then $(\mathcal{Z}^m, \mathcal{N}_{\mathcal{Z}})$ is a c-FNS where $\mathcal{Z}^m= \mathcal{Z} \times \mathcal{Z} \times\ldots\times \mathcal{Z}$ (m-times) and $\mathcal{N}_{\mathcal{Z}}[(y_1, y_2, \ldots,y_m)]= \delta_1\mathcal{N}(y_1) +\delta_2\, \mathcal{N}(y_2) + \ldots +\delta_k\, \mathcal{N}(y_m)$ for all $(y_1, y_2, \ldots,y_m) \in \mathcal{Z}$, where $\delta_1, \delta_2, \ldots,\delta_k \in[0, 1]$ with $\delta_1+ \delta_2+ \ldots+\delta_k=1$.

Corollary 2.7:
If $(\mathcal{Z}_1, \mathcal{N}_1)$, $(\mathcal{Z}_2, \mathcal{N}_2)$, …, $(\mathcal{Z}_m, \mathcal{N}_m)$ are c-FNS, then $(\mathcal{Z}, \mathcal{N})$ is c-FNS where $\mathcal{Z}= \mathcal{Z}_1 \times \mathcal{Z}_2 \times \ldots \times \mathcal{Z}_m$ and $\mathcal{N}[(y_1, y_2, \ldots, y_m)]= \delta_1\mathcal{N}_1(y_1) +\delta_2\, \mathcal{N}_2(y_2) + \ldots +\delta_k\, \mathcal{N}_m(y_m)$ for all $(y_1, y_2, \ldots,y_m) \in \mathcal{Z}$ where $\delta_1, \delta_2, \ldots,\delta_k \in[0, 1]$ with $\delta_1+ \delta_2+ \ldots+\delta_3=1$.

Definition 2.8:
Suppose that $(\mathcal{Z}, \mathcal{N})$ is c-FNS.
(i) cfb$(y, \alpha)= \{q \in \mathcal{Z}: \mathcal{N}(y - q) < \alpha\}$.
(ii) cfb$[y, \alpha]= \{q \in \mathcal{U}: \mathfrak{N}(y - q) < \alpha\}$.
Then (i) is a **convex open fuzzy ball** and (ii) is **convex closed fuzzy ball** with centre $y \in \mathcal{Z}$ and radius α, where $0< \alpha < 1$.

Definition 2.9:
If $(\mathcal{Z}, \mathcal{N})$ be c-FNS then (u_k) is **convex fuzzy Cauchy sequence** in $\mathcal{Z}$ if for all $\varepsilon \in (0, 1)$ there exists N satisfies $\mathfrak{N}(u_j - u_i) < \varepsilon$, for all j, i ≥N.

Definitions 2.10:
If $(\mathcal{Z}, \mathcal{N})$ is c-FNS and $\mathcal{W} \subseteq \mathcal{Z}$. Then
(1) $\mathcal{W}$ is **convex fuzzy open set** if cfb$(w,\alpha) \subseteq \mathcal{W}$ for all $w \in \mathcal{W}$ with $0< \alpha < 1$**.**
(2) Again $\mathcal{F} \subseteq \mathcal{Z}$ is **convex fuzzy closed set** whenever $\mathcal{F}^C$ is convex fuzzy open.
(3) Furthermore, $\bar{\mathcal{P}} =\cap\{\mathcal{F}$ is convex fuzzy closed in $\mathcal{Z}$ and $\mathcal{P} \subseteq \mathcal{F}\}$is the convex fuzzy closure of $\mathcal{P}$.
(4) If $\bar{\mathcal{P}}=\mathcal{Z}$ where $\mathcal{P} \subseteq \mathcal{Z}$ then $\mathcal{P}$ is **convex fuzzy dense set** in $\mathcal{Z}$.

Theorem 2.11:
If $u_k \to y \in \mathcal{Z}$ then (u_k) is convex fuzzy Cauchy in the c-FNS $(\mathcal{Z}, \mathcal{N})$.

Definition 2.12:
If $u_k \to y \in \mathcal{Z}$, for all (u_k) in $\mathcal{Z}$ which is convex fuzzy Cauchy then the c-FNS $(\mathcal{Z}, \mathcal{N})$ is

convex fuzzy complete set.
Theorem 2.13: [9]
In c-FNS $(\mathcal{Z}, \mathcal{N})$ if $\mathcal{W} \subset \mathcal{U}$ then $w \in \overline{\mathcal{W}}$ if and only if there exists $(w_k) \in \mathcal{W}$ with $w_k \to w$.

3. Results and discussion

In this section we proved the main results for convex fuzzy compact spaces.

Definitions 3.1:
If $(\mathcal{Z},\mathcal{N})$ is a c-FNS and $\Gamma = \{\mathfrak{B}_k$: k∈K where $\mathfrak{B}_k$ is convex fuzzy open} so
(1) If $\mathcal{Z} \subseteq \cup_{\mathfrak{B}_k \in \Gamma} \mathfrak{B}_k$ then Γ is called a **convex fuzzy open cover** of $\mathcal{Z}$.
(2) If $\mathcal{Z} = \cup_{j=1}^{k} \mathfrak{B}_j$ then $\{\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \ldots, \mathfrak{B}_k\} \subset \Gamma$ is called a **finite convex fuzzy open sub cover.**

Definition 3.2:
A c-FNS $(\mathcal{Z},\mathcal{N})$ is called **convex fuzzy compact space** if $\mathcal{Z} \subseteq \cup_{\mathfrak{B}_k \in \Gamma} \mathfrak{B}_k$, then there is $\{\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \ldots, \mathfrak{B}_k\}$ such that $\mathcal{Z} = \cup_{j=1}^{k} \mathfrak{B}_j$.

Example 3.3:
An open interval (0, 1) in a c-FNS $(\mathbb{R}, \mathcal{A}_{|.|})$, where $\mathcal{A}_{|.|}(\sigma) = \frac{|\sigma|}{1+|\sigma|}$ is not a convex fuzzy compact. Since $\Gamma = \{(\frac{1}{m}, 1): 2 \leq m\}$ is a convex fuzzy open covering of (0, 1), but $(0, 1) \neq \cup_{m=2}^{n} \left(\frac{1}{m}, 1\right)$, where $n \in \mathbb{N}$.
Example 3.4:
If $\mathcal{P} = \{p_1, p_2, \ldots, p_k\} \subseteq \mathcal{Z}$ where $(\mathcal{Z},\mathcal{N})$ is a c-FNS then $\mathcal{P}$ is a convex fuzzy compact.

Definition 3.5:
Let $\Sigma = \{A_j$:j∈J} be a family of subsets of a space $\mathcal{Z}$. The family Σ is **centred** if for any finite number of sets $A_1, A_2, \ldots, A_k \in \Sigma$ we have $\cap_{j=1}^{k} A_j \neq \emptyset$.

Theorem 3.6:
The following two statements are equivalent where $(\mathcal{Z},\mathcal{N})$ is a c-FNS:
(1) $\mathcal{Z}$ is convex fuzzy compact;
(2) We have $\cap_{A \in \Sigma} A \neq \emptyset$ for any centred family Σ of convex fuzzy closed subset of Σ.
Proof:
(2)⟹(1)
Let $\Sigma = \{A_j$:j∈J} be a fuzzy open cover of $\mathcal{Z}$. We need to show that Σ has a finite subcover. For j∈J, define $G_j = \mathcal{Z} - A_j$ this gives a family $\mathcal{G} = \{G_j$:j∈J} of convex fuzzy closed sets in $\mathcal{Z}$. We have $\cap_{j \in J} G_j = \cap_{j \in J}[\mathcal{Z} - A_j] = \mathcal{Z} - [\cup_{j \in J} A_j] = \mathcal{Z} - \mathcal{Z} = \emptyset$, since $\mathcal{Z} = \cup_{j \in J} A_j$.
This implies that $\mathcal{G}$ is not centered family, so there exists $G_1, G_2, \ldots, G_k \in \mathcal{G}$ such that $\cap_{j=1}^{k} G_j = \emptyset$. This gives $\emptyset = \cap_{j=1}^{k} G_j = \cap_{j=1}^{k} [\mathcal{Z} - A_j] = \mathcal{Z} - [\cup_{j=1}^{k} A_j]$.
Therefore, $\mathcal{Z} = \cup_{j=1}^{k} A_j$ and so $\mathcal{Z}$ is a convex fuzzy compact since $\{A_j$:j=1, 2, …, k} is a finite subcover of Σ.
(1)⟹(2)
Can be established following similar technique.

Definitions 3.7:
Let $\mathcal{Y}$ be a subset of c-FNS $(\mathcal{Z}, \mathcal{N})$ and let $0< \alpha <1$.
(1) The set $\mathcal{B}_\alpha \subseteq \mathcal{Y}$ is an **convex α-fuzzy net** for $\mathcal{Y}$ whenever $y \in \mathcal{Y}$ $\exists$ $b \in \mathcal{B}_\alpha$ satisfying $\mathcal{N}(y - b)< \alpha$.
(2) If for all $0< \alpha <1$, there exists a convex α-fuzzy net $\mathcal{B}_\alpha=\{b_1, b_2, \dots, b_k\}\subseteq \mathcal{Y}$ then $\mathcal{Y}$ is called **totally convex fuzzy bounded.**

Theorem 3.8:
The following two statements are equivalent whenever $(\mathcal{Z},\mathcal{N})$ is a c-FNS and $\mathcal{Y} \subseteq \mathcal{Z}$.
(1)$\mathcal{Y}$ is totally convex fuzzy bounded;
(2) if for all (p_k) in $\mathcal{Y}$ contains (p_{k_j}) as a fuzzy Cauchy.

Proof: (1)⟹(2)
Choose a finite convex $\frac{1}{2}$ -fuzzy net $\mathcal{B}_{\frac{1}{2}}$ in $\mathcal{Y}$. Then one of the cfb$(b_j, \frac{1}{2})$ for $b_j \in \mathcal{B}_{\frac{1}{2}}$ contains infinitely many elements of the members of (p_k). Let $(p_{1\mathrm{k}}) \subseteq (p_k)$ denote these members. Again, pick a finite convex fuzzy $\frac{1}{4}$ –net $\mathcal{B}_{\frac{1}{4}}$ in $\mathcal{Y}$. Then one of the cfb$(b_j, \frac{1}{2})$ for $b_j \in \mathcal{B}_{\frac{1}{4}}$ contains infinitely many elements of the members of $(p_{1\mathrm{k}})$.
Let $(p_{2\mathrm{k}}) \subseteq (p_{1\mathrm{k}})$ denote these members. After finite steps, we have $(p_{1\mathrm{k}}) \supseteq (p_{2\mathrm{k}}) \supseteq\dots$ so that at the jth stage, the terms (p_{jk}) lie in cfb$(b_j, \frac{1}{2^{\mathrm{j}}})$ for $b_j \in \mathcal{B}_{\frac{1}{2^{\mathrm{j}}}}$.
Hence, (p_{jj}) is a subsequence of (p_{j}). Thus when $\mathrm{m} > \mathrm{j} > \mathrm{N}$, we have

$$\mathcal{N}(p_{\mathrm{jj}} - p_{\mathrm{mm}}) \leq \delta_j \mathcal{N}(p_{\mathrm{jj}} - p_{\mathrm{j+1j+1}})+ \delta_{j+1}\, \mathcal{N}(p_{\mathrm{j+1j+1}} - p_{\mathrm{j+2j+2}}) +\dots + \delta_m \mathcal{N}(p_{\mathrm{m-1m-1}} - p_{\mathrm{mm}}),$$

where $\delta_j, \delta_{j+1},\dots, \delta_m \in[0, 1]$ with $\delta_j+ \delta_{j+1}+\dots+ \delta_m=1$. It follows that

$$\mathcal{N}(p_{\mathrm{jj}} - p_{\mathrm{mm}}) \leq \delta_j(\tfrac{1}{2^{\mathrm{j}}}) +\delta_{j+1}(\tfrac{1}{2^{\mathrm{j+1}}}) + \dots + \delta_m(\tfrac{1}{2^{\mathrm{m-j}}}).$$

Choose σ, $0< \sigma <1$, with $\delta_j(\frac{1}{2^{\mathrm{j}}}) +\delta_{j+1}(\frac{1}{2^{\mathrm{j+1}}}) + \dots + \delta_m(\frac{1}{2^{\mathrm{m-j}}}) < \sigma$.
Therefore, $\mathcal{N}(p_{\mathrm{jj}} - p_{\mathrm{mm}}) < \sigma$. Thus the sequence (p_{kk}) is convex fuzzy Cauchy sequence.
(2)⟹(1)
Assume that for all (p_k) in $\mathcal{Y}$ contains (p_{k_j}) as a fuzzy Cauchy and suppose that $0 < \alpha < 1$. Let $p_1 \in \mathcal{Y}$. If $\mathcal{Y}$ −cfb$(p_1,\alpha) = \emptyset$, we get a convex α-fuzzy net, namely, the set $\{p_1\}$. Otherwise choose $p_2 \in \mathcal{Y}$ −cfb(p_1, α). If $\mathcal{Y}$ −[cfb$(p_1, \alpha) \cup$ cfb(p_2, α)] = $\emptyset$. We get a convex α-fuzzy net, namely the set $\{p_1,p_2\}$. If it does not stop after finite steps, it will be construct (p_{k}) satisfying $\mathfrak{N}(p_{\mathrm{j}} - p_{\mathrm{m}}) \geq \alpha$, $\mathrm{j} \neq \mathrm{m}$. Hence, (p_{k}) does not contains (p_{k_j}) as a fuzzy Cauchy, this a contradict our assumption.

Proposition 3.9:
Let $(\mathcal{Z},\mathcal{N})$ be a c-FNS. If $\mathcal{Z}$ is convex fuzzy compact then $\mathcal{Z}$ is totally convex fuzzy bounded.
Proof:
Since $\mathcal{Z}=\cup_{y\in\mathcal{Z}}$ cfb(y, α), so for all $0 < \alpha < 1$, the collection Λ =\{cfb(y, α) : $y \in \mathcal{Z}$\} is a convex fuzzy open cover of $\mathcal{Z}$. Using our assumption $\mathcal{Z}$ is convex fuzzy compact we have Λ contains \{cfb(p_1, α_1), cfb(p_2, α_2), …, cfb(p_k, α_k)\}. Hence, $\mathcal{Z}$ is convex fuzzy totally bounded.

Theorem 3.10:
Let $(\mathcal{Z},\mathcal{N})$ be a c-FNS. If $\mathcal{Z}$ is a convex fuzzy compact then $\mathcal{Z}$ is convex fuzzy complete.

Proof:

Assume that $\mathcal{Z}$ is not convex fuzzy complete. Then there exists convex fuzzy Cauchy sequence $(u_k) \in \mathcal{Z}$ with $u_k \nrightarrow y$ in$\mathcal{Z}$. If $y \in \mathcal{Z}$, since $u_k \nrightarrow y$ so there exists $0 < \alpha < 1$ satisfying $\mathcal{N}(u_k - y) \geq \alpha$ for infinitely k $\in \mathbb{N}$ since (u_k) is convex fuzzy Cauchy, there exists N$\in \mathbb{N}$ satisfy j, $\ell \geq$ N so $\mathcal{N}(u_j - u_\ell) < \alpha$. Consider $\ell \geq$ N for which $\mathcal{N}(u_\ell - y) < \alpha$. So, cfb(y, α) contains u_k for only finite values of k. In this case, with each $y \in \mathcal{Z}$ we can associate cfb(y, $\alpha(y)$) ,where $0 < \alpha(y) < 1$ depends on y, and cfb(y, $\alpha(y)$) contains u_k with finite values of k. Notice that
$\mathcal{Z} = \cup_{y \in \mathcal{Z}}$cfb($y$, $\alpha(y)$) that is Λ={cfb(y, $\alpha(y)$): $y \in \mathcal{U}$} is a convex fuzzy open cover of $\mathcal{Z}$. Using $\mathcal{Z}$ is convex fuzzy compacts we have {cfb(y_j, $\alpha(y_j)$), j = 1, 2, …, k}$\subseteq \Lambda$ such that
$\mathcal{Z} = \cup_{j=1}^{k}$cfb($y_j$, $\alpha(y_j)$), Since each cfb(y_j, $\alpha(y_j)$) contains u_k for a finite values of k, but $\mathcal{Z}$ $= \cup_{j=1}^{k}$ cfb(y_j, $\alpha(y_j)$) and also $\mathcal{Z}$ is finite but $\mathcal{Z}$ is not finite. Therefore, $\mathcal{Z}$ is convex fuzzy complete.

Lemma 3.11:
Let $(\mathcal{Z},\mathcal{N})$ be a c-FNS and $\mathcal{Y} \subseteq \mathcal{Z}$. If $\mathcal{Z}$ is totally convex fuzzy bounded then $\mathcal{Y}$ is totally convex fuzzy bounded.
Proof:
When $\mathcal{Y} = \emptyset$ is clear. Let $\mathcal{Y} \neq \emptyset$ and given $0 < \alpha < 1$, there exists a finite α_1-fuzzy net B_{α_1}= $\{b_1, b_2, \dots, b_k\} \subseteq \mathcal{Z}$ for $\mathcal{Y}$ where $\alpha_1 = \frac{\alpha}{2}$. Thus $\mathcal{Y} \subseteq \cup_{j=1}^{k}$ cfb(b_j, α_1). Let B_j= cfb(b_j, α_1) for j=1,2,…,k, which intersect $\mathcal{Y}$. Let $z_j \in (\mathcal{Y} \cap B_j)$. Thus B_α={z_1, z_2, …, z_k} $\subseteq \mathcal{Y}$ is convex α-fuzzy net for $\mathcal{Y}$ since for all $y \in \mathcal{Y}$ there exists B_j where $y \in B_j$ satisfying
$\mathfrak{N}(z - z_j) \leq \delta_1 \mathfrak{N}(z - b_j) + \delta_2 \mathfrak{N}(b_j - z_j) \leq \delta_1 \alpha_1 + \delta_2 \alpha_1 = (\delta_1 + \delta_2)\, \alpha_1 = \alpha_1 < \alpha.$
Since δ_1 , $\delta_2 \in [0, 1]$ with $\delta_1 + \delta_2 = 1$.

Theorem 3.12:
Suppose that $(\mathcal{Z},\mathcal{N})$ be a c-FNS. If $\mathcal{Z}$ is totally convex fuzzy bounded and convex fuzzy complete, then $\mathcal{Z}$ is a convex fuzzy compact

Proof:
Assume that $\mathcal{Z}$ is not convex fuzzy compact. Then there exists $\Pi = \{\mathfrak{B}_\mu: \mu \in \Gamma\}$ such that $\mathcal{Z} = \cup_{\mu \in \Gamma}$ $\mathfrak{B}_\mu$ that does not has {$\mathfrak{B}_1$, $\mathfrak{B}_2$, …,$\mathfrak{B}_k$} such that $\mathcal{Z} = \cup_{j=1}^{k} \mathfrak{B}_j$. Since $\mathcal{Z}$ is convex fuzzy totally bounded, it is convex fuzzy bounded by Lemma 3.11, thus for some $0 < \sigma < 1$ and some p$\in$ $\mathcal{Z}$, we have $\mathcal{Z} \subseteq$ cfb(p, σ). Observe that $\mathcal{Z} \subseteq$ cfb(p, σ) implies $\mathcal{Z}$ = cfb(p, σ).
Let $\alpha_k = \frac{\sigma}{2^k}$. Since $\mathcal{Z}$ is a convex fuzzy totally bounded $\mathcal{Z} = \cup_{j=1}^{m}$cfb($p_j$, α_j). Thus there exists cfb(p_1, α_1) such that cfb(p_1, α_1) $\neq \cup_{j=1}^{k} \mathfrak{B}_j$. But cfb($p_1$, α_1) is itself convex fuzzy totally bounded [if $\mathcal{Y} \neq \emptyset \subset \mathcal{Z}$ where $\mathcal{Z}$ is convex fuzzy totally bounded set then so is $\mathcal{Y}$], so there exists $p_2 \in$ cfb(p_1, α_1), such that cfb(p_2, α_2) $\neq \cup_{j=1}^{k} \mathfrak{B}_j$. After many steps, we construct a sequence (p_k) satisfying the condition that for each k, cfb(p_k, α_k) $\neq \cup_{j=1}^{k} \mathfrak{B}_j$ and p_{k+1} $\in$fb(p_k,α_k). We next show that the sequence (p_k) is convex fuzzy convergent.
Since $p_{k+1} \in$cfb(p_k,α_k) it follows that $\mathcal{N}(p_k - p_{k+1}) < \alpha_k$. Hence,
$\mathcal{N}(p_k - p_m) \leq \delta_k \mathcal{N}(p_k - p_{k+1}) + \dots\dots + \delta_m \mathcal{N}(p_{m-1} - p_m)$,
where $\delta_k + \delta_{k+1} + \dots + \delta_m = 1$. Now, $\mathcal{N}(p_k - p_m) \leq \delta_k \alpha_k + \dots\dots + \delta_m \alpha_m$.
Choose ε, $0 < \varepsilon < 1$ with $\delta_k \alpha_k + \dots\dots + \delta_m \alpha_m < \varepsilon$, (for ε=min{1, $\delta_k \alpha_k + \dots\dots + \delta_m \alpha_m$).
It follows that $\mathcal{N}(p_k - p_m) < \varepsilon$.
So, (p_k) is a convex fuzzy Cauchy in $\mathcal{Z}$, but $\mathcal{Z}$ is a convex fuzzy complete, $p_k \to p \in \mathcal{Z}$. Since $p \in \mathcal{Z}$ there exists $\mu_0 \in \Gamma$ satisfying $p \in \mathfrak{B}_{\mu_0}$. Using $\mathfrak{B}_{\mu_0}$ is convex fuzzy open it contains cfb(p,

δ) for $\delta \in (0, 1)$. Pick N so large that, $\mathcal{N}(p_k - p) < \mu$ so for all $y \in \mathcal{Z}$ satisfies $\mathcal{N}(y - p_k) < \alpha_k$. Therefore, $\mathcal{N}(y - p) \leq \gamma_1 \mathcal{N}(y - p_k) + \gamma_2 \mathcal{N}(p_k - p)$ where $\gamma_1 + \gamma_2 = 1$. This implies that $\mathcal{N}(y - p) \leq \gamma_1 \alpha_k + \gamma_2 \mu < \sigma$. So that cfb($p_k$, α_k) $\subseteq$ cfb(p, σ). Therefore, cfb(p_k,α_k) = $\cup_{\mu_0} \mathfrak{B}_{\mu_0}$ by a finite sets $\mathfrak{B}_{\mu_0}$. But cfb(p_k, α_k) $\neq \cup_{j=1}^{k} \mathfrak{B}_j$ which is a contradiction. By adding Theorem3.10 and Theorem3.12 we obtain

Theorem 3.13:
The following two statements are equivalent when ($\mathcal{Z}$,$\mathcal{N}$) is a c-FNS
(1) $\mathcal{Z}$is convex fuzzy compact;
(2) $\mathcal{Z}$ is convex fuzzy complete and totally convex fuzzy bounded.

Proposition 3.14:
Consider that ($\mathcal{Z}$,$\mathfrak{N}$) is a c-FNS. Then for all $\{u_1, u_2,\ldots,\}$ in $\mathcal{Z}$ has at least one limit point in $\mathcal{Z}$ if and only if every sequence (p_k) in $\mathcal{Z}$ contains a convex fuzzy convergent subsequence.

Proof:
Let $(u_k) \in \mathcal{Z}$, and suppose that for all $\{u_1, u_2,\ldots,\}$ in $\mathcal{Z}$ has at least one limit point in $\mathcal{Z}$. If the set $\{u_1, u_2,\ldots,\}$ is finite, then choose any member, say u_j hence $(u_j, u_j, \ldots)$ is a subsequence of (u_k), which $(u_j, u_j, \ldots) \to u_j$. Suppose that the set $\{u_1, u_2,\ldots,\}$ is not finite.
Now, the set $\{u_1, u_2,\ldots,\}$ has limit point $u \in \mathcal{Z}$. Let k_1 be any integer such that $\mathfrak{N}(u_{k_1} - u) > 0$. Define k_j, $k_1 < k_2 < \ldots < k_j < k_{j+1} < \ldots$, and $\mathfrak{N}(u_{k_{j+1}} - u) < \frac{1}{(j+1)}$. Thus, $u_{k_j} \to u$.
For the converse let for all (p_k) in $\mathcal{Z}$ contains (p_{k_i}) such that $p_{k_i} \to p \in \mathcal{Z}$. Put S=$\{p_1, p_2, \ldots\} \subseteq \mathcal{Z}$, so there exists ($p_k$) in $\mathcal{Z}$ of distinct terms. Now, (p_k) has (p_{k_i}) of distinct terms $p_{k_i} \to p \in .\mathcal{Z}$ Hence every cfb(p,α) contains p_{k_1}, p_{k_2}, ... of (p_{k_i}). But the terms are distinct hence every cfb(p,α) contains p_{k_1}, p_{k_2}, ... of S. Therefore, S has a limit point $p \in \mathcal{Z}$.

Theorem 3.15:
The following two statements are equivalent when ($\mathcal{Z}$,$\mathcal{N}$) is c-FNS
(1) $\mathcal{Z}$ is convex fuzzy compact;
(2) For all (u_k) in $\mathcal{Z}$ has a subsequence (u_{k_i}) such that $u_{k_i} \to u \in \mathcal{Z}$.

Proof: (1)$\Longrightarrow$ (2)
Let $\mathcal{Z}$ be convex fuzzy compact (that is totally convex fuzzy bounded and convex fuzzy complete see Theorem 3.10. Let $(u_k) \in \mathcal{Z}$. But $\mathcal{Z}$ is a convex fuzzy totally bounded by using Theorem 3.13 we have (u_k) contains a convex fuzzy Cauchy subsequence (u_{k_i}). But $u_{k_i} \to u \in \mathcal{Z}$ because $\mathcal{Z}$ is a fuzzy complete. Thus, if $\mathcal{Z}$ is a fuzzy compact, then for all (u_k) in $\mathcal{Z}$ has a subsequence (u_{k_i}) such that $u_{k_i} \to u \in \mathcal{Z}$.
(2)$\Longrightarrow$ (1) For all (u_k) in $\mathcal{Z}$ has a subsequence (u_{k_i}) such that $u_{k_i} \to u \in \mathcal{Z}$ and by Theorem 3.10, $\mathcal{Z}$ is totally convex fuzzy bounded. To show that $\mathcal{Z}$ is a convex fuzzy complete, let (p_k) be a convex fuzzy Cauchy sequence in $\mathcal{Z}$. By assumption (p_k) has a subsequence (p_{k_i}) and $p_{k_i} \to p \in \mathcal{Z}$. We shall show that $p_k \to p$. Let $0 < \alpha < 1$. Now, $p_{k_i} \to p$, so there exists N_1 satisfying $\mathcal{N}(p_{k_i} - p) < \alpha$ for all $k_i \geq N_1$. But (p_k) is a convex fuzzy Cauchy so there exists N_2 satisfying $\mathcal{N}(p_j - p_m) < \alpha$ for all $m, j \geq N_2$.
Consider $N = \min\{N_1, N_2\}$, then $\mathcal{N}(p_k - p) \leq \delta_1 \mathcal{N}(p_k - p_{k_i}) + \delta_2 \mathcal{N}(p_{k_i} - p)$
where $\delta_1, \delta_2 \in [0, 1]$ with $\delta_1 + \delta_2 = 1$.
It follows that $\mathcal{N}(p_k - p) \leq \delta_1 \alpha + \delta_2 \alpha = \alpha$ for all $k \geq N$. This complete the proof.

Definition 3.16:
Let $(\mathcal{Z},\mathcal{N})$ be a c-FNS and $\mathcal{D} \subseteq \mathcal{Z}$. Then $\mathcal{D}$ is called a **relatively convex fuzzy compact** if $\overline{\mathcal{D}}$ is a convex fuzzy compact.

Proposition 3.17:
Let $(\mathcal{U},\mathfrak{N})$ be a c-FNS and $\mathcal{D} \subseteq \mathcal{U}$. If $\mathcal{D}$ is a relatively convex fuzzy compact then $\mathcal{D}$ is totally convex fuzzy bounded.

Proof:
If $\mathcal{D} =\emptyset$ then $\emptyset$ is a convex α-fuzzy net for $\mathcal{D}$. Assume that $\mathcal{D} \neq \emptyset$ we select $d_1 \in \mathcal{D}$ if $\mathfrak{N}(d_1 - d)< \alpha$ for all $d \in \mathcal{D}$ then $\{d_1\}$ is a convex α-fuzzy net for $\mathcal{D}$. Otherwise let $d_2 \in \mathcal{D}$ be such that $\mathfrak{N}(d_1 - d_2)\geq \alpha$. If for all $d \in \mathcal{D}$, $\mathfrak{N}(d_2 - d)< \alpha$ and $\mathfrak{N}(d_1 - d)< \alpha$ then $\{ d_1, d_2\}$ is a convex α-fuzzy net for $\mathcal{D}$.
Otherwise let $d_3 \in \mathcal{D}$ be such that $\mathfrak{N}(d_1 - d_3)\geq \alpha$, $\mathfrak{N}(d_2 - d_3)\geq \alpha$. If for all $d \in \mathcal{D}$, $\mathfrak{N}(d - d)< \alpha$ for j=1,2,3. Thus $\{d_1, d_2, d_3\}$ is a convex α-fuzzy net for $\mathcal{D}$. By Continue this process we select an $d_4 \in \mathcal{D}$, …, etc. After k steps the set $\{d_1, d_2, d_3, \ldots, d_k\}$ is a convex α-fuzzy net for $\mathcal{D}$. If this process does not stop this will give a sequence (d_j) satisfying $\mathfrak{N}(d_m - d_i)\geq \alpha$ for all m≠i. Therefore, (d_j) does not contain a fuzzy Cauchy subsequence. Hence (d_j) could not have (d_{j_k}) with $d_{j_k} \to d \in \mathcal{D}$. Since $\mathcal{D}$ is relatively convex fuzzy compact this a contradiction. Hence there must be a finite convex α-fuzzy net for $\mathcal{D}$. Thus $\mathcal{D}$ is totally convex fuzzy bounded since $\alpha \in(0, 1)$ was arbitrary.

Theorem 3.18:
The set $\mathcal{Y}$ is relatively convex fuzzy compact if $\mathcal{Y}$ is totally convex fuzzy bounded and $\mathcal{Z}$ is convex fuzzy complete when $(\mathcal{Z},\mathfrak{N})$ be a c-FNS and $\mathcal{Y} \subseteq \mathcal{Z}$.

Proof:
Consider (b_j) in $\mathcal{Y}$ hence $\mathcal{Y} \subseteq \cup_{j=1}^{k}$ cfb(w_j, 1) where B_1=$\{w_1, w_2, \ldots, w_k\}$. We can choose $\mathfrak{B}_1$=cfb(w_1, α =1) which contains b_1, b_2,... of (b_j). Let $(b_j^{(1)}) \subset (b_j)$ and $(b_j^{(1)}) \subset \mathfrak{B}_1$. Again we can choose $\mathfrak{B}_2$=cfb(w_2,α=$\frac{1}{2}$) which contains infinitely many terms $(b_j^{(1)})$. Let be $(b_j^{(2)})$ of the subsequence $(b_j^{(1)})$ which lies in $\mathfrak{B}_2$.
By continue this process by using an induction selecting for α=$\frac{1}{3}$, α=$\frac{1}{4}$, …, and putting y_k=$b_k^{(k)}$.
Thus, for all $0< \alpha <1$ there exists N satisfying all y_k with k>N are in cfb(w_k,α).
Therefore, (y_j) is fuzzy Cauchy in $\mathcal{Z}$ thus $y_j \to$y $\in \mathcal{Z}$ since $\mathcal{Z}$ is a convex fuzzy complete as well as $y_j \in \mathcal{Y}$ so y$\in \mathcal{Y}$.
For all (z_k) in $\bar{\mathcal{Y}}$ $\exists$ (w_k) in $\mathcal{Y}$ which satisfies $\mathcal{N}(z_k - w_k)\leq \frac{1}{k}$ for all k. But (w_k) in $\mathcal{Y}$, so it has $(w_{k_j}) \subset (w_k)$ such that $w_{k_j} \to$w $\in \bar{\mathcal{Y}}$. Therefore, (z_k) also has $(z_{k_j}) \subset (z_k)$ such that $z_{k_j} \to$z $\in \bar{\mathcal{Y}}$. But $\mathcal{N}(z_k - w_k)\leq \frac{1}{k}$ thus $\bar{\mathcal{Y}}$ is a convex fuzzy compact and $\mathcal{Y}$ is a relatively convex fuzzy compact.

Proposition 3.19:
The set $\mathcal{Y}$ is separable if $\mathcal{Y}$ is totally convex fuzzy bounded whenever $(\mathcal{Z},\mathcal{N})$ be a c-FNS and $\mathcal{Y} \subseteq \mathcal{Z}$.

Proof:
If $\mathcal{Y}$ is totally convex fuzzy bounded then by Lemma 3.11, $\mathcal{Y} \subseteq \cup_{k=1}^{j}$cfb($w_j$, $\alpha = \alpha_k = \frac{1}{k}$), $B_\alpha = \{w_1, w_2, \ldots, w_j\}$ for itself where $\alpha = \alpha_k = \frac{1}{k}$, k=1,2, … .
Put $\mathcal{B} = \cup_{k=1}^{\infty} B_{\alpha_k}$ then $\mathcal{B}$ is countable and fuzzy dense in $\mathcal{Y}$. It is clear that for all $\alpha \in (0, 1)$ there exists k satisfying $\frac{1}{k} < \alpha$, thus for all $y \in \mathcal{Y}$ there exists $b \in B_{\frac{1}{k}} \subseteq \mathcal{B}$ satisfying $\mathcal{N}(y - b) < \alpha$. Therefore, $\mathcal{Y}$ is separable.

Proposition 3.20:
The set $\mathcal{Z}$ is convex fuzzy bounded if the c-FNS $(\mathcal{Z}, \mathcal{N})$ is a totally convex fuzzy bounded.

Proof:
For all $0 < \alpha < 1$ there exists a finite convex α-fuzzy net B_α for $\mathcal{Z}$. Put $\mathcal{N}(B_\alpha) = \sup\{\mathcal{N}(b): b \in B_\alpha\}$. By considering $u_1 \in \mathcal{Z}$ then there exists $b \in B_\alpha$ satisfying $\mathcal{N}(u_1 - b) < \alpha$. Now, $\mathcal{N}(u_1) = \mathcal{N}(u_1 - b + b) \leq \delta_1 \mathcal{N}(u_1 - b) + \delta_2 \mathcal{N}(b)$, where δ_1, $\delta_2 \in [0, 1]$ with $\delta_1 + \delta_2 = 1$. Hence, $\mathcal{N}(u_1) \leq \delta_1 \alpha + \delta_2 \mathcal{N}(B_\alpha)$. Thus, we can find σ, $0 < \sigma < 1$ where $\delta_1 \alpha + \delta_2 \mathcal{N}(B_\alpha) < \sigma$.
It follows that $\mathcal{N}(u_1) < \sigma$. Therefore, $\mathcal{Z}$ is a convex fuzzy bounded.

Theorem 3.21:
If $(\mathcal{Z}_1, \mathcal{N}_1)$ and $(\mathcal{Z}_2, \mathcal{N}_2)$ are c-FNS then $(\mathcal{Z}_1, \mathcal{N}_1)$ and $(\mathcal{Z}_2, \mathcal{N}_2)$ are convex fuzzy compact if and only if $(\mathcal{Z}, \mathcal{N})$ is a convex fuzzy compact where $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$ and $\mathcal{N}[(z_1, z_2)] = \alpha\, \mathcal{N}_1(z_1) + \beta\, \mathcal{N}_2(z_2)$ for all $(z_1, z_2) \in \mathcal{Z}$ where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

Proof:
Let $(\mathcal{Z}_1, \mathcal{N}_1)$ and $(\mathcal{Z}_2, \mathcal{N}_2)$ be two convex fuzzy compact c-FNS. Let $(u_k) \in \mathcal{Z}$ then $(u_k) = (u_{1k}, u_{2k})$ where $(u_{1k}) \in \mathcal{Z}_1$ and $(u_{2k}) \in \mathcal{Z}_2$. Hence, (u_{1k}) has a convex fuzzy convergent subsequence (u_{1k_j}) that is there is $u_1 \in \mathcal{Z}_1$such that $\mathcal{N}_1(z_{1k_j} - z_1)$ convex fuzzy converges to zero as $k_j \to \infty$. Similarly, (z_{2k}) has a convex fuzzy convergent subsequence (u_{2k_j}) that means there is $u_2 \in \mathcal{Z}_2$such that $\mathcal{N}_2(u_{2k_j} - u_2)$ convex fuzzy converges to zero as $k_j \to \infty$. Now, (u_k) has a subsequence (u_{1k_j}, u_{2k_j}) which is convex fuzzy converge to $(z_1, z_2) \in \mathcal{Z}$ since
$\mathcal{N}[((z_{1k_j}, z_{2k_j}) - ((z_1, z_2)] = [\alpha \mathcal{N}_1(z_{1k_j} - z_1) + \beta \mathcal{N}_2(z_{2k_j} - z_2)]$
convex fuzzy converges to zero as $k_j \to \infty$. Therefore, $(\mathcal{Z}, \mathcal{N})$ is a fuzzy compact.
Conversely, let $(\mathcal{Z}, \mathcal{N})$ be fuzzy compact, (u_{1k}) be a sequence in $(\mathcal{Z}_1, \mathcal{N}_1)$, and (u_{2k}) be sequence in $(\mathcal{Z}_2, \mathcal{N}_2)$. Then (u_{1k}, u_{2k}) is a sequence in $\mathcal{Z}$. Now, as $\mathcal{Z}$ is a fuzzy compact so (u_{1k}, u_{2k}) has a convex fuzzy convergent subsequence (u_{1k_j}, u_{2k_j}) that means there is $(u_1, u_2) \in \mathcal{Z}$ such that
$0 = \lim_{k_j \to \infty} \mathcal{N}[((u_{1k_j}, u_{2k_j}) - ((u_1, u_2)]$.
$0 = \alpha \lim_{k_j \to \infty} \mathcal{N}_1(u_{1k_j} - u_1) + \beta \lim_{k_j \to \infty} \mathcal{N}_2(u_{2k_j} - u_2)]$.
This implies that $\lim_{k_j \to \infty} \mathcal{N}_1(u_{1k_j} - u_1) = 0$ since $\alpha \neq 0$ and $\lim_{k_j \to \infty} \mathcal{N}_2(u_{2k_j} - u_2)$ that means (u_{1k}) has a convex fuzzy convergent subsequence (u_{1k_j}), and (u_{2k}) has a convex fuzzy convergent subsequence (u_{2k_j}). Thus, $(\mathcal{Z}_1, \mathcal{N}_1)$ and $(\mathcal{Z}_2, \mathcal{N}_2)$ are convex fuzzy compact.
The next result we can prove it in a similar way.

Corollary 3.22:
If $(\mathcal{Z}_1, \mathcal{N}_1)$, $(\mathcal{Z}_2, \mathcal{N}_2)$, …, $(\mathcal{Z}_k, \mathcal{N}_k)$ are c-FNS then the following two statements are equivalent
(1) $(\mathcal{Z}, \mathcal{N})$ is convex fuzzy compact;
(2) $(\mathcal{Z}_1, \mathcal{N}_1)$, $(\mathcal{Z}_2, \mathcal{N}_2)$, …, $(\mathcal{Z}_k, \mathcal{N}_k)$ are convex fuzzy compact,
where $\mathcal{Z}= \mathcal{Z}_1 \times \mathcal{Z}_2 \times \ldots \times \mathcal{Z}_k$ and $\mathcal{N}[(y_1, y_2, \ldots, y_k)]=\delta_1\mathcal{N}_1(y_1) +\delta_2 \mathcal{N}_2(y_2) + \ldots, +\delta_k \mathcal{N}_k(y_k)$ for all $(y_1, y_2, \ldots, y_k) \in \mathcal{Z}$ where $\delta_1, \delta_2, \ldots, \delta_k \in[0, 1]$ with $\delta_1+ \delta_2+\ldots+ \delta_k=1$.

In view of the Theorem 3.21, we state the following result without proof

Theorem 3.23:
If $(\mathcal{Z}, \mathcal{N}_\mathcal{Z})$ is a c-FNS then $(\mathcal{Z}^2, \mathcal{N}_2)$ is convex fuzzy compact if and only if $(\mathcal{Z}, \mathcal{N}_\mathcal{Z})$ is a convex fuzzy compact where $\mathcal{Z}^2= \mathcal{Z} \times \mathcal{Z}$ and $\mathcal{N}_2[(u_1, u_2)]=\alpha\ \mathcal{N}_\mathcal{Z}(u_1) +\beta\ \mathcal{N}_\mathcal{Z}(u_2)$ for all $(u_1, u_2) \in \mathcal{U}^2$ where α , $\beta \in[0, 1]$ with $\alpha + \beta=1$.

In similar way we can prove the next result

Corollary 3.24:
If $(\mathcal{Z}, \mathcal{N}_\mathcal{Z})$ is a c-FNS then the following two statements are equivalent
(1) $(\mathcal{Z}^m,\mathcal{N})$ is convex fuzzy compact;
(2) $(\mathcal{Z}, \mathcal{N}_\mathcal{Z})$ are fuzzy compact,
where $\mathcal{Z}^m = \mathcal{Z} \times \mathcal{Z} \times \ldots \times \mathcal{Z}$ [m-times], m $\in \mathbb{N}$ and
$\mathcal{N}[(u_1, u_2, \ldots, u_m)]=\delta_1\ \mathcal{N}_\mathcal{Z}(u_1) + \delta_2\mathcal{N}_\mathcal{Z}(u_2) + \ldots +\delta_k\ \mathcal{N}_\mathcal{Z}(u_m)$,
for all $(u_1, u_2, \ldots, u_m) \in \mathcal{Z}^m$ where $\delta_1, \delta_2, \ldots, \delta_k \in[0, 1]$ with $\delta_1+ \delta_2+ \ldots+\delta_k=1$.

Corollary 3.25:
The set $\mathcal{Y}$ is a convex fuzzy compact if $\mathcal{Y}$ is a convex fuzzy closed and $\mathcal{Z}$ is a convex fuzzy compact whenever $(\mathcal{Z}, \mathcal{N})$ is a c-FNS and $\mathcal{Y} \subseteq \mathcal{Z}$.

Proof:
If $(p_k) \in \mathcal{Y}$, so $(p_k) \in \mathcal{Z}$ and $(p_{k_j}) \subseteq (p_k)$, also $p_{k_j} \to$ u$\in \mathcal{Z}$. But u$\in \mathcal{Y}$ because $\mathcal{Y}$ is a convex fuzzy closed. Thus, for all $(p_k) \in \mathcal{Y}$ has $(p_{k_j}) \subseteq (p_k)$ also $p_{k_j} \to$ u$\in \mathcal{Y}$. Therefore, $\mathcal{Y}$ is a convex fuzzy compact by Theorem 3.10.

Theorem 3.26:
The set $\mathcal{Y}$ is a convex fuzzy closed subset of $\mathcal{Z}$ if $\mathcal{Y}$ is a convex fuzzy compact whenever $\mathcal{Y}$ is a subset of c-FNS $(\mathcal{Z}, \mathcal{N})$.

Proof:
If $y \in \bar{\mathcal{Y}}$, so there exists (y_k) in $\mathcal{Y}$ sstisfying $y_k \to y$. Hence, (y_k) is a convex fuzzy Cauchy in $\mathcal{Y}$. Since $\mathcal{Y}$ is convex fuzzy complete, $y_k \to d$ in $\mathcal{Y}$. Thus $d= y$ and so $\in \mathcal{Y}$. Hence, $\bar{\mathcal{Y}} \subset \mathcal{Y}$ and therefore $\mathcal{Y}$ is convex fuzzy closed.

Theorem 3.27:
The following two statements are equivalent
(1)$\mathcal{W}$ is convex fuzzy complete;
(2) $\mathcal{W}$ is convex fuzzy closed.
whenever $(\mathcal{U}, \mathfrak{N})$ is c-FNS and convex fuzzy complete with $\mathcal{W} \subseteq \mathcal{U}$.

Proof:
(1)⟹ (2);
Suppose that $\mathcal{W}$ is convex fuzzy complete then for all z ∈ $\overline{\mathcal{W}}$ there exists $(z_k) \in \mathcal{W}$ satisfying z_k →z.
Since (z_k) is fuzzy Cauchy in $\mathcal{W}$, but $\mathcal{W}$ is a convex fuzzy complete so (z_k) is a convex fuzzy converge in $\mathcal{W}$. Since limit is unique so z ∈ $\mathcal{W}$ this implies that $\overline{\mathcal{W}} \subseteq \mathcal{W}$, but $\mathcal{W} \subseteq \overline{\mathcal{W}}$. Hence, $\mathcal{W} = \overline{\mathcal{W}}$ so $\mathcal{W}$ is a convex fuzzy closed.
(2)⟹ (1);
Suppose that $\mathcal{W}$ is a convex fuzzy closed and let (z_k) be convex fuzzy Cauchy sequence in $\mathcal{W}$, so (z_k) be convex fuzzy Cauchy sequence in $\mathcal{U}$, but $\mathcal{U}$ is a convex fuzzy complete so z_k →z ∈ $\mathcal{U}$, but $\mathcal{W}$ is convex fuzzy closed so z∈ $\mathcal{W}$. Hence, $\mathcal{W}$ is a convex fuzzy complete.

4. Conclusions

In this paper our space is ordinary vector space but our norm is fuzzy, also this paper is depending on the concepts and theorems of the reference [29]. Here we introduce the notion of convex fuzzy compact space after we proved properties of this space. Thus, as a future work the authors can be use the convex fuzzy norm to introduce other notions.

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