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RESEARCH ARTICLE - MATHEMATICS

A numerical method for solving for nonlinear fuzzy integral equation by using modified decomposition method

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Abstract

In this paper, non-linear fuzzy Volterra integral equation of the second kind (NFVIEK2) is considered. The modified decomposition method will be used to solve it. Some numerical examples are prepared to show the efficiency and simplicity of the method.

Keywords: Fuzzy integral; Fuzzy number and sets; Modified decomposition method; non-linear fuzzy Volterra integral equation of second kind.

1. Introduction

The solutions of integral equations have a major role in the field of science and engineering. Since few of these equations can be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [1, 2]. There are several numerical methods for solving linear Volterra integral equation [3] and system of nonlinear Volterra integral equations [4], used a collocation method to solve the Volterra-integral equation numerically, obtained a numerical solution of nonlinear Fredholm integral equations of the second kind. The concept of fuzzy numbers and fuzzy arithmetic operations were first intro. We refer the reader to [5] for more information on fuzzy numbers and fuzzy arithmetic. The topics of fuzzy integral [6] and fuzzy integral equations (FIE) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. The fuzzy mapping function was introduced. Later, presented an elementary fuzzy calculus based on the extension principle also the concept of integration of fuzzy functions was first introduced obtained a numerical solution of linear Fredholm fuzzy integral equations of the second kind, while finding an approximate solution for the fuzzy nonlinear kinds.

$u(x) = f(x) + \lambda \int_a^x k\big(x,t,u(t)\big)\, dt$, is more difficult and a numerical method in this case can be found in [7]. In this paper, we present a novel and very simply numerical method (Modifying decomposition method) for solving fuzzy nonlinear Volterra integral equation of the second kinds.

Definition 1.1 [8]

- Any functional equation in which the unknown function appears under sign of integration is called an integral equation.

The general nonlinear integral equation can be presented in the form:

$$h(x)u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k\big(x,t,u(t)\big)dt, \tag{1}$$

then (1) is called nonlinear integral equation, wherever the indefinite function $u(x)$ to be determined appears under one or several integral signs. Also, if:

$k\big(x,t,u(t)\big) = k(x,t)u(t)$,

then (1) is called nonlinear integral equation.

Note: suppose $\lambda = 1$:

$$h(x)u(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)\, u(t)dt, \qquad (2)$$

here the function $h(\mathrm{x}), f(\mathrm{x})$ and $k(x,t)$ are prescribed while $u(x)$ unknown function to be determined and λ is a scalar parameter named the eigenvalue of the integral equation.

- The integral equation (1) is called Homogeneous if $f(\mathrm{x}) = 0$,

$$h(x)u(x) = \int_{a(x)}^{b(x)} k\big(x,t,u(t)\big)dt,$$

Otherwise, it is non homogenous.

- The integral equation (1) is said to of the first kind if unidentified function u, only appears under the integral sign that is $h(x) = 0$, such that:

$$f(x) = \int_{a(x)}^{b(x)} k\big(x,t,u(t)\big)dt, \qquad (3)$$

- The integral equation (1) is said to be a second kind if $h(x) = 1$, hence:

$$u(x) = f(x) + \int_{a(x)}^{b(x)} k\big(x,t,u(t)\big)dt, \qquad (4)$$

- The integral equation (1) is called Fredholm integral equation of the limit of the integral are say that is, such that $b(\mathrm{x}) = b$ and $a(x) = a$,

$$u(x) = f(x) + \int_{a}^{b} k\big(x,t,u(t)\big)dt, \qquad (5)$$

- The integral equation (1) is called Volterra integral equation if the limit $b(x) = x$ and $\mathrm{a}(\mathrm{x}) = 0$:

$$u(x) = f(x) + \int_{0}^{x} k(x,t), u(t)\, dt, \qquad (6)$$

Definition 1.2 [9]

The membership function of a fuzzy set A is well-defined by $\mu_{\tilde{A}}: X \to [0,1]$, the value of $\mu_{\tilde{A}}(x)$ is called the membership degree of x in X, defined by:

$$\tilde{A} = \big\{ \big(x\, , \mu_{\tilde{A}}(x)\big) : x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1 \big\}$$

The collection of all fuzzy sets in X will be denoted by I^X, that is $I^X = \{\tilde{A} : \tilde{A}$ is a fuzzy set in $X\}$.

Example 1.1 [9]

Let set $\tilde{A} = \{\, (x, \mu_{\tilde{A}}(x)\, , x \in X\, \}$ be a fuzzy set $\tilde{A} = \{real\ number\ near\ 0\}$ and the membership function $\mu_{\tilde{A}}(x) = \frac{1}{1+x^2}$, explanted as the following:

$$\tilde{A} = \{\, (x, \mu_{\tilde{A}}(x)\,) = \{(0,1), (1, 0.5), (2, 0.2), (3, 0.10), \dots \dots\}$$

As a memberships function of fuzzy set ($real\ number\ near\ 0$).

Example 1.2 [9]

Consider a finite set $X = \{a, b, c\}$ and $\mu_{\tilde{A}}: X \to I$. Then $\tilde{A} = \{(a, 0.5), (b, 0.9), (c, 0.8)\}$ is a fuzzy subset of X.

Example 1.3 [9]

We will suppose a possible membership function for the fuzzy set of real numbers close zero as follows, $\mu_{\tilde{A}}: \mathbb{R} \to [0,1]$ where $\mu_{\tilde{A}}(x) = \frac{1}{1+10x^2}$, $\forall x \in \mathbb{R}$.

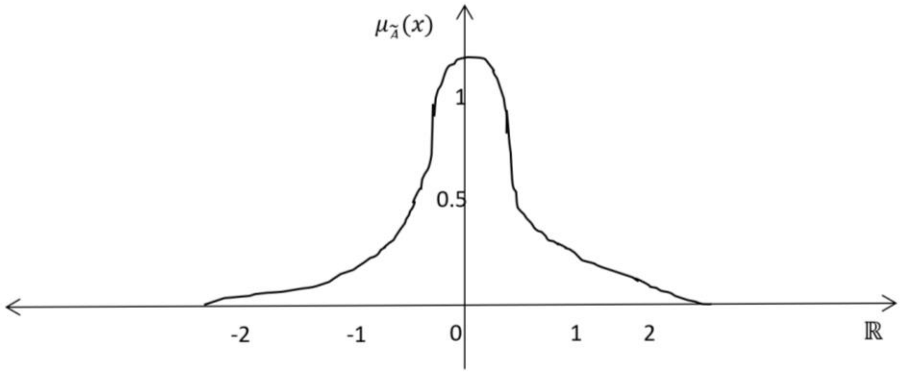

Fig. 1. Relationship between real numbers and fuzzy set

2. Some fundamental fuzzy set concepts

Let X be space of objects, let $\tilde{A}$ be a fuzzy set in X. Then one can express the concept associated to a fuzzy subset $\tilde{A}$ of X.

- Let $\tilde{A}$ be a fuzzy set in X, the support of $\tilde{A}$ denoted $Supp(\tilde{A})$ is the crisp set of X whose elements all have nonzero membership grades in $\tilde{A}$ such that:

$$Supp(\tilde{A}) = \{x |\ \mu_{\tilde{A}}(X) > 0 : x \in X\}$$

Notation 2.1

A fuzzy set $\tilde{A}$ is said to be a finite fuzzy set if and only if $Supp(\tilde{A})$ is a finite set.

Proposition 2.1

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets of X and $\{\tilde{A}_\lambda\}_{\lambda \in \wedge} \subseteq I^X$ be a family of a fuzzy sets in X then:

a. $Supp(\tilde{A} \cap \tilde{B}) = Supp(\tilde{A}) \cap Supp(\tilde{B})$.
b. $Supp(\bigcup_{\lambda \in \wedge} \tilde{A}_\lambda) = \bigcup_{\lambda \in \wedge} Supp(\tilde{A}_\lambda)$.
c. $[Supp(\tilde{A})]^c \subseteq Supp(\tilde{A}^c)$.

- The set of every point $x \in X$ is fuzzy set $\tilde{A}$ is core.

$$\mu_{\tilde{A}}(x) = 1$$

- A fuzzy set $\tilde{A}$ is height is the largest membership grade above X.

$$hgt(\tilde{A}) = sup_{x \in X} \mu_{\tilde{A}}(x)$$

- The point in X is the crossover point of a fuzzy set $\tilde{A}$ and the membership in $\tilde{A} = 0.5$.
- A fuzzy singleton is a fuzzy set with a single point of support in X

$$\mu_{\tilde{A}}(x) = \alpha, \alpha \in (0,1]$$

- Fuzzy set $\tilde{A}$ is named the normalized if it is height =1, otherwise it is subnormal such that $hgt(\tilde{A}) < 1$.

Notation 2.2 [8]

A fuzzy set that is not empty $\tilde{A}$ may always be normal by the following:

$$\mu^*_{\tilde{A}}(x) = \frac{\mu_{\tilde{A}(x)}}{sup\ \mu_{\tilde{A}(x)}}.$$

- Empty fuzzy set ($\tilde{A} = \widetilde{\emptyset}$) if and only if $\mu_{\tilde{A}}(x) = 0$, for each $x \in X$.
- Universal fuzzy set ($\tilde{A} = X$) iff $\mu_{\tilde{A}}(x) = 1$, for each $\in X$.

Example 2.1 [8]

Let $X = [-5,1], Y = [-5,12]$ and $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of X and Y respectively with membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x}{3} + \frac{5}{3}, & -5 \le x \le -2, \\ \frac{-x}{3} + \frac{1}{3}, & -2 \le x \le 1. \end{cases}$$

$$\mu_{\tilde{B}}(x) = \begin{cases} 0, & -5 \le y \le -3, \\ \frac{y}{7} + \frac{3}{7}, & -3 \le y \le 4, \\ \frac{-y}{8} + \frac{12}{8}, & 4 \le y \le 12. \end{cases}$$

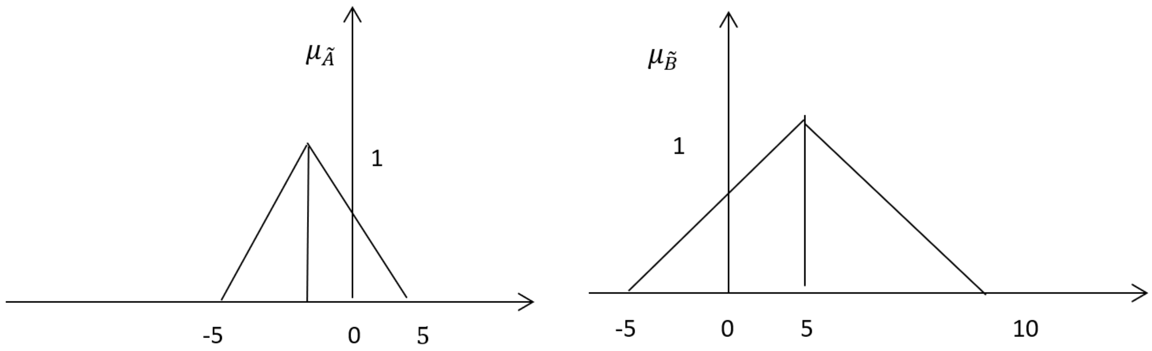

Fig. 2. The associated membership function

- $\widetilde{A} = \widetilde{B}$ if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \forall x \in X$.
- $\widetilde{A} \subseteq \widetilde{B}$ if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x),\ x \in X$.
- $\widetilde{A}^{\,c}$ *or* $\tilde{A}$` is the complement of $\tilde{A}$ with membership function

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x), \qquad \forall\, x \in X$$

- $\tilde{C} = \tilde{A} \cup \tilde{B}$ is a fuzzy set by membership function:

$$\mu_{\tilde{C}}(x) = max\ \{\, \mu_{\tilde{A}}(x), \qquad \mu_{\tilde{B}}\ (x)\}.$$

- $\tilde{C} = \tilde{A} \cap \tilde{B}$ is a fuzzy set by membership function:

$$\mu_{\tilde{C}}(x) = min\{\mu_{\tilde{A}}(x), \qquad \mu_{\tilde{B}}(x)\}$$

More generally for a fuzzy subset of I^x , $\tilde{A} = \{\tilde{A}_i \mid i \in J$, where J is a set$\}$ then the union $\tilde{C} = \cup_i \tilde{A}_i$, and the intersection $\tilde{C} = \cap_i \tilde{A}_i$ are fuzzy sets with the membership functions defined by:

$$\mu_{\tilde{C}}(x) = sup_i\{\, \mu_{\tilde{A}_i}(x) : x \in X\, , i \in J\}.$$

$\mu_{\tilde{C}}(x) = inf_i\{ \mu_{\tilde{A}_i}(x) : x \in X , i \in J\}$.

Example 2.2

Let $X = \{a, b, c\}$ and $\tilde{A}, \tilde{B}, \tilde{C}$ are fuzzy subsets of X where:

$\tilde{A} = \{(a, 0.3) , (b, 0.3) , (c, 0.6)\}$,

$\tilde{B} = \{(a, 0.4), (b, 0.4), (c, 0.5)\}$,

$\tilde{C} = \{(a, 0.3), (b, 0.3), (c, 0.3)\}$.

Then:

$\tilde{A} = \{(a, 0.3), (b, 0.3), (c, 0.6)\} \cap \tilde{B} = \{(a, 0.4), (b, 0.4), (c, 0.5)\} = \{(a, 0.3), (b, 0.3), (c, 0.5)\}$.

And

$\tilde{A} = \{(a, 0.3), (b, 0.3), (c, 0.6)\} \cup \tilde{B} = \{(a, 0.4), (b, 0.4), (c, 0.6)\} = \{(a, 0.4), (b, 0.4), (c, 0.6)\}$.

$\tilde{A}^c = 1 - \mu_A(x), \tilde{A}^c = \{(a, 0.7), (b, 0.7), (c, 0.4)\}$.

Notation 2.3 [9]

Here are some properties $\cup, \cap$ and complementation:

- Commutativity: $\tilde{A}\cup\tilde{B} = \tilde{B}\cup\tilde{A}$ and $\tilde{A}\cap\tilde{B} = \tilde{B}\cap\tilde{A}$.
- Associativity: $(\tilde{A}\cup\tilde{B})\cup\tilde{C} = \tilde{A}\cup(\tilde{B}\cup\tilde{C})$ and $(\tilde{A}\cap\tilde{B})\cap\tilde{C} = \tilde{A}\cap(\tilde{B}\cap\tilde{C})$.
- Idempotency: $\tilde{B}\cap\tilde{B} = \tilde{B}$ and $\tilde{B}\cup\tilde{B} = \tilde{B}$.
- Distributivity: $\tilde{A}\cup(\tilde{B}\cap\tilde{C}) = (\tilde{A}\cup\tilde{B})\cap(\tilde{A}\cup\tilde{C})$ and $\tilde{A}\cap(\tilde{B}\cup\tilde{C}) = (\tilde{A}\cap\tilde{B})\cup(\tilde{A}\cap\tilde{C})$.
- $\tilde{A}\cap\tilde{\emptyset} = \tilde{\emptyset}$ and $\tilde{A}\cup\tilde{X} = \tilde{X}$.
- Identity: $\tilde{A}\cup\tilde{\emptyset} = \tilde{A}$ and $\tilde{A}\cap\tilde{X} = \tilde{A}$.
- Absorption: $\tilde{A}\cup(\tilde{A}\cap\tilde{B}) = \tilde{A}$ and $\tilde{A}\cap(\tilde{A}\cup\tilde{B}) = \tilde{A}$.
- Demorgans law: $(\tilde{A}\cup\tilde{B})^c = \tilde{A}^c\cap\tilde{B}^c$ and $(\tilde{A}\cap\tilde{B})^c = \tilde{A}^c\cup\tilde{B}^c$.
- Involution: $\tilde{A}^{c^c} = \tilde{A}$.
- Equivalence formula: $(\tilde{A}^c\cup\tilde{B}) \cap (\tilde{A}\cup\tilde{B}^c) = (\tilde{A}^c\cap\tilde{B}^c)\cup(\tilde{A}\cap\tilde{B})$.
- Symmetrical difference: $(\tilde{A}^c\cap\tilde{B})\cup(\tilde{A}\cap\tilde{B}^c) = (\tilde{A}^c\cup\tilde{B}^c)\cap(\tilde{A}\cup\tilde{B})$.

Notation 2.4 [9]

The only lower for the contradiction, $A\cup A^c = X$, and the lower of A $\cap A^c = \emptyset$. Both laws are broken for the fuzzy set because $\tilde{A} \cup \tilde{A}^{\,c} \neq X$ and $\tilde{A} \cap \tilde{A}^{\,c} \neq \tilde{\emptyset}$ in deed $\forall x \in \tilde{A}$ such that $\mu_{\tilde{A}}(x) = \alpha$, then according to the point (7), we have $\mu_{\tilde{A}\cup\tilde{B}}(x) = max\{\alpha , 1 - \alpha\} \neq 1$ and $\mu_{\tilde{A}\cap\tilde{B}}(x) = min\{\alpha , 1 - \alpha\} \neq 0$.

- The cartesian product of a fuzzy set is well-defined by, suppose $\tilde{A}_1, \tilde{A}_2 , \dots , \tilde{A}_n$ be a fuzzy set in $\tilde{X}_1 , \tilde{X}_2 , \dots, \tilde{X}_n$. The cartesian product is then a fuzzy set in the product space $X_1 \times X_2 \times \dots \times X_n$ with the membership function:

$\mu_{(\tilde{A}_1 \times \dots \times \tilde{A}_n)}(x) = min\{ \mu_{\tilde{A}_i}(x_i) \mid x = (x_1 , \dots , x_n), x_i \in X_i \}$.

- The m^{th} power for the fuzzy set $\tilde{A}$ is a fuzzy set by membership function $\mu_{\tilde{R}^m}(x) = [\mu_{\tilde{A}}(x)]^m, \forall\ x \in X$.
- The algebraic sum $\tilde{C} = \tilde{A} + \tilde{B}$ the defined by:

$\tilde{C} = \{(x , \mu_{\tilde{A}+\tilde{B}}(x) \mid x \in X\}$,

then:

$\mu_{\tilde{A}+\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x)\,\mu_{\tilde{B}}(x)$.

- The bounded sum, $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is defined as:

$\tilde{C} = \left\{ \left(x, \mu_{\tilde{A}\oplus\tilde{B}}(x) \right) \middle| \; x \in X \right\}$,

where:

$\mu_{\tilde{A}\oplus\tilde{B}}(x) = min\{1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)\}$.

- The bounded difference $\tilde{C} = \tilde{A} \circledast \tilde{B}$ is defined by:

$\tilde{C} = \left\{ \left(x, \mu_{\tilde{A}\circledast\tilde{B}}(x) \right) \middle| \, x \in X \right\}$

Then:

$\mu_{\tilde{A}\circledast\tilde{B}}(x) = max\{ 0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\}$.

- Two fuzzy sets' algebraic product $\tilde{C} = \tilde{A} \odot \tilde{B}$ is defined as:

$\tilde{C} = \{ (x, \mu_{\tilde{A}}(x)\,\mu_{\tilde{B}}(x)) \,|\, x \in X \}$.

Example 2.3 [9]

Let $\tilde{A} = \{(3, 0.4), (5, 2), (7, 0.5)\}$ and $\tilde{B} = \{(3, 2), (5,0.5)\}$.

Then:

$\tilde{A} \times \tilde{B} = \{((3,3),0.4), ((5,3),2), ((7,3),0.5), ((3,5),0.4), ((5,5),0.5), ((7,5), 0.5)\}$.

$\tilde{A}^2 = \{ (3, 0.16), (5, 2), (7, 0.25)\}$.

$\tilde{A} + \tilde{B} = \{ (3, 2), (5, 2), (7, 0.5)\}$.

$\tilde{A} \oplus \tilde{B} = \{(3, 2), (5, 2), (7, 0.5)\}$.

$\tilde{A} \circledast \tilde{B} = \{ (3, 0.4), (5, 0.5)\}$.

$\tilde{A} . \tilde{B} = \{ (3, 0.4), (5, 0.5)\}$.

α_cut or α_level [10]

The among the basic notions of a fuzzy set is the concept of the $\alpha -$level or $\alpha -$level set. And its variant strong $\alpha -$level or (strong $\alpha -$level set). Given a fuzzy set $\tilde{A}$ defined on X, and any number $\alpha \in [0,1]$ the $\alpha -$cut A_α is (the crisp set) that the contain all the elements of the universal set X whose the membership grades in $\tilde{A}$ are greater than or equal to the specified value of $A_\alpha = \{ x \in X : \mu_{\tilde{A}}(x) \geq \alpha \}, \forall\, x \in X$ while $A_{\alpha^+} = \{ x \epsilon X \; : \; \mu_{\tilde{A}}(x) > \alpha \} \forall\, x \in X$ is the called "strong α_cuts". The following properties, are satisfied for all $\alpha \in [0,1]$:

- If α_1, $\alpha_2 \in [0,1]$ and $\alpha_1 \leq \alpha_2$, then $A_{\alpha_1} \supseteq A_{\alpha_2}$ if A is convex.
- $(\tilde{A} \cup \tilde{B})_\alpha = A_\alpha \cup B_\alpha$.
- $(\tilde{A} \cap \tilde{B})_\alpha = A_\alpha \cap B_\alpha$.
- $(\tilde{A} \subseteq \tilde{B})_\alpha$ gives $A_\alpha \subseteq B_\alpha$.
- $\tilde{A} = \tilde{B}$ if and only if $A_\alpha = B_\alpha$, $\forall\, \alpha \in [0,1]$.

Notation 2.5

If $A_{\alpha_1} = B_{\alpha_2}$ then it is not necessary that $\tilde{A} = \tilde{B}$, for different α_1 and α_2.

Notation 2.6 [10]

- The set of all level $\alpha \in [0,1]$, that represent distinct $\alpha - cuts$ of a given fuzzy set $\widetilde{A}$ is named a level set of A.

$\wedge\left(\widetilde{A}\right) = \{\alpha \mid \mu_{\widetilde{A}}(x) = \alpha, for\ \ some\ x \in X\}$.

- The support for $\tilde{A}$ is just like the strong α −cut of $\widetilde{A}$ for $\alpha = 0$,

$A_{0+} = Supp\left(\tilde{A}\right)$.

- The core of $\widetilde{A}$ is exactly, the same as the α −cut of $\widetilde{A}$ for $\alpha = 1$, that is:

$A_1 = core\left(\tilde{A}\right)$.

- It is also possible to view the height of $\tilde{A}$ as the supermom of the α −level for which $A_\alpha \neq \emptyset A$.
- The membership function of a fuzzy set $\tilde{A}$ might be described in terms of the quality function for is α −cuts of based on the formula:

$\mu_{\widetilde{A}}(x) = sup_{\alpha\in[0,1]} Min\{\ \alpha\ , \mu_{A^\alpha}(x)\ \}$,

where:

$$\mu_{A^\alpha}(x) = \begin{cases} 1 & if\ \ x \in A_\alpha, \\ 0, & otherwise. \end{cases}$$

If the universal set X is specified in R, then we can generalize the notion of convexity to fuzzy set. The fuzzy set with these α _cut sets is convex if all α _ cuts are convex.

Definition 2.1 [10]

A fuzzy set $\tilde{A}$ on $\mathbb{R}$ is convex if $\mu_{\widetilde{A}}(\lambda\, x_1 + (1-\lambda)x_2) \geq min\{\ \mu_{\widetilde{A}}(x_1),\ \mu_{\widetilde{A}}(x_2)\}$, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$.

Notation 2.7 [8]

A_α is a convex for any $\alpha \in [0,1]$.

Definition 2.2 [11]

Let $f: X \to Y$ and $\widetilde{\mathrm{B}}$ be a fuzzy set definite on X, after that can obtain a fuzzy set $\widetilde{\mathrm{B}}$ in Y by f and $\tilde{B}$ for all $y \in Y$ that's:

$$\mu_{f(\widetilde{B})}(y) = \begin{cases} sup\{\ \mu_{\tilde{B}}(x)\ if\ f^{-1}(y) \neq 0\ , \forall x \in X, \qquad y = f(x)\}, \\ \qquad\qquad 0 \qquad\qquad if\ f^{-1}(y) = 0. \end{cases}$$

The generalization of the per explained extension of fuzzy set in above definition as follows, let $X = X_1, X_2, \dots, X_r$ and $\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_r$ be r_fuzzy set in the universal, f is a function form X to a universe $Y\big(y = f(x_1, x_2, \dots, x_r)\big)$, then a fuzzy set $\tilde{C}$ in Y is defined by:

$\tilde{C} = \left\{\big(y, \mu_{\tilde{C}}(y)\big) \,\middle|\, y = f(x_1, x_2, \dots, x_r), f(x_1, x_2, \dots, x_r \in X\right\}$,

Where:

$$\mu_{\tilde{C}}(y) = \begin{cases} sup_{(x_1,x_2,\dots,x_r)\in f^{-1}(y)}\ min\{\mu_{\tilde{C}}(x_1), \dots, \mu_{\tilde{C}}(x_r) \quad if\ \ f^{-1}(y) \neq 0\}, \\ \qquad\qquad 0 \qquad\qquad if\ f^{-1}\,(y) = 0. \end{cases}$$

Where f^{-1} is an inverse of f.

Definition 2.3 [12]

The fuzzy function if $\tilde{F}: X \to \widetilde{P}\ (x)$, with $\mu_{\tilde{F}}\ (x): X \to I$ for any $\alpha \in [0,1]$. So can define the α_cut of $\tilde{F}$ denoted, by F_α as:

$\forall\ x \in X\ , F_\alpha\ (x) = \{y \mid \mu_{\tilde{F}\ (x)}(y) \geq \alpha\}$.

For a fuzzy set of function $\widetilde{F}$ with $\mu_{\tilde{F}}\ (x)$: $R^x \to I$, the α −cut of F, F_α is as:

$F_\alpha = \{f\ : X \to R\ , x \in X\ , f(\ x) \in \tilde{F}_\alpha(x)\}$
and
$\{f: X \to R\ \left| Inf_{x \in X} \mu_{\tilde{F}\ (x)} \left(f(x)\right) = \mu_F\ (f) \geq \alpha\ \right\}$.

Notation 2.8 [13]

- A fuzzy function having a one curve named normalized fuzzy mapping.
- A continuous fuzzy function is a fuzzy function $\tilde{F}(x)$ that is $\mu_{\tilde{F}(x)}(y)$ is a continuous $\forall\ x \in I \subset R$ and $\forall\ y \in R$.
- The concept of fuzzy interval is a convex normalized fuzzy set of R whose membership function is a continuous.

Fuzzy Number 2.1 [5]

A fuzzy number $\widetilde{N}$ is the convex normalized, fuzzy set $\widetilde{N}$ for the realline $\mathbb{R}$, that is:

- There exist exactly one $x_0 \in \mathbb{R}$ with $\mu_{\widetilde{N}}\ (x_0) = 1$ (x_0 is named mean value of $\widetilde{N}$).
- $\mu_{\widetilde{N}}(x)$ is continuous function.

Definition 2.4 [14]

A fuzzy number $\widetilde{N}$ is called positive (negative) if it is membership function define by: $\mu_{\widetilde{N}}(x) = 0, \forall\ x < 0$.

Definition 2.5 [15]

A fuzzy number is a fuzzy set which is a map $\tilde{u}: R \to [a, b]$, that satisfies:

- $\tilde{u}$ is upper semi continuous function.
- $\tilde{u}(x) = 0$ outside some interval $[a, d]$.
- There are real numbers b, c such that $a \leq b \leq c \leq d$ that is:

a. $\tilde{u}(x)$ is a monotonic increasing function on $[a, b]$.
b. $\tilde{u}(x)$ is a monotonic decreasing function on $[c, d]$.
c. $\tilde{u}(x) = 1, \forall\ x \in [b, c]$.

Definition 2.6 [16]

A fuzzy number $\tilde{u}$ is a parametric form is a pair $(\underline{u}\ , \overline{u})$ of function $\underline{u}(\alpha), \overline{u}\ (\alpha)$, $0 \leq\ \alpha \leq 1$ the following conditions by:

- $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function on [0,1].
- $\overline{u}(\alpha)$ is a bounded left continuous nonincreasing function on [0,1].
- $\underline{u}\ (\alpha) \leq \overline{u}(\alpha),\ 0 \leq \alpha \leq 1$.

Definition 2.7 [16]

For any arbitrary fuzzy number $\tilde{u} = \left(\underline{u}(\alpha), \overline{u}(\alpha)\right)$ and $\tilde{v} = \left(\underline{v}(\alpha), \overline{v}\,(\alpha)\right)$ and K is scalar. The ensuing characteristics are met for all $\alpha \in [0, 1]$.

- $(\underline{u+v})\,(\alpha) = \left(\underline{u}(\alpha) + \underline{v}(\alpha)\right)$ and $(\overline{u+v})\,(\alpha) = \left(\overline{u}(\alpha) + \overline{v}(\alpha)\right)$.
- $K\tilde{u} = (k\underline{u}(\alpha), k\,\overline{u}(\alpha))$.
- $\tilde{u}\,.\,\tilde{v} = \left\{\,\underline{u}(\alpha)\,\underline{v}\,(\alpha), \underline{u}(\alpha)\,\overline{v}\,(\alpha), \overline{u}(\alpha)\,\underline{v}\,(\alpha), \overline{v}(\alpha)\,\overline{u}(\alpha)\right\}$.

Definition 2.8 [17]

For any arbitrary, a fuzzy number $\tilde{u}, \tilde{v} \in E^1$

$$D(\tilde{u}, \tilde{v}) = max\left\{sup_{\alpha\in[0,1]}\left|\underline{u}\,(\alpha) - \underline{v}\,(\alpha)\right|, sup_{\alpha\in[0,1]}\left|\overline{u}(\alpha) - \overline{v}(\alpha)\right|\right\}$$

Denoted the distance between $\tilde{u}$ and $\tilde{v}$, also (E^1, D) is a complete metric space.

Theorem 2.1 [18]

(E^1, D) is a metric space.

Definition 2.9 [18]

Let $\{\tilde{a}_n\} \subset E^1$ and $\tilde{a} \in E^1$ the sequence $\{\tilde{a}_n\}$ is said to be convergence to $\tilde{a}$ in distance denoted by if $\lim_{n\to\infty}\tilde{a}_n = \tilde{a}$ if any given $\varepsilon > 0$ there's for integral $N > 0$ such that $D(\tilde{a}_n, \tilde{a}) < \varepsilon$ for $n \geq N$. A sequence $\{\tilde{a}_n\}$ in E^1 is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists an integral $N > 0$ such that $D\,(\tilde{a}_n, \tilde{a}_m) < \varepsilon$ for $n, m > N$. A fuzzy metric space (E^1, D) is called the complete metric space if every Cauchy sequence in E^1 is a convergence.

Theorem 2.2 [18]

The sequence $\{\tilde{a}_n\}$ in E^1 is a convergence in the metric D iff $\{\tilde{a}_n\}$ is a cauchy sequence.

Definition 2.10 [18]

The distance between two fuzzy numbers $\tilde{a}, \tilde{b} \in E^1$ is given by:

$$D\left(\tilde{a}, \tilde{b}\right) = \sup_{0\leq\alpha\leq1}\{max\{\sup_{a\in[a^-{}_\alpha, a^+{}_\alpha]}\ \inf_{b\in[b^-{}_\alpha, b^+{}_\alpha]}|a-b|, \sup_{b\in[b^-{}_\alpha, b^+{}_\alpha]}\ \inf_{a\in[a^-{}_\alpha, a^+{}_\alpha]}|a-b|\}\}$$

$$D\left(\tilde{a}, \tilde{b}\right) = \sup_{0\leq\alpha\leq1}\{max\,\{|a^-{}_\alpha - b^-{}_\alpha|\,, |a^+{}_\alpha - b^+{}_\alpha|\}\}.$$

Definition 2.11 [18]

A fuzzy function $\tilde{f}: X \times X \to E^1$ is a called level wise continuous at point $(x_0, t_0) \in X \times X$ provided for any fixed $\alpha \in [0,1]$ and for any arbitrary $\varepsilon > 0$ there's $\delta(\varepsilon, \alpha) > 0$ then:

$$D\left(\left|\tilde{f}(x,t)\right|\alpha\,, \left|\tilde{f}(x_0, t_0)\right|\alpha\right) < \varepsilon,$$

whenever $|t - t_0| < \delta$ and $|x - x_0| < \delta\ \forall\ x, t \in X$.

Definition 2.12 [8]

Let R be the set of real number and $\tilde{P}(R)$ all a fuzzy subset defined on R defined the fuzzy number $\tilde{a} \in E^1$ as follows by:

- $\tilde{a}$ is a normal that is there exists $x \in R$ that is $\mu_{\tilde{a}}(x) = 1$.
- For every $\alpha \in [0,1], a_\alpha = \{x: \mu_{\tilde{a}}(x) \geq \alpha\}$ is a closed interval denoted by $[a^-{}_\alpha, a^+{}_\alpha]$.

Using Zadeh notation $\tilde{a} \in F(R)$ is the fuzzy set on R defined by:

$\tilde{a} = \cup_{\alpha\in[0,1]}\ a_\alpha = \cup_{\alpha\epsilon[0,1]}\ \alpha[a^-{}_\alpha, a^+{}_\alpha]$.

Definition 2.13 [11]

A function $F: I \to E^n$ is called a bounded if there exists a constant $M > 0$ that is $D(F(x), \tilde{0}) \leq M$ for all $\in$ I.

Definition 2.14 [19]

Let $\tilde{A} \subset F(R)$,

- If there is $\widetilde{M} \in E^1$ that is $\tilde{a} \subseteq \widetilde{M}$ for all $\tilde{a} \in \tilde{A}$ then $\tilde{A}$ is said to have an upper bound $\widetilde{M}$.
- If there is $\widetilde{M} \in E^1$ that is $\widetilde{m} \subseteq \tilde{a}$ for all $\tilde{a} \in \tilde{A}$ then $\tilde{A}$ is called lower bound $\widetilde{m}$.
- $\tilde{A}$ is said to be bound if $\tilde{A}$ has both upper and lower bounds.
- A sequence $\{\tilde{a}_n\} \subseteq E^1$ is said to be bound if the set $\{\tilde{a}_n | n \in N\}$ is bound.

Definition 2.15 [20]

The family of E^n denotes nonempty compact, the convex a fuzzy subset of R^n. Let $I = [a, b]$ be compact interval $E^n = \{p: R^n \to I\}$. That is p satisfies the following:

- p is " normal ".
- p is " fuzzy convex ".
- p is " upper semi continuous" such that the $\alpha\_$ cuts sets $[p]_\alpha$ are closed for each $\alpha \in [0, 1]$.
- $[p]^0 = cl\{ x \in R^n | p(x) > 0\}$ the is compact where $\alpha_$cut sets $[p]^\alpha$ is defined by $[p]^\alpha = \{ x \in R^n | p(x) \geq \alpha \}$ for $0 < \alpha < 1$ and $[p]^0$ for $\alpha = 0$.

Then from (1)-(4) it is follows that is $[p]^\alpha \in E^n$ for all $0 \leq \alpha \leq 1$.

Definition 2.16 [11]

A function $F: \mathrm{I} \to E^n$ is said to be continuous if $x_0 \in \mathrm{I}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ then $D\big(F(x), F(x_0)\big) < \varepsilon$.

Definition 2.17 [18]

Let $\tilde{f}(x)$ be a closed and bounded a fuzzy function on $[a, b]$ suppose the $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^R_\alpha(x)$ are the Riemann integral on $[a, b]$ for every $\alpha \in [0,1]$. Let:

$$B_\alpha = \left[\int_a^b \tilde{f}^L_\alpha(x)dx, \int_a^b \tilde{f}^R_\alpha(x)dx \right]$$

Then we say that $\tilde{f}(x)$ is a fuzzy Riemann integral of $[a, b]$ and the membership function of $\int_a^b \tilde{f}(x)dx$ is defined by:

$$M_{\int_a^b \tilde{f}(x)dx}(\alpha) = \sup_{0\le\alpha\le1} \alpha \,.\, 1_{B_\alpha}(r), for \ \in B_0.$$

Definition Triangular Fuzzy Number 2.18 [18]

It is fuzzy number represented with three points as follows by: $\tilde{B} = [b_1, b_2, b_3]$. This representation is interpreted as in the following membership function:

$$\mu_{\tilde{B}}(x) = \begin{cases} 0 & , x < b_1, \\ \dfrac{x - b_1}{b_2 - b_1} & , b_1 \le x \le b_2, \\ \dfrac{b_3 - x}{b_3 - b_2} & , \ b_2 \le x \le b_3, \\ 0 & , \ x > b_3. \end{cases}$$

Now, if we get crisp interval by α_cut operation interval B_α shall be obtained as follows for every $\alpha \in [0,1]$

$$B_\alpha = \left[b_1^{(\alpha)}, b_3^{(\alpha)}\right] = [\ (b_2 - b_1)\alpha + b_1\ , -(b_3 - b_2)\alpha + b_3\ ].$$

Definition Trapezoidal Fuzzy Number 2.19 [8]

We can explain the trapezoidal fuzzy number B as follows: $\tilde{B} = [b_1, b_2, b_3, b_4]$. The membership function of this fuzzy number will be interpreted as follows:

$$\mu_{\tilde{B}}(x) = \begin{cases} 0 & if \quad x < b_1 \ or\ x > b_4, \\ \dfrac{x - b_1}{b_2 - b_1} & if \quad b_1 \le x \le b_2, \\ \dfrac{b_4 - x}{b_4 - b_3} & if \quad b_3 \le x \le b_4, \\ 1 & if \quad b_2 \le \ x \le b_3. \end{cases}$$

The α −level interval for this shape is written as:

$$\forall\alpha \in [0,1], B_\alpha = [(b_2 - b_1)\alpha + b_1, -(b_4 - b_3)\alpha + b_4].$$

Definition 2.20 [21]

Let $\tilde{u}$ be a fuzzy set on R then $\tilde{u}$ is called a fuzzy interval if it satisfies:

- $\tilde{u}$ is normal there exsist $x_0 \in R$ .i.e. $u(x_0) = 1$.
- $\tilde{u}$ is convex for every $x, t \in R, 0 \le \lambda \le 1$ it holds that $\mu(\lambda x + (1 - \lambda)y \ge min\{u(x), u(y)\}$.
- $\tilde{u}$ is upper semi continuous.
- $[u]^0 = cl\{x \in R: u(x) > 0\}$ is a compact subset of R the α_cuts of a fuzzy interval u with $0 \le \alpha \le 1$ is the crisp set $[u]^\alpha = \{x \in R, u(x) \ge \alpha\}$. For a fuzzy interval $\tilde{u}$ is α_cuts are closed interval in R let denoted by $[u]^\alpha = \left[\underline{u}(\alpha)\ , \overline{u}(\alpha)\right]$.

Definition 2.21 [20]

A fuzzy number $\widetilde{N}$ is of LR −type if there's functions L(named the left-function) and R(named the right-function) so $L(x) \le \mu_{\widetilde{N}}\ (x) \le R(x), \forall\ x \in X$, and scalars $a > 0, b > 0$ with:

$$\mu_{\widetilde{N}}\ (x) = \begin{cases} L\ \left(\frac{n - x}{a}\right), & for\ x \le n, \\ R\ \left(\frac{x - n}{b}\right), & for\ x \ge n. \end{cases}$$

The mean value of $\widetilde{N}$ is denoted by n, while the left and right spreads of n are denoted by a and b, respectively. $\widetilde{N}$ is represented symbolically as $(n, a, b)_{LR}$.

$$\mu_B(x) = \begin{cases} x - 1 & a \leq x \leq b, \\ \dfrac{-1}{2}x + 2 & b \leq x \leq c. \end{cases}$$

$$\mu_B(x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{b-a}, & a \leq x \leq b, \\ \dfrac{c-x}{d-c}, & c \leq x \leq d. \end{cases}$$

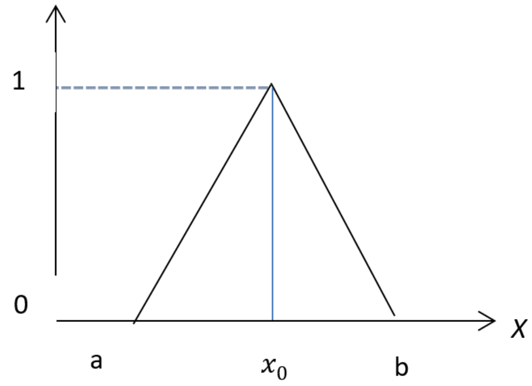

Fig. 3. The triangular membership function

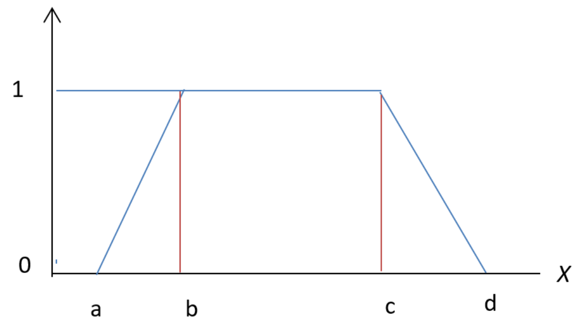

Fig. 4. The trapezoidal function

Definition 2.22 [20]

Any fuzzy number can be described by:

$$\mu_{\widetilde{N}}(x) = \begin{cases} L\left(\dfrac{a-x}{\alpha}\right), & for\; x \in [a-\alpha\,, a], \\ 1, & for\; x \in [a, b], \\ R\left(\dfrac{x-b}{\beta}\right), & for\; x \in [b\,, b+\beta], \\ 0, & otherwise. \end{cases}$$

Where $[a, b]$ is the core of A and $[a, b] \rightarrow [0,1], R: [0,1] \rightarrow [0,1]$ are shape function (named briefly S shape) that is continuous and nonincreasing so, $L(0) = R(0) = 1, L(1) = R(1) = 0$.

When the fuzzy function is considered to be of the LR type, determining the integration becomes somewhat easier. We shall assume that fuzzifying function $\tilde{f}(u) = (f(u), s(u), t(u))_{LR}$ is a fuzzy number in LR −typr for every $x \in [a, b]$ which mean that there exists a reference functions $L: R^+ \rightarrow [0,1]$ and $R: R^+ \rightarrow [0,1], f: \mathrm{I} \rightarrow \mathrm{R}$ and $s: \mathrm{I} \rightarrow \mathrm{R}^+$ and $t: \mathrm{I} \rightarrow R^+$ that is for every $u \in [a, b]$:

$$\mu_{\tilde{f}(u)}(v) = \begin{cases} L\left(\dfrac{f(u)-v}{s(u)}\right), & for\ all\ v \leq f(u), \\ \\ R\left(\dfrac{v-f(u)}{t(u)}\right), & for\ all\ v \geq f(u). \end{cases}$$

Where $f(u)$ is the mean value of $\tilde{f}(u)$ and $s(u), t(u)$ are the spread functions and the reference functions L and R are such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ for each $x \in$ I.

Definition 2.23 [8]

Let $F: I \to E^n$ be an integral of I which is levelwise continuous is denoted by $\int_I F(x)dx$ or $\int_a^b F(x)dx$ also $[\int_I F(x)dx]^\alpha = \int_I F(x)_\alpha dx \left\{\int_I f(x)dx \middle| f: I \to R^n\right.$ is a measurable function for $F(x)_\alpha$ for every $0 \leq \alpha \leq 1\}$.

Notation 2.9 [20]

- For any a fuzzy function $\widetilde{f}$ we have $\int_I \widetilde{f} = \int_a^b \widetilde{f} = -\int_b^a \widetilde{f}$, with the membership function $\mu_{\int_a^b \widetilde{f}}(u) = \mu_{\int_a^b \widetilde{f}}(-u)$.
- To integrate of LR fuzzifying Func over a non-fuzzy interval $[a, b]$, it is a sufficient to integrate the mean value and spread function over $[a, b]$, the result is an LR a fuzzy number.
- Commutative condition for $\int_I \widetilde{f}$ if for all $\alpha \in [0,1]$ is $(\int_I \tilde{f})_\alpha = \int_I \widetilde{f}_\alpha$.

Fuzzy Integral Equation 2.1 [8]

The fuzzy nonlinear Volterra in integral equation of the second kind may be represented as follows by:

$$\widetilde{U}(x) = \tilde{f}(x) + \lambda \int_a^x k\left(x, t, \tilde{u}(t)\right)dt, \quad (7)$$

Where $\lambda > 0$ and k is arbitrary given kernel function f is given function of $x \in [a, b]$. If $f(x)$ is a crisp function then the solution of above equation is crisp as well. If $f(x)$ is a fuzzy function this equation only possesses fuzzy solving the sufficient condition for the existence of the solving of the equation of the second kind, for solution (7) we may replace (7) by the equivalent system:

$$\underline{U}(x) = \underline{f}(x) + \lambda \int_a^x \underline{k}\left(x, t, F(u(t)\right)dt,$$

$$\overline{U}(x) = \overline{f}(x) + \lambda \int_a^x \overline{k}\left(x, t, F(u(t)\right)dt. \quad (8)$$

Which possesses a unique solving $(\underline{U}, \overline{U}) \in B$, which is a fuzzy function, such that for each x. The pair $\left(\underline{U}(x, \alpha), \overline{U}(x, \alpha)\right)$ is a fuzzy number. Let $F(x, t, u, v)$ be the function F of (8), where u and v are constants and $u \leq v$. In other word $F(x, t, u, v)$ are obtained by substituting $U = (u, v)$ in (8). The domain where F indeed by:

$\Delta = \{(x, t, u, v) | a \leq x, t \leq b, -\infty \leq v \leq \infty, -\infty \leq u \leq v\}$

The parametric form of (8) is given by:

$\underline{U}(x, \alpha) = \underline{f}(x, \alpha) + \lambda \int_a^x k\left(x, t, F\left(\underline{U}(t, \alpha)\right)\right)dt,$

$$\overline{U}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_a^x k\left(x,t,F\left(\overline{U}(t,\alpha)\right)\right)dt. \qquad (9)$$

For $\alpha \in [0,1]$. In most cases, however analytic solution to (9) may not be found and a numerical approach must be considered.

3. The modified decomposition method (MDM)

As shown before the Adomian decomposition method provides the solution in an infinite series of components. The components $u_j, j \geq 0$ are easily computed if the inhomogenous term $f(x)$ in the Volterra integral equation:

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \qquad (10)$$

Consists of a polynomial. However, if the function $f(x)$ consists of a combination of two or more of polynomials, trigonmetric functions, hyperbolic function, and others, the evaluation of the components $u_j, j \geq 0$ requires cumbersome work. A reliable modification of the Adomian decomposition method was developed by Wazwaz and presented. The modified decomposition method will facilitate the computational process and further accelerate the convergence of the series solution. The modified decomposition method will be applied, wherever it is appropriate, to all integral equation and differential equation of any order. It is interesting to note that the modified decomposition method depends mainly on splitting the function $f(x)$ into two parts, therefore it cannot be used if the function $f(x)$ consists of only one term. The modified decomposition method will be outlined and employed in this section and in other chapters as well. To give a clear description of the technique, we recall that the standard Adomian decomposition method admits use of the recurrence relation:

$$u_0(x) = f(x),$$

$$u_{k+1}(x) = \lambda \int_0^x k(x,t)\, u_k(t)dt, \quad k \geq 0, \qquad (11)$$

Where the solution $u(x)$ is expressed by an infinite sum of components defined before by:

$$u(x) = \sum_{n=0}^{\infty} u_n\,(x). \qquad (12)$$

In view of (8), the components $u_n(x), n \geq 0$ can be easily evaluated. The modified decomposition method introduces a slight variation to the recurrence relation (8) that will lead to the determination of the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x). \qquad (13)$$

In view of (10), we introduce a introduce a qualitative change in the formation of the recurrence relation (7). To minimize the size of calculations, we identify the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ among other terms. In other words, the modified decomposition method introduces the modified recurrence relation:

$$u_0(x) = f_1(x),$$

$$u_2(x) = f_2(x) + \lambda \int_0^x k(x,t)\, u_0\,(t)\, dt, \qquad (14)$$

$$u_{k+1}(x) = \lambda \int_0^x k(x,t)u_k\,(t)dt, k \geq 1.$$

This shows that the difference between the standard recurrence relation (10) and the modified recurrence relation (14) rests only in the formation of the first two components $u_0(x)$ and $u_1(x)$ only. The other components $u_j\,, j \geq 2$ remain the same in the two recurrence relations. Although this variation in the formation of $u_0(x)$ and $u_1(x)$ is slight, however it plays a major role in accelerating the convergence of the solution and in minimizing the size of computational work. Moreover, reducing the number of terms in $f_1(x)$ affects not only the component $u_1(x)$, but also the other components as well. This result was confirmed by several research works.

Two important remarks related to the modified method can be made here. First, by proper selection of the functions $f_1\ (x)$ and $f_2(x)$, the exact solution $u(x)$ may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the proper choice of $f_1(x)$ and $f_2(x)$, and this can be made through trials only. A rule that may help for the proper choice of $f_1(x)$ and $f_2(x)$ could not be found yet. Second if $f(x)$ consists of one term only, the standard decomposition method can be used in this case.

It is worth mentioning that the modified decomposition method will be used for Volterra and Fredholm integral, equations linear and nonlinear equation. The modified decomposition method will be illustrated by discussing the following examples.

Example:

Solve fuzzy Volterra nonlinear integral equation:

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \int_0^x t^2\ \underline{u}(t,\alpha)^2\ dt$$

$$x^2\alpha = \underline{f}(x,\alpha) + \int_0^x t^2\ \ (t^2\ \alpha)^2\ dt$$

$$x^2\alpha = \underline{f}(x,\alpha) + \int_0^x t^6\ \ \alpha^2\ \ dt$$

$$x^2\alpha = \underline{f}(x,\alpha) + |\frac{t^7}{7}\ \ \alpha^2|_0^x$$

$$x^2\alpha = \underline{f}(x,\alpha) + \frac{x^7}{7}\ \ \alpha^2$$

$$\underline{f}(x,\alpha) = x^2\alpha - \frac{x^7}{7}\ \ x^2$$

$$\underline{u}_1(x,\alpha) = -\frac{x^7}{7}\ \ x^2 + \int_0^x t^2\ \underline{u}_0\ (t,\alpha)\ dt$$

$$\underline{u}_1(x,\alpha) = -\frac{x^7}{7}\ \ x^2 + \int_0^x t^2\ \ (t^2\alpha)^2 dt$$

$$\underline{u}_1(x,\alpha) = -\frac{x^7}{7}\ \ x^2 + \int_0^x t^2\ \ (t^4\alpha^2) dt$$

$$\underline{u}_1(x,\alpha) = -\frac{x^7}{7}\ \ x^2 + |\frac{t^7}{7}\ \ x^2|_0^x$$

$$\underline{u}_1(x,\alpha) = -\frac{x^7}{7}\ \ x^2 + \frac{x^7}{7}\ \ x^2 = 0$$

$$\underline{f_1}(x,\alpha) + \underline{f_2}(x,\alpha) = x^2\alpha$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) \quad + \int_0^x t^2 \ u(t,\alpha)^2 dt$$

$$\overline{u_0}(x,\alpha) = \overline{f}(x,\alpha) \quad + \int_0^x t^2 \ u(t,\alpha)^2 dt$$

$$x^2(2-\alpha) = \overline{f}(x,\alpha) + \int_0^x t^2(t^2\,(2-\alpha)^2)\,dt$$

$$x^2(2-\alpha) = \overline{f}(x,\alpha) + \int_0^x t^6(2-\alpha)^2\,dt$$

$$x^2(2-\alpha) = \overline{f}(x,\alpha) + |\frac{t^7}{7}\ (2-\alpha)^2\,|_0^x$$

$$\overline{f}(x,\alpha) = x^2(2-\alpha) - \frac{x^7}{7}(2-\alpha)^2$$

$$\overline{u}_0(x,\alpha) = x^2(2-\alpha)$$

$$\overline{u}_1(x,\alpha) = -\frac{x^7}{7}(2-\alpha)^2 + \int_0^x t^2\ u_0\,(t,\alpha)dt$$

$$\overline{u}_1(x,\alpha) = -\frac{x^7}{7}(2-\alpha)^2 + \int_0^x t^2\,(t^2(2-\alpha)^2\ dt$$

$$\overline{u}_1(x,\alpha) = -\frac{x^7}{7}(2-\alpha)^2 + \int_0^x t^6\,(t^2(2-\alpha)^2\ dt$$

$$\overline{u}_1(x,\alpha) = -\frac{x^7}{7}(2-\alpha)^2 +|\frac{t^7}{7}(2-\alpha)^2|_0^x$$

$$\overline{u}_1(x,\alpha) = -\frac{x^7}{7}(2-\alpha)^2 + \frac{x^7}{7}(2-\alpha)^2$$

$$\overline{u}_1(x,\alpha) = 0$$

$$\overline{f}_1(x,\alpha) + \overline{f}_2(x,\alpha) = x^2(2-\alpha)$$

Table 1. Comparison between the Exact Solution and the Modified Decomposition Method for Upper with Different Level α and Finding the Absolute Error, $\underline{f}_1(x,\alpha) + \underline{f}_2(x,\alpha) = x^2\alpha$

		$\underline{u}(x,\alpha)$				MDM $\underline{u}$				Absolute error $\underline{u}$			
X	α	0.1	0.3	0.5	0.7	0.1	0.3	0.5	0.7	0.1	0.3	0.5	0.7
0		0	0	0	0	0	0	0	0	0	0	0	0
0.2		0.004	0.012	0.02	0.028	0.004	0.012	0.02	0.028	0	0	0	0
0.4		0.016	0.048	0.08	0.112	0.016	0.048	0.08	0.112	0	0	0	0
0.6		0.036	0.108	0.18	0.252	0.036	0.108	0.18	0.252	0	0	0	0

Table 2. Comparison between the Exact Solution and the Modified Decomposition Method for Upper with Different Level α and Finding the Absolute Error, $\overline{f}_1(x,\alpha) + \overline{f}_2(x,\alpha) = x^2(2-\alpha)$

$\overline{u}(x,\alpha)$	MDM $\overline{u}$	Absolute error $\overline{u}$

x \ α	0.1	0.3	0.5	0.7	0.1	0.3	0.5	0.7	0.1	0.3	0.5	0.7
0	0	0	0	0	0	0	0	0	0	0	0	0
0.2	0.076	0.068	0.06	0.052	0.076	0.068	0.06	0.052	0	0	0	0
0.4	0.304	0.272	0.24	0.208	0.304	0.272	0.24	0.208	0	0	0	0
0.6	0.684	0.612	0.54	0.408	0.684	0.612	0.54	0.468	0	0	0	0

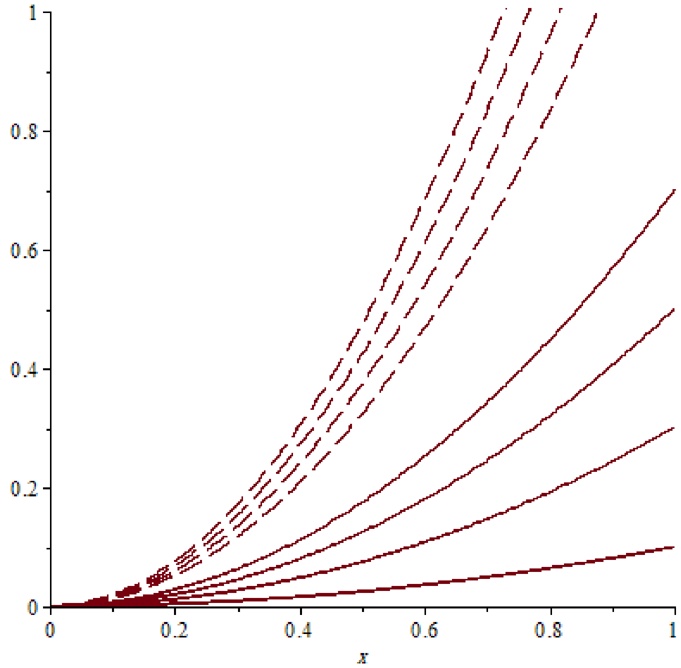

Fig. 5. Comparison between the exact solution and the modified decomposition method for upper and lower with different level alpha

4. Conclusion

This work presents the use of the reliable modified decomposition method for solving non-linear fuzzy Volterra integral equations of second kind. The modified decomposition method is implemented in a straight forward manner and provided significant improvement by requiring only two iterations to obtain the exact solution. Accelerating convergence of the modified Admian method requires that the exact solution must be a part of $\tilde{f}(x,r)$.

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