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The Non-Zero Divisor Graph of Prime Ring

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Abstract

Suppose that R be a ring, the non-zero divisor graph of R denoted by $\Phi(R)$ and defined as a simple graph has vertex set $R \setminus \{0, \pm 1\}$ and different vertices $\alpha, \beta \in V(\Phi(R))$ are adjacent if and only if $\alpha\beta \neq 0$ or $\beta\alpha \neq 0$. For the prime ring R , we aim to investigate the structure of the graph $\Phi(R)$. This involved the behavior of the nilpotent elements of the prime ring inside the graph as well as proving the graph is connected with girth of three. Furthermore, characterize certain universal vertex in the graph.

Keywords: The non-zero divisor graph, prime ring, connected graph, girth.

البيان المقسوم عليه الغير الصفري للحلقة الأولية

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الخلاصة :

لنفرض ان R حلقة، البيان المقسوم عليه الغير الصفري للحلقة R يرمز له بالرمز $\Phi(R)$ ويعرف كبيان بسيط يمتلك مجموعة رؤوس تكون $R \setminus \{0, \pm 1\}$ مع رؤوس مختلفة $\alpha, \beta \in V(\Phi(R))$ متجاورة اذا فقط اذا كان $\alpha\beta \neq 0$ او $\beta\alpha \neq 0$. للحلقة الاولى R ، نحن نهدف الى دراسة هيكل البيان $\Phi(R)$. هذا تضمن سلوك العناصر المدعومة للحلقة الاولى داخل البيان بالإضافة إلى برهان ان البيان مرتبط مع مقاس يساوي ثلاثة. علاوة على ذلك، تميز رأس شامل المعينة في الرسم البياني.

1. Introduction

One of the most common ways of comprehending algebraic structure is by observing how algebraic structure acts on a graph. Numerous articles have been conducted to examine the algebraic characteristics of groups or rings by assigning a graph to them; some of these studies are [1-5]. Given a ring R , the non-zero divisor graph denoted by $\Phi(R)$ has a vertex set $V(\Phi(R)) = R \setminus \{0, \pm 1\}$, whose vertices $\alpha, \beta \in V(\Phi(R))$ are linked to each other if and only if $\alpha \neq \beta$ and $\alpha\beta \neq 0$ or $\beta\alpha \neq 0$. Certainly, when R without 1, then we have $V(\Phi(R)) = R \setminus \{0\}$, this graph was initially created by Kadem et al. [6]. They figuring out the diameter of the connected graph and analyzing the structure of $\Phi(R)$, when R is the ring of integers modulo n , Z_n , reduced ring, and matrix ring [6]. A ring R is a domain if and only if $\Phi(R)$ is a complete

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graph, meaning that any two different vertices are adjacent. As well as R is a field if it is a finite commutative ring with one such that $\Phi(R)$ is complete graph. The aforementioned graphs are built on base on the zero-divisor graph. It is described as the directed graph $\Gamma(R)$ with vertices that are all non-zero divisors of R and edges that can be any two distinct vertices, α and β such that $\alpha\beta=0$. A number of fundamental findings about the zero divisor graph are presented by the study contributors in, [7-14]. Examining the theoretical characteristics of $\Phi(R)$ when R is a prime ring is the primary goal of this work. We consider all graphs to be simple graphs, that is, graphs without more than one edge or loop. Through use of terminology, we represent the vertices of a graph $\Phi(R)$ by $V(\Phi(R))$. It is also important to note that if a path connects any two different vertices in $V(\Phi(R))$, then a graph $\Phi(R)$ is connected. Let $\wp(\alpha, \beta)$ represents the length of the shortest path between vertices α and β of $V(\Phi(R))$. In the event that such a path does not exist, we may allow $\wp(\alpha, \beta) = \infty$ and $\wp(\alpha, \alpha) = 0$. Also, $N(\alpha) = \{ \beta \in V(\Phi(R)) : \wp(\alpha, \beta) = 1 \}$ the neighbor of α in $\Phi(R)$. Moreover, the least number of vertices needed to form a cycle in the graph is known as the girth of the graph and denoted by $\omega(\Phi(R))$. Finally, the diameter and the radius of $\Phi(R)$ are define as, $\text{Diam}(\Phi(R)) = \max \{ \wp(\alpha, \beta) : \alpha, \beta \in V(\Phi(R)) \}$ and, $r(\Phi(R)) = \min \{ \wp(\alpha, \beta) : \alpha, \beta \in V(\Phi(R)) \}$. For further information, on this context we refer to [15] and [16]. This work contributes to providing crucial findings for the non-zero divisor graph of prime rings, such as the connectivity and the girth the graph. Moreover, identify a specific universal vertex in the graph, also demonstrate the effect of the nilpotent elements of the prime ring within the non-zero divisor graph.

2. Preliminaries

This section displays a general result related to the non-zero divisor graph. We begin with defining the prime ring and the non-zero divisor graph, followed by a relevant instance to serve as an explanation:

Definition 2.1 [6]: Let R be a ring, a simple graph with a vertex set $R \setminus \{0, \pm 1\}$ such that distinct vertices α, β linked to each other whenever $\alpha\beta \neq 0$ or $\beta\alpha \neq 0$. This graph is called the non-zero divisor graph of the ring R , represented as $\Phi(R)$.

Definition 2.2 [17]: A ring R is a prime ring if for any $\alpha, \beta \in R$ such that $\alpha R \beta = 0$, then either $\alpha = 0$ or $\beta = 0$.

Example 2.3 : Consider the matrix ring over the ring of integer mode 2, $M(Z_2)$. So that $M(Z_2)$ is a prime ring and the non-zero divisor graph $\Phi(M(Z_2)) \cong K_{14}$ (the complete graph of degree 14). Thus, we conclude that $M(Z_2)$ is a domain.

The results on the non-zero divisor graph of specific rings, as stated in [6], are given in the following theorem:

Theorem 2.4: If we assume that R is a ring, then the following characteristics of the non-zero divisor graph $\Phi(R)$ exists:

- i- $\Phi(R)$ is connected when R has an invertible element.
- ii- $\Phi(Z_n)$ is connected if and only if $n \notin \{1, 3, 2, 6\}$.
- iii- For any ring R we have $\text{Diam}(\Phi(R)) \leq 2$ if $R \not\cong Z_2 \times Z_4$ and $\Phi(R)$ is a connected.
- iv- $\Phi(R)$ is connected if R is isomorphic to reduced ring and such that $|V(\Phi(R))| > 3$.
- v- Boolean rings form a connected non-zero divisor graph.
- vii- Let R be an identity-free ring such that for any $\beta \in V(\Phi(R))$, $\deg(\beta) > 0$. Thus, the graph $\Phi(M_n(R))$ is considered to be connected.
- viii- $\Phi(M_n(R))$ is connected whenever $\Phi(R)$ is connected.

3. Main results

We take R to be a prime ring for the intent of this section and we investigate both cases when R with and without one. First, we prove that the graph $\Phi(R)$ is connected. Then, we give the requirement that the vertex $\beta \in V(\Phi(R))$, be a universal vertex and hence a dominant set. Lastly, we demonstrate that $\Phi(R)$ has a girth of almost 3.

In the next, we demonstrate that the non-zero divisor graph of prime ring is always connected graph.

Theorem 3.1: Let R be a prime ring, then the non-zero divisor graph $\Phi(R)$ is connected. **Proof:** Suppose that α, β are arbitrary vertices in $\Phi(R)$, note that the proof is accomplished when $\wp(\alpha, \beta)=1$. Therefore, we ought to have $\alpha\beta=\beta\alpha=0$. Thus, we will divide the proof into the following subcases:

Case (1): If $\alpha R \beta = 0$ then the primeness of R leads to either $\alpha = 0$ or $\beta = 0$ which is impossible as $\alpha, \beta \in V(\Phi(R))$.

Case (2): If $\alpha R \beta \neq 0$ then there exists $0 \neq r \in R$, such that $\alpha r \beta \neq 0$, our aim is to show that $r \in V(\Phi(R))$. To prove that, we assume the opposite where $r \in \{-1, 1\}$, then we have $\alpha r \beta = -\alpha \beta = \alpha \beta = 0$, which is not true and hence we obtain $r \in V(\Phi(R))$. Now, we claim that $r \in N(\alpha) \cap N(\beta)$, if this is not true then $r \beta = \alpha r = 0$, so that $\alpha r \beta = 0$, which is a contradiction. Then we get $r \in N(\alpha) \cap N(\beta)$ and we done. Therefore, for above cases we conclude that $\Phi(R)$ is connected when R is a prime ring.

The next result follow from that the ideal of any prime ring is also a prime ring.

Corollary 3.2: Let R be a prime ring then the non-zero divisor graph of any ideal of R is connected.

The next corollary can be proved by Theorem 2.4 and Corollary 3.2.

Corollary 3.3: Let R be a prime ring then the diameter of $\Phi(R)$ is less than or equal to 2.

As we know that every simple ring R is a prime ring then $\Phi(R)$ is a connected graph where $\text{Diam}(\Phi(R)) \leq 2$. Generally, we have the following result:

Corollary 3.4: The non-zero divisor graph of a left or a right primitive ring is connected with diameter less than or equal to 2.

The dominating vertex (universal vertex) define as a vertex has an edge with all other vertices in the graph. For a prime ring R , in the next result we determine certain universal vertex in R .

Proposition 3.5: The primness of R along with the condition $|V(\Phi(R))| > 1$, ensure that all element belongs to $\text{Cent}(R) \setminus \{0, \pm 1\}$ is dominating vertex (universal vertex).

Proof: We have $\Phi(R)$ is connected. Now, let $0 \neq \alpha \in \text{Cent } R \cap V(\Phi(R))$ and suppose that $\beta \in V(\Phi(R))$ such that $\alpha \neq \beta, \alpha\beta = \beta\alpha = 0$. Since $\alpha \in \text{Cent } R$, then $\alpha R \beta = 0$, but R is a prime ring, that means either $\alpha = 0$ or $\beta = 0$ which is a contradiction with the fact that both $\alpha, \beta \in V(\Phi(R))$. Therefore, both $\alpha\beta \neq 0$ and $\beta\alpha \neq 0$ which means $V(\Phi(R)) \setminus \{\alpha\} = N(\alpha)$. Thus, α is a universal vertex.

Proposition 3.6: No commutative prime ring R exists that satisfies the conditions $|V(\Phi(R))| > 1$ and $\text{g}(\Phi(R))=2$.

Proof: Assume R is commutative then $\text{Cent}(R) \setminus \{0, \pm 1\} = V(\Phi(R))$. By applying Proposition 3.5 we obtain that $\wp(\alpha, \beta) = 1$, for all $\alpha, \beta \in V(\Phi(R))$. Consequently we get $\text{g}(\Phi(R))=1$ which is a contradiction. Therefore, R is non-commutative ring.

We should note that, if R is a commutative prime ring then $\Phi(R)$ is a complete graph, thus we get $\text{g}(\Phi(R))=1$. However, when $\text{g}(\Phi(R))=1$ it is not always true that R is a commutative ring, see Example 2.3.

The following result examines the behavior of the nilpotent elements inside the prime ring.

Proposition 3.7: Consider the graph $\Phi(R)$ of a prime ring R with condition $|V(\Phi(R))| > 1$. If $\alpha \in V(\Phi(R))$ be a nilpotent element. Then there exists $u \in N(\alpha)$ satisfying $\alpha \mp u \in V(\Phi(R))$.

Proof: Since $|V(\Phi(R))| > 1$, then there is $\beta \neq \alpha \in V(\Phi(R))$, also by Theorem 3.1 we have $\Phi(R)$ is connected which means there is a path of vertices connected α with β , thus $N(\alpha) \neq \emptyset$. Also, as α is a nilpotent element then there exists a positive integer n such that $\alpha^n = 0$. Now, if for all $u \in N(\alpha)$ we have $\alpha \mp u \in \{0, \pm 1\}$, then by multiply both side by α^{n-1} we get $\alpha^{n-1} u \alpha^{n-1} = 0$. Also, $\alpha^{n-1} v \alpha^{n-1} = 0$ for all $v \notin N(\alpha)$ and $\alpha^{n-1} w \alpha^{n-1} = 0$ if $w \in \{0, \pm 1\}$. Thus we have $\alpha^{n-1} R \alpha^{n-1} = 0$, but R is a prime this means $\alpha^{n-1} = 0$ which is impossible then there exists $u \in N(\alpha)$ such that $\alpha \mp u \in V(\Phi(R))$.

Proposition 3.8: Let R be a prime ring with $|V(\Phi(R))| > 1$, $\alpha \in V(\Phi(R))$ be a nilpotent element that is ($\alpha^n = 0$ for some positive integer n). Then there exists $u \in N(\alpha)$ such that $\alpha \mp u \in N(\alpha^{n-1})$.

Proof: By applying Proposition 3.7, there exists $u \in N(\alpha)$ such that $\alpha \mp u \in V(\Phi(R))$. We aim to show that $\alpha \mp u \in N(\alpha^{n-1})$. Assume the opposite, thus for all $u \in N(\alpha)$ such that $\alpha \mp u \in V(\Phi(R))$, we have $\alpha^{n-1} u = u \alpha^{n-1} = 0$, which leads to $\alpha^{n-1} u \alpha^{n-1} = 0$. Also, when $u \in N(\alpha)$ with $\alpha \mp u \notin V(\Phi(R))$, we have $\alpha \mp u \in \{0, \pm 1\}$ so that as the same as above we obtain $\alpha^{n-1} u \alpha^{n-1} = 0$. Moreover, $\alpha^{n-1} r \alpha^{n-1} = 0$ for all $r \in R \setminus \{N(\alpha)\}$. Therefore, we get $\alpha^{n-1} R \alpha^{n-1} = 0$ and by the primness of R we have $\alpha^{n-1} = 0$ and this is a contradiction. Thus, we must have $\alpha^{n-1} u \neq 0$ and $u \alpha^{n-1} \neq 0$ and from this we conclude that $\alpha \mp u \in N(\alpha^{n-1})$.

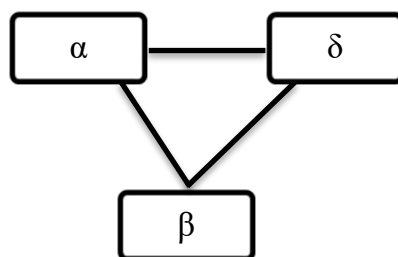
Let R be a prime ring then by Theorem 2.4 part iii we have under some conditions that $\Phi(R)$ assigned a diameter that had to be lower than or equal to two when the graph is connected. In this paper we will show that the girth of the non-zero divisor graph over a prime ring R is equal to 3. We will divide the proof into two parts based on the number of vertices in the graph.

Lemma 3.9: Let R be a prime ring with characteristic ($\text{Ch}R \neq 2$). Then if there exists $\alpha \in V(\Phi(R))$ such that $|N(\alpha)| > 1$ then $\omega(\Phi(R)) = 3$.

Proof: Since R is a prime with $\text{Ch}R \neq 2$ then R is a 2-torsion free thus $\alpha \neq -\alpha$. Since $N(\alpha) = N(-\alpha)$ and because $|N(\alpha)| > 1$ then there exists $y \notin \{\alpha, -\alpha\}$ such $y \in N(\alpha) = N(-\alpha)$. To complete the proof, a number of subcases should be put into regard:

Case1: If $\alpha^2 = 0$.

If $\alpha^2 = 0$, then $\alpha \notin N(\mp\alpha)$ and there exist $\beta, \delta \notin \{\alpha, -\alpha\}$ such $\beta, \delta \in N(\mp\alpha)$. If $\beta \in N(\delta)$, we get a cyclic of length three.

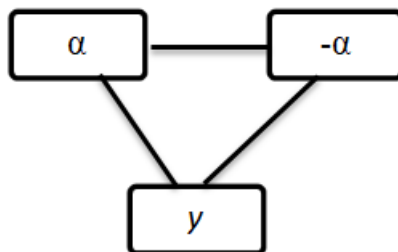


In the absence of such condition, we have $\beta\delta = \delta\beta = 0$. In this case consider $\delta + \alpha \in R$, first we show that $\delta + \alpha \in V(\Phi(R))$. Suppose that this not the case, so that either $\delta\alpha = \alpha\delta = 0$ which is impossible or $\delta\alpha = \mp\alpha = \alpha\delta$, but since $\beta \in N(\mp\alpha)$, so that with no sacrificing generality, we can presume that $\beta\alpha \neq 0$, thus we get $0 = \beta\delta\alpha = \beta\alpha \neq 0$, then a contradiction is shown. If the vertex $\delta + \alpha \neq \beta$ then $\delta + \alpha \in N(\beta) \cap N(\mp\alpha)$. Now we claim that $\delta - \alpha \neq \mp 1$ else we have $\delta\alpha = \mp\alpha = \alpha\delta$ this is a contradiction as $\beta \in N(\mp\alpha)$, as a result we obtain that $\delta - \alpha \in V(\Phi(R))$. If the vertex $\delta - \alpha \neq \beta$, then $\delta - \alpha \in N(\beta) \cap N(\mp\alpha)$.

Finally, if $\delta - \alpha = \delta + \alpha = \beta$ then $\alpha = -\alpha$ which is impossible as $\text{ChR} \neq 2$, therefore the girth equal 3.

Case2: If $\alpha^2 \neq 0$.

Assume that $\alpha^2 \neq 0$ then $\alpha \in N(-\alpha)$, so that we obtain



And therefore, we get $\omega(\emptyset(R)) = 3$.

Lemma 3.10: Let R be a prime ring, then if there exists $\alpha \in V(\emptyset(R))$ such that $|N(\alpha)| > 2$ we have $\omega(\emptyset(R)) = 3$.

Proof: If $\text{ChR} \neq 2$ then by Lemma 3.9 we get $\omega(\emptyset(R)) = 3$. So that, let $\text{ChR} \neq 2$ that is $x = -x$ for x in R , also since $|N(\alpha)| > 2$, then there exist $\beta, \delta, \gamma \in N(\alpha)$. Now, if there is an edge between the vertices $\{\beta, \delta, \gamma\}$ then we are done. Therefore, assume that $\beta\delta = \delta\beta = \beta\gamma = \gamma\beta = \delta\gamma = \gamma\delta = 0$. Thus, we consider the following subcases:

Case1: If $\alpha^2 = 0$.

Consider $\beta + \alpha \neq 0$, if $\beta + \alpha = 1$, then $\delta\alpha = \alpha\delta = \delta$, since $\delta \in N(\alpha)$ so we assume without loss of generality that $\delta\alpha \neq 0$. Then we get $0 = \delta\alpha^2 = \delta\alpha$ which is a contradiction, thus $\beta + \alpha \in V(\emptyset(R))$. Furthermore, since $\beta + \alpha \neq \delta$, we obtain $\alpha\gamma = \gamma\alpha = 0$ which is impossible as $\gamma \in N(\alpha)$. Therefore, we can see immediately that $\beta + \alpha \in N(\alpha) \cap N(\delta)$, then we conclude that $\omega(\emptyset(R)) = 3$.

Case2: If $\alpha^2 \neq 0$.

Since α, β, δ and γ are all distinct vertices in ring with $\text{ChR} \neq 2$, then the elements $\beta + \alpha, \delta + \alpha$ and $\gamma + \alpha$, are all non-zero. To complete the proof, we take into account the next subcases:

i-If $1 \notin \{\beta + \alpha, \delta + \alpha, \gamma + \alpha\}$.

Then as $\text{ChR} \neq 2$, we get $\beta + \alpha, \delta + \alpha$ and $\gamma + \alpha$ are all in $V(\emptyset(R))$. Furthermore, since $\{\beta, \delta, \gamma\} \subseteq N(\alpha)$, then $\beta + \alpha \notin \{\delta, \gamma\}$, thus we conclude that $\beta + \alpha \in N(\gamma) \cap N(\delta), \delta + \alpha \in N(\gamma) \cap N(\beta)$, and $\gamma + \alpha \in N(\beta) \cap N(\delta)$. Now, we can achieve our target if one of these vertices is adjacent with α . Otherwise, let $\beta + \alpha, \delta + \alpha$ and $\gamma + \alpha$ are all not in $N(\alpha)$, thus $\alpha^2 + \delta\alpha = \alpha^2 + \alpha\delta = \alpha^2 + \gamma\alpha = \alpha^2 + \alpha\gamma = \alpha^2 + \alpha\beta = \alpha^2 + \beta\alpha = 0$. However, since $\beta \in N(\alpha)$ we may assume without loss of generality that $\beta\alpha \neq 0$. Therefore, $(\beta + \alpha)(\gamma + \alpha) = \beta\gamma + \beta\alpha + \alpha\gamma + \alpha^2 = \beta\alpha \neq 0$. Thus, the vertices $\{\beta + \alpha, \gamma + \alpha, \delta\}$ form a cycle of length 3, thus we obtain $\omega(\emptyset(R)) = 3$.

ii-If $1 \in \{\beta + \alpha, \delta + \alpha, \gamma + \alpha\}$.

Since all the elements $\{\beta + \alpha, \delta + \alpha, \gamma + \alpha\}$ are distinct, then only one of these elements is equal to 1. Thus, we may assume without loss of generality that $\beta + \alpha = 1$. Therefore, we get $\delta + \alpha, \gamma + \alpha \in V(\emptyset(R))$. Furthermore, $\delta + \alpha \in N(\gamma) \cap N(\beta)$, and $\gamma + \alpha \in N(\beta) \cap N(\delta)$. Now, if one of $\delta + \alpha$ and $\gamma + \alpha$ in $N(\alpha)$, we can achieve our target. Therefore, assume that $\{\delta + \alpha, \gamma + \alpha\} \not\subseteq N(\alpha)$, then similar as above we can show that $\gamma + \alpha \in N(\delta + \alpha)$. Thus, we end up with a cycle of three vertices, consequently we obtain that $\omega(\emptyset(R)) = 3$.

In the following theory, which deals with counting the girth of $\Phi(R)$. The structure of the rings and their behavior on the graph $\Phi(R)$ will be ascertained in cases of finite rings by applying the Gap program [18]. Applying the Gap codes number small rings and small ring, accordingly, will assist to achieve the aforementioned objective. Moreover, it is important to take into account that an in-depth classification of rings of order p^2 , for a given prime number p , was made in [19].

Theorem 3.11 The primness of R along with the condition $|V(\Phi(R))| > 2$, guaranty that $\omega(\Phi(R)) = 3$.

Proof: The proof will be partitioned as follows based on $|V(\Phi(R))|$:

i. If $|V(\Phi(R))| = 3$.

Then we look at all finite prime rings such that $|V(\Phi(R))| = 3$. There are only three possibilities of a finite rings without identity has vertex set of size 3. However, no one of these rings is prime ring. Furthermore, for the rings with one, we have only the ring Z_6 has vertex set with size 3, nevertheless Z_6 not prime ring.

ii. If $|V(\Phi(R))| = 4$.

For the rings without identity, there is only one ring say R with vertex set of size 4. However, there exists an element $\alpha \neq 0$ in R such that $\alpha R = 0$, thus R not prime ring. On the other hand, the field Z_7 is the only ring with identity such that $|V(\Phi(Z_7))| = 4$, since $\Phi(Z_7)$ is a complete graph, thus $\omega(\Phi(Z_7)) = 3$.

iii. $|V(\Phi(R))| = 5$.

When R without identity, we can find only three rings with vertex set of size 5. Moreover, these rings cannot be a prime because they have non-zero element α satisfy $\alpha\beta = 0$, for all β of these rings. Furthermore, the only ring with one and vertex set of size 5 is Z_8 , also in Z_8 the vertices $\{\bar{2}, \bar{3}, \bar{5}\}$ form a cycle of length 3, thus $\omega(\Phi(Z_8)) = 3$.

iv. $|V(\Phi(R))| > 5$, because the diameter size for $\Phi(R)$ is under or equal two, then there exists α in $V(\Phi(R))$, such $|N(\alpha)| > 2$, then by Lemma 3.10 we have $\omega(\Phi(R)) = 3$. Assume that no such α exists, that is for all $\alpha \in V(\Phi(R))$, we have $|N(\alpha)| \leq 2$. Let $\alpha \in V(\Phi(R))$ and $\alpha_1, \alpha_2 \in N(\alpha)$, if $\wp(\alpha_1, \alpha_2) = 1$ then have $\omega(\Phi(R)) = 3$. So that suppose that $\alpha_1 \notin N(\alpha_2)$, and since $|V(\Phi(R))| > 5$, thus there are $\alpha_3, \alpha_4 \in V(\Phi(R))$ such that $\wp(\alpha_1, \alpha_3) = 1 = \wp(\alpha_2, \alpha_4)$. Furthermore, the graph $\Phi(R)$ is connected with $|V(\Phi(R))| > 5$, then there exist $\alpha_5 \in V(\Phi(R))$ such that $\alpha_5 \in N(\alpha_3) \cap N(\alpha_4)$, thus $\wp(\alpha_5, \alpha) = 1$ which is a contradiction as $\text{Diam}(\Phi(R)) \leq 2$. Finally, if $\wp(\alpha_3, \alpha_4) = 1$, and because of the neighbor size of the vertices $\{\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ equal two, then this vertices cannot be connected to any other vertex in the graph. Thus, we get a contradiction as $\Phi(R)$ is connected with $|V(\Phi(R))| > 5$.

Based on all of the aforementioned subcases, we can infer that $\omega(\Phi(R)) = 3$.

4. Conclusions

The structure of the non-zero divisor graph of prime rings have been examined in this work. Several conclusions were made, including the demonstration that the graph is connected with girth equal to three. Moreover, a specific universal vertex throughout the graph is recognized. In addition, the action of the nilpotent elements within the prime ring is examined.

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