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On Rad_μ-GP*property

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Abstract
Throughout this paper we depend on the concept of μ- small submodule to generalized the Jacobson radical called μ-Radical that lead us to present a generalization of (GP*) property.

Keywords: μ-Radical of X, Rad_μ -GP* property, N- Rad_μ -GP* property.

حول خاصية*GPلجاكوبسون من النمط −μ

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الخلاصة
في هذا البحث سوف نعتمد تعريف المقاسات الجزئية الصغيرة من النمط −μ− لأجل اعطاء تعميم لجذر جاكوبسون ونسميه جذر جاكوبسون من النمط −μ وهذا التعميم سوف يقودنا إلى اعطاء تعميم للخاصية من النمط*P.

1. Introduction

In this work every module is a unital left R-module, and all rings are associative with the identity element. The sets K and U are submodules of X, where X is an R-module. A sub module U is said to be small submodule of X if whenever $K + U = X$ for some submodule K of X, then $X = K$,indicates that $U << X$, see [1]. Let U and W be submodules of X, U is a supplement of W in X if U is minimal with respect to the property $X = U + K$. Equivalently, $X = U + K$ and $U \cap K << U$, [1]. The module X is called ⊕-supplemented module if each submodule of X has a supplement of a direct summand of X [2, 3]. There are many generalizations of ⊕-supplemented module were presented, see [4-6].

Many authors generalized small submodule see [7-11]. In [12] it was defined the submodule Z^* (X) as a dual of singular submodule to be the set of all elements x in X such that Rx is a small module. The set $Z^*(X) = \{x \text{ in } X: Rx \ll_\mu E(X)\}$, where $E(X)$ is an injective hull of X. The set X is called cosingular (non-cosingular) module if $Z^*(X) = X$, ($Z^*(X) = 0$). W. Khalid and E. Mustafa presented the notion of μ- small submodules in [7] as a generalization of small submodules. Any submodule U of X that has X/L cosingular whenever X=U+L and X = L is referred to as μ-small submodule of X (denoted by $U \ll_\mu X$). If there is a $L \leq X$ such

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that X= U+L and U ∩ L $\ll_\mu$ K, then X is said to be μ- supplemented. If L is a direct summand of X, then X is called a $\oplus_\mu$- supplemented module [13]. If for each submodule U of X, there exists direct summand L of X such that L ≤ U and $U/L \ll_\mu X/L$, then M is said to be lifting [14].

A module X is called μ- lifting if for each submodule U of X there exists H of U such that X= H⊕H', where H' in X and U ∩ H' $\ll_\mu$ H'. Equivalently, for each submodule U of X there exists $L \leq^{\oplus} X$ such that L ≤ U, and $U/L \ll_\mu X/L$ [15]. A module X is called has the property (P^*) if for each submodule U of X , $\exists\, L \leq^{\oplus} X$ such that L ≤ U and U/L << (X/L), [16]. A module X is said to has property (GP^*) if every $\lambda \in End_R$ (X) there exists a direct summand H of X such that $H \subseteq \mathrm{Im}(\lambda)$ and $\frac{\mathrm{Im}(\lambda)}{H} << Rad(\frac{X}{H})$, see [17].

According to the property (GP^*) and μ- lifting module we generalize μ- lifting module to be (Rμ-P*) and (N- Rμ-GP*) properties. We present a generalization to Rad (X) called Rad_μ (X), to define (Rμ-P*) property. It is evident that the μ- lifting module possesses the property (Rμ-P*), and each module that possesses this feature is Rad_μ - ⊕- supplemented.

In our work we learn some information on the properties (Rμ-GP*) with (N- Rμ-GP*). We demonstrate that each direct summand of a module with (Rμ-GP*) has also the property (Rμ-GP*) by proving various results on these properties.

2. μ-Radical of X:

This section defines and establishes some of the fundamental features of X is μ-maximal submodule and μ-Radical. Among many generalizations of Rad (X) [18], we present the following:

Definition2.1: Let X be an R-module and let K be submodule of X, then K is called μ-maximal submodule of X if K is maximal and $Z^*(\frac{X}{K})= \frac{X}{K}$.

Examples and remarks 2.2:

1- While every μ-maximal is maximal, this is not always the case.

For example, Z_6 as Z_6 -module $<\bar{2}>$ is maximal but not μ-maximal since $Z^*\left(\frac{Z_6}{<\bar{2}>}\right) \cong Z^*(Z_2) = 0 \neq \frac{Z_6}{<\bar{2}>}$.

2- If Z^* (X) =X, then every maximal is μ-maximal. To show that, let N be maximal and N≠ X and consider the neutral epimorphism.

$f : X \to \frac{X}{N}$ since $f\big(Z^*(X)\big) \leq Z^*\left(\frac{X}{N}\right), f \quad (X)\, Z^*\left(\frac{X}{N}\right)$, but $f(X) = \frac{X}{N}$.

Then $Z^*\left(\frac{X}{N}\right) = \left(\frac{X}{N}\right)$. Hence, N is μ-maximal.

3- The module Z_4 as Z – module, by (2) $<\bar{2}>$ is μ-maximal.

4- The module Z as Z – module: since $Z^*(Z) = Z$, then by (2) for all P prime, $< P >$is maximal if and only if <P> is μ-maximal.

5- If 0 is maximal and *M* is cosingular, then 0 is μ-maximal, by (2).

6- The module Q as Z- module since Q has no maximal. So, by (2) Q has no μ-maximal submodule.

Definition 2.3: If X is an R-module, then the intersection of all of its μ-maximal submodules is known as X is μ-Radical, and it is represented as (Rad_μ (X)). The Rad_μ (X) = X if there is no μ-maximal submodule for X.

Note: The fact that Rad (X) ≤ Rad_μ (X), is obvious, but the inverse is typically not the case. The Rad_μ (Z_6) = Z_6 as Z_6 - module, but Rad (Z_6) = 0.

Proposition 2.4: Let X be R – module and a ∈ Rad_μ (X) if and only if Ra $<<_\mu$ X.

Proof: Let Ra $<<_\mu$ X and suppose that a $\notin$ Radμ (X), then there exists μ - maximal L and a $\notin$ L, so Z* $(\frac{X}{L}) = \frac{X}{L}$, and Ra + L = X. Since Ra $<<_\mu$ X, then X = L, which is a contradiction. Therefore, a $\in$ Rad_μ (M).
Conversely, suppose that a $\in$ Rad_μ (X) and Ra is not μ-small in X.
Let F = {N $\subset$ X such that $Z^*(\frac{X}{L}) = \frac{X}{L}$ and N + Ra= X} $\neq \phi$.
Since Ra is not μ- small in X, then let $\{C_\alpha\}_{\alpha\in\wedge}$ be chain in *F*. To show that $\cup_{\alpha\in\wedge} C_\alpha \in F$, let $C_\alpha \in F$, for each $\alpha \in \wedge$. Since $C_\alpha \in F$, then Z* $\frac{X}{C} = \frac{X}{C}$ and Ra+ C_α = X, for all α∈∧. Since Z* $(\frac{X}{C})= (\frac{X}{C})$, then Z* $(\frac{X}{\cup_{a\in\wedge} C\alpha}) = (\frac{X}{\cup_{a\in\wedge} C\alpha})$, by [3] and Ra + $\cup_{\alpha\in\wedge} C_\alpha$ =X. To show $\cup_{\alpha\in\wedge} C_\alpha \subsetneq X$, if $\cup_{\alpha\in\wedge} C_\alpha$ = X = Ra + H, a $\in \cup_{\alpha\in\wedge} C_\alpha$ then there exists $\alpha_0 \in \wedge$ such that $\alpha \in C_{\alpha 0}$, (since $\{C_a\}_{a\in\wedge}$is chain). Then Ra ≤ Cα and Ra + $C_{\alpha 0}$ = $C_{\alpha 0}$ = X, which is a contradiction, then $\cup_{\alpha\in\wedge} C_\alpha \subset F$. By Zorn's Lemma, *F* has a maximal element say N_0. Since $N_0 \subset X$ and $Z^*(\frac{X}{N_0}) = \frac{C}{N_0}$ and Ra + N_0 = X. To show N_0 is a maximal submodule suppose that $N_0 \subset$ N, then Ra+ N = X, and since $Z^*(\frac{X}{N_0}) = \frac{X}{N_0}$, then by [7], we get $Z^*(\frac{X}{N}) = \frac{X}{N}$, then N ∈ *F*, a contradiction with the maximality of N_0 , a $\notin N_0$, hence a $\notin$ Rad_μ (X), and that is a contradiction. Therefore, Ra$<<_\mu$ X.

Corollary 2.5: $Rad_\mu(X) = \Sigma_{A\alpha <<\mu X} A_\alpha$.
Proof: $A_\alpha <<_\mu$ X, then $\forall_\alpha \in A\alpha$, Ra <<α X, by Proposition 2.4, a ∈ $Rad_\mu(X)$, for all a ∈ A_α , which leads to $A_\alpha \leq Rad_\mu$ (X) , then $\Sigma_{A\alpha <<\mu M} A_\alpha \leq Rad_\mu$ (X).
Conversely, let a ∈ $Rad_\mu(X)$, then Ra $<<_\mu$ X, thus a ∈ $\Sigma_{A\alpha <<\mu M} A_\alpha$.
So, $Rad_\mu(X) \leq \Sigma_{A\alpha <<\mu X} A_\alpha$, hence Rad_μ (X) = $\Sigma_{A\alpha <<\mu X} A_\alpha$.

Theorem 2.6:
Let Φ: X⟶ N be an R- homomorphism, then $\Phi(Rad_\mu(X)) \leq Rad_\mu(N)$.
Proof: Let U $<<_\mu$ X, then by [7], Φ(U) $<<_\mu$ N, and Φ(U) ≤ Rad_μ (N), for all U $<<_\mu$ X. Therefore, Φ (Rad_μ (X)) ≤ Rad_μ(N).

Corollary 2.7:
Let X be any R- module:
a. If C ≤ X, then $Rad_\mu(C) \leq Rad_\mu(X)$.
b. Let $\{X_i\}_{i\in I}$ be a family of R- modules, so if $X = \oplus_{i\in I} X_i$, then $Rad_\mu(X) = \oplus_{i\in I} Rad_\mu(X_i), i \in I$.
c. If $X = \oplus_{i\in I} X_i$, then $\frac{X}{Rad_\mu(X)} \cong \oplus_{i\in I} \frac{X_i}{Rad_\mu(X_i)}$.
Proof:
a. Let $i: C \to X$ be the Inclusion map, then by Corollary 2.5, we have $i\left(Rad_\mu(C)\right) \leq Rad_\mu(X)$. Therefore $Rad_\mu(C) \leq Rad_\mu(X)$.
b. For all $i \in I, X_i \leq X$, by (a) we have $Rad_\mu(X_i)\ Rad_\mu(X), \forall i \in I$, then $\oplus_{i\in I} Rad_\mu(X_i) \leq Rad_\mu(X)$. (1)
For all $i \in I$, let $P_i : \oplus_{i\in I} X_i \longrightarrow X_i$, defined by $P_i(\sum_{i\in I}^{n} X_i) = X_i$.
Let x ∈ $Rad_\mu(X)$ X, then $x = \sum_{i\in I}^{n} m_i$, $m_i \in X_i$, $m_i \neq 0$ for at most a finite number of $i \in I$.
For all $i \in I$, $P_i(x) = P_i\ (\sum_{i\in I}^{n} m_i) = m_j$, $P_i\ (Rad_\mu(X_j))\ Rad_\mu(X_j)$, then $m_i \in Rad_\mu(X_i)$, then

$$x = \sum_{i\in I}^{n} m_i \in \oplus\ Rad_\mu(X_i).$$

Then,

$$\mathrm{Rad}_{\mu}(\mathrm{X})\left(\mathrm{Rad}_{\mu}(\mathrm{X}_i)\right). \quad (2)$$

Form (1) and (2) we get $Rad_{\mu}(\mathrm{X}) = \oplus_{i\in I}(Rad_{\mu}(\mathrm{X}_i))$.

c. Let Φ: $\frac{X}{Rad\mu(X)} \longrightarrow \oplus_{i\in I}\left(\frac{X_i}{Rad_{\mu}(X_i)}\right)$.

Let $\Sigma \mathrm{m}_i \in \oplus \mathrm{X}_i$ with $\mathrm{m}_i \in \oplus \mathrm{X}_i$ be an arbitrary element from X, and define $\Phi(\Sigma \mathrm{m}_i) + Rad_{\mu}(\mathrm{X}) := \Sigma(\mathrm{m}_i + Rad_{\mu}(\mathrm{X}_i)) \in \oplus_{i\in I}\left(\frac{\mathrm{X}_i}{\mathrm{Rad}_{\mu}(\mathrm{X_i})}\right)$. To show that Φ is well define: Let $\sum(\mathrm{m}_i) + Rad_{\mu}(\mathrm{X}) := \sum(\mathrm{m}_i') + Rad_{\mu}(X)$with $\mathrm{m}_i, \mathrm{m}_i' \in \mathrm{X}_i$.
$\Sigma(\mathrm{m}_i) - \Sigma(\mathrm{m}_i') = Rad_{\mu}(\mathrm{X})$, then $\sum(\mathrm{m}_i - \mathrm{m}_i') \in Rad_{\mu}(\mathrm{X})$, hence $\mathrm{m}_i - \mathrm{m}_i' \in Rad_{\mu}(\mathrm{X}_i)$. It follows that $\mathrm{m}_i + Rad_{\mu}(\mathrm{X}_i) = \mathrm{m}_i' + Rad_{\mu}(\mathrm{X}_i)$, thus $\Sigma(\mathrm{m}_i + Rad_{\mu}(\mathrm{X})) = \Sigma(\mathrm{m}_i' + \mathrm{Rad}_{\mu}(\mathrm{X}_i))$, then Φ is well-define

Now, let $\Sigma(\mathrm{m}_i + Rad_{\mu}(X) \in \mathrm{Ker}\Phi$, then $\Phi(\Sigma(\mathrm{m}_i + Rad_{\mu}(\mathrm{X})) = \Sigma(\mathrm{m}_i + Rad_{\mu}(\mathrm{X}_i)) = 0$, hence $\mathrm{m}_i + Rad_{\mu}(\mathrm{X}_i) = 0, \forall i$.

Then it follows that $\mathrm{m}_i \in \mathrm{Rad}_{\mu}(\mathrm{X}_i)$for all occurring m_i, since $\mathrm{X}_i \leq \mathrm{X}$. Therefore, $Rad_{\mu}(\mathrm{X}_i) \leq Rad_{\mu}(\mathrm{X})$, we deduce that $\Sigma(\mathrm{m}_i) + \mathrm{Rad}_{\mu}(X) = Rad_{\mu}(\mathrm{X})$, hence $\ker\Phi = 0$.

Let $\mathrm{m}_i \in \mathrm{X}_i$ then $\mathrm{m}_i \in Rad_{\mu}(\mathrm{X}_i)$, then $\mathrm{m}_i + Rad_{\mu}(\mathrm{X}_i)) \in \oplus_{i\in I}\left(\frac{\mathrm{X}_i}{\mathrm{Rad}_{\mu}(\mathrm{X}i)}\right)$,then $\Sigma(\mathrm{m}_i + Rad_{\mu}(\mathrm{X}_i)) \in \oplus_{i\in I}\left(\frac{\mathrm{X}_i}{\mathrm{Rad}_{\mu}(\mathrm{X}_i)}\right)$, and since $\mathrm{X} = \oplus_{i\in I}\mathrm{X}_i$, then $\Sigma(\mathrm{m}_i) \oplus_{i\in I}\mathrm{X}_i = \mathrm{X}$, and $\Sigma(\mathrm{m}_i) + Rad_{\mu}(\mathrm{X}) \in \frac{X}{\mathrm{Rad}_{\mu}(\mathrm{X})}$. Then $\Phi(\Sigma(\mathrm{m}_i) + Rad_{\mu}(\mathrm{X})) = \Sigma(\mathrm{m}_i + Rad_{\mu}(\mathrm{X}))$.

Remark 2.8:
If X is an R-module and N is a submodule of X, then $\mathrm{Rad}_{\mu}(\mathrm{N}) \leq \mathrm{Rad}_{\mu}(\mathrm{X}) \cap \mathrm{N}$.
Proof:
Since $\mathrm{Rad}_{\mu}(\mathrm{N}) \cap \mathrm{N}$ and $\mathrm{Rad}_{\mu}(\mathrm{N}) \cap \mathrm{Rad}_{\mu}(\mathrm{X})$ by Corollary 2.7, then $\mathrm{Rad}_{\mu}(\mathrm{N}) \leq \mathrm{N} \cap \mathrm{Rad}_{\mu}(\mathrm{X})$and $\mathrm{Rad}_{\mu}(\mathrm{N}) \leq \mathrm{Rad}_{\mu}(\mathrm{X}) \cap \mathrm{N}$.

However, the converse of Corollary2.8 is not true in general, for example, Z_8 as Z- module. The set $<\bar{2}>$ is maximal in Z_8. $\bar{0} \leq <\bar{4}> \leq <\bar{2}> \leq Z_8$. Since $Z^*(Z_8) = Z_8$, then $<\bar{2}>$ is μ - maximal in Z_8. Also, since $\mathrm{Rad}_{\mu}(Z_8) = <\bar{2}>$ and $\mathrm{Rad}_{\mu}(<\bar{2}>) = <\bar{4}>$.
Thus $\mathrm{Rad}_{\mu}(<\bar{2}>) \leq \mathrm{Rad}_{\mu}(Z_8) \cap <\bar{2}>$, but $\mathrm{Rad}_{\mu}(Z_8) \cap <\bar{2}>=<\bar{2}> \nleq \mathrm{Rad}_{\mu}(<\bar{2}>) = <\bar{4}>$, then $\mathrm{Rad}_{\mu}(\mathrm{X}) \cap \mathrm{N} \nleq \mathrm{Rad}_{\mu}(\mathrm{N})$.

Note 2.9: For $\mathrm{N} \leq \mathrm{X}$, $\mathrm{Rad}_{\mu}(\mathrm{X}) \cap \mathrm{N} = \mathrm{Rad}_{\mu}(\mathrm{N})$, if N is the direct summand of X.
Proof: By Remark 2.8 $\mathrm{Rad}_{\mu}(\mathrm{N}) \leq \mathrm{Rad}_{\mu}(\mathrm{X}) \cap \mathrm{N}$.
Conversely, let $w \in \mathrm{Rad}_{\mu}(\mathrm{X}) \cap \mathrm{N}$, then $w \in \mathrm{Rad}_{\mu}(\mathrm{X})$, hence $w = \sum_{i=1}^{n} a_i, a_i \in A_i, A_i \ll_{\mu} X$ and $a_i \ll_{\mu} X$, thus $Rw = Ra_1 + Ra_2 + \ldots + Ra_n$and $Ra_i \ll_{\mu} X$, for each i = 1,2,…,n , thus $R\mathrm{w} \ll_{\mu} \mathrm{X}$, since N is a direct summand of X , then by [3], $R\mathrm{w} \ll_{\mu} N$, therefore $R\mathrm{w} <<_{\mu} \mathrm{Rad}_{\mu}(\mathrm{N})$ and $\mathrm{w} \leq \mathrm{Rad}_{\mu}(\mathrm{N})$.

Proposition 2.10:
If L and K are submodule of module X, then $\mathrm{Rad}_{\mu}(\mathrm{L}) + \mathrm{Rad}_{\mu}(\mathrm{K}) \leq \mathrm{Rad}_{\mu}(\mathrm{L} + \mathrm{K})$.
Proof:
Demonstrating that $\mathrm{Rad}_{\mu}(\mathrm{L}) \leq \mathrm{Rad}_{\mu}(\mathrm{L} + \mathrm{K})$ is sufficient. Since K is a μ-small submodule of L+K according to [3], $\mathrm{K} \leq \mathrm{Rad}_{\mu}(\mathrm{L}+\mathrm{K})$. Let K be a μ-small sub module of X. Thus $\mathrm{Rad}_{\mu}(\mathrm{K}) \leq \mathrm{Rad}_{\mu}(\mathrm{L} + \mathrm{K})$. Similarly, $\mathrm{Rad}_{\mu}(\mathrm{L}) \leq \mathrm{Rad}_{\mu}(\mathrm{L}+\mathrm{K})$, therefore $\mathrm{Rad}_{\mu}(\mathrm{L}) + \mathrm{Rad}_{\mu}(\mathrm{K}) \leq \mathrm{Rad}_{\mu}(\mathrm{L} + \mathrm{K})$.

3. A generalization of modules with the property (GP*)

It is known that X is said to have the property (P*) if for each submodule U of X, there exists a direct summand K of X such, that K ≤ U. and $U/K \subseteq Rad(X/K)$ [16].

Definition 3.1: If for every element λ in End_R (X), there exists a direct summand, such that $N \subseteq Im(\lambda)$ with $\frac{Im(\lambda)}{N} \subseteq \mathrm{Rad}_\mu\left(\frac{X}{N}\right)$, then the module X has the property (Rad_μ-GP*).

Remark 3.2: It is evident that each module satisfies (Radμ-GP*) by having the (P*) property. But generally speaking, the opposite is not true; for example, F can be any field. Take into consideration the direct product of F_i, where $F_i = F$, and the commutative ring R. Therefore, R is not semisimple; rather, it is a regular ring [17]. The property (GP*) is held by [16]; nevertheless, since Rad(X) ⊆ Rad_μ (X), R as R-module satisfies (Rad_μ-GP*). The not satisfy (P*) property.

Proposition 3.3: For a module X, the corresponding conditions are equivalent.
1. The module X has the property (Rad_μ -GP*);
2. Every $\lambda \in End_R$ (X)has a decomposition $X = X_1 \oplus X_2$ such that $X_1 \subseteq Im(\lambda)$ and $X_2 \cap Im(\lambda) \subseteq \mathrm{Rad}_\mu(X_2)$;
3. The representation of $Im(\lambda)$ for each $\lambda \in End_R(X)$ is $Im(\lambda) = U \oplus U'$, whereas U is the direct sum of X and $U' \subseteq \mathrm{Rad}_\mu(X)$.
Proof: 1) → 2) By assumption, if there exists direct summands X_1, X_2 of X such that $X_1 \subseteq Im(\lambda)$, $X = X_1 \oplus X_2$ and $\frac{Im(\lambda)}{X_1} \subseteq \mathrm{Rad}_\mu\left(\frac{X}{X_1}\right)$. Since X_2 is a Rad_μ-supplement of X_1 in X, $\mathrm{Rad}_\mu\left(\frac{X}{X_1}\right) = \frac{\mathrm{Rad}_\mu(X)+X_1}{X_1}$. Then $\frac{Im(\lambda)}{X_1} \subseteq \frac{\mathrm{Rad}_\mu(X)+X_1}{X_1}$. So, we have $Im(\lambda) \subseteq \mathrm{Rad}_\mu(X_2) + X_1$, and, $X_2 \cap Im(\lambda) \subseteq \mathrm{Rad}_\mu(X_2)$.
2) → 3) For every $\lambda \in End_R$ (X),there exists a decomposition $X = X_1 \oplus X_2$ such that$X_1 \subseteq Im(\lambda)$ and $X_2 \cap Im(\lambda) \subseteq \mathrm{Rad}_\mu(X_2)$. So, $Im(\lambda) = X_1 \oplus (Im(\lambda) \cap X_2)$, by the modular law. Say $U = X_1$ in addition that $U' = Im(\lambda) \cap X_2$. Therefore, $Im(\lambda) = U \oplus U'$, where U is a direct summand of X and $U' \subseteq \mathrm{Rad}_\mu$ (X).
3) →1) Given X the assumption states that for any λ in End_R (X), $Im(\lambda) = U \oplus U'$, where U is the direct summand of X and U′ is a subset of $\mathrm{Rad}_\mu(X)$. Therefore, U is absolutely greater than Im(λ) since it is a direct summand of X. As a result, we may write it as $\frac{Im(\lambda)}{U} = \frac{U \oplus U'}{U} \subseteq \frac{U+\mathrm{Rad}_\mu(X)}{U} \subseteq \mathrm{Rad}_\mu\left(\frac{X}{U}\right)$.

Definition 3.4: The property (N – Rad_μ –GP*) applies to a module X if, for any homomorphism λ: X → N, then there exists a direct summand L of N, such that $\frac{Im(\lambda)}{L} \subseteq \mathrm{Rad}_\mu\left(\frac{N}{L}\right)$ with $L \subseteq Im(\lambda)$.
It is evident that if and only if X possesses the property (X–Rad_μ-GP*), then X has the property (Rad_μ-GP*).
Remember that if a submodule N of X lacks a valid submodule K such that K ⊂ N, then N is said to be μ-coclosed in X. The module X is μ-coclosed in all direct summands, as is evident [14].

Theorem3.5: Given two R-modules X and U. The module X possesses the property (U –Rad_μ –GP*). If and only if X' has the property (U′−Rad_μ -GP*), for each direct summand X' and a μ-coclosed submodule U′ of U.
Proof: (⇒) For each $e^2 = e \in End_R(X)$, let U′ be a μ-coclosed submodule of U, and let X' = eX. Presume that Hom (X′, U′) ∈ α. $U = U_1 \oplus U_2$ can be decomposed so that $U_1 \subseteq Im$ (α(e)) and $U_2 \cap Im$ (α(e)) ⊆ $\mathrm{Rad}_\mu(X_2) \subseteq \mathrm{Rad}_\mu(U)$, since α (eX) = α (X′) ⊆ U′ ⊆ U and X satisfies the

condition (U – Rad_μ –GP*). The modular law therefore, gives us $U' = U_1 \oplus (U_2 \cap U')$. $Rad_\mu(U') = Rad_\mu(U) \cap U'$ by [4], where U′ is a μ-coclosed sub module of U. Therefore, $U_2 \cap U' \cap Im(\alpha) \subseteq Rad_\mu(U_2 \cap U')$ is what we obtain. Hence, X' possesses the property (U′ −Rad_μ -GP*).
(⇐) Clear

Corollary 3.6: For a module X, the following criteria are equivalent.
1- The module X has the property (Rad_μ -GP*);
2- For each a direct summand L of X owns the property (U-Rad_μ-GP*) for any a μ-coclosed sub module U of X.

Corollary 3.7:
It is true that all direct summands of modules with the property (Rad_μ -GP*) also have this property.

Proposition 3.8: An indecomposable module X is defined here. Suppose for any δ in End_R (X), the fact that $Im(\delta) \subseteq Rad_\mu(X)$ means that δ = 0. The property (Rad_μ -GP*) is applicable to X if and only if each non-zero endomorphism $\delta \in End_R$ (X) is an epimorphism.
Proof. Let $0 \neq \delta \in End_R(X)$. Since X has the property (Rad_μ -GP*), there exists a decomposition $X = X_1 \oplus X_2$ with $X_1 \subseteq Im(\delta)$ and $X_2 \cap Im(\delta) \subseteq Rad_\mu(X_2)$. Since X is indecomposable, $X_1 = 0$ or $X_1 = X$. If $X_1 = 0$, then $Im(\delta) \subseteq Rad_\mu(X)$. By assumption δ = 0; which is a contradiction. Thus, $X_1 = X$ and hence δ is epimorphism.
The opposite is clear.

Remember that if there is an isomorphism for each surjective endomorphism of module X, then that module is said to be Hopfian [18, 19].

Proposition3.9: Consider X as a Noetherian module with the following property: (Rad_μ -GP*). $Im(\lambda) \subseteq Rad_\mu(X)$ means that $\lambda = 0$ if every endomorphism λ of X exists. Thereafter, $X = X_1 \oplus X_2 \oplus \ldots \oplus X_n$ has a decomposition, where X_i is an indecomposable noetherian module with the condition (Rad μ -GP*), and $End_R(X_i)$ is a division ring for it.
Proof: X has a finite decomposition of Noetherian direct summands since it is a Noetherian. All direct summand has the property (Radμ -GP*), by Corollary 3.7, and every decomposable direct summand has a division ring by [16] because every Noetherian module is Hopfian, by [20].

4.Conclusions
We confirm the following outcomes:
1.While every μ-maximal is maximal
2. If Z* (X) =X, then every maximal is μ-maximal.
3. Let Φ: X→ N be an R- homomorphism, then $\Phi(Rad_\mu(X)) \leq Rad_\mu(N)$.
4. If X is an R-module and N is a submodule of X, then $Rad_\mu(N) \leq Rad_\mu(X) \cap N$.
5. For N ≤ X, $Rad_\mu(X) \cap N = Rad_\mu(N)$, if N is the direct summand of X.

Conflict of Interest
The authors declare that they have no conflicts of interest.

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