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# Resize result involution graphs of some finite groups

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## ABSTRACT

The resize result involution graph of a finite group $G$, is a simple undirected graph whose vertices are the conjugacy classes of $G$ and two distinct vertices are adjacent if their representative product is a non-trivial involution. In this paper, we describe an algorithm to obtain the resize result involution graphs. Also, we prove that the resize result involution graphs for $PGL(2,q)$, $PSL(2,q)$, the Janko groups $J_1$, $J_2$, the Held group $He$ and the exceptional group ${}^2F4(2)'$ are connected with diameter at most $3$ and girth $3$. We find some properties of the resize result involution graphs. The properties of these graphs are obtained by using GAP and YAGs packages.

**Mathematics Subject Classification (2010):** Primary 13F15, Secondary 13M05

http://www.ams.org/msc/

## 1.Introduction

Due to graph theory, a graph is consisting of vertices and edges that join the vertices. Graphs can be used to describing events, representing relationship between items and additionally for analysis and problem solving. The relation between graph theory and group theory has been studied in publications for many years. Certain academics represent groups as graphs to study structure of a group. In effect, the result involution graph of a finite group first appeared in the paper of Jund and Mohammed salih (Jund and Mohammed Salih 2021) . They study this graph for the symmetric and alternating groups. The structures of the result involution graph are found for all Mathiau groups in (Abd Aubad and Mohammed Salih 2023). There are several other papers studying some simple sporadic groups, such as (Aubaid and Arkan Meteab 2023, Abd Aubad and Arkan Meteab 2024) . Many other authors associated groups with graphs such as (Devillers and Giudici 2008, Everett and Rowley 2020). In this paper, we consider simple graphs which are undirected, with no loops or multiple edges. Also, we denote the sets of vertices and edges of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree of a vertex $v$ in $\Gamma$ is the number of edges incident to $v$ and denoted by $\deg(v)$. For any graph $\Gamma$, a subset $Y$ of the vertices of $\Gamma$ is called a clique if the induced subgraph on $Y$ is a complete graph. The maximum size of a clique in a graph $\Gamma$ is called the clique number and denoted by $\omega(\Gamma)$. A path $P$ is a sequence of distinct vertices $x_0, \dots, x_k$ whose terms are alternately adjacent vertices and the number $k$ is called the length of $P$. In addition, if $x_0$ and $x_k$ are adjacent in $\Gamma$, then $x_0, x_1, \dots, x_k x_0$ is called a cycle of the length $k+1$. The length of the shortest cycle in $\Gamma$ is called girth of $\Gamma$ and denoted by $girth(\Gamma)$. A graph $\Gamma$ is connected if there is a path between each pair of distinct vertices of $\Gamma$, otherwise $\Gamma$ is said to be disconnected. If $x$ and $y$ are vertices in $\Gamma$, then the distance between $x$ and $y$, denoted by $d(x,y)$, is the length of the shortest path between $x$ and $y$, if such a path exists; otherwise, $d(v,w) = \infty$. The diameter of a graph $\Gamma$ is $diam(\Gamma) = \max\{d(u,v) | u \text{ and } v \text{ are distinct vertices of } \Gamma\}$. Throughout this paper, we assume that the multi-set of conjugacy classes is $C = \{C_1, \dots, C_l\}$ of $G$. Let $\mathbb{F}_q$ be a field with $q = p^n$ elements and $p$ a prime number. We will write $GL(2,q)$ for $GL(2,\mathbb{F}_q)$, the group $2 \times 2$ invertible matrices over $\mathbb{F}_q$. The projective general linear group $PGL(2,q)$ is the quotient $GL(2,q)/Z(GL(2,q))$. We consider the result involution graph whose vertex set consists of all elements of a group $G$ and adjacency is defined by a product of two elements of $G$ is a non-trivial involution. The groups under the consideration are too big. For this purpose, we take the conjugacy classes as a vertex set instead of the group elements. That is, we consider the resize result involution graph of $G$ whose vertex set consists of all conjugacy classes of $G$ where two distinct conjugacy classes, $C_i$ and $C_j$ are adjacent if there exist some elements $x \in C_i$ and $y \in C_j$ such that $xy$ is a non-trivial involution.

The paper is organized as follows: In the next section, we give some necessary results for other sections. In Section 3 we describe an algorithm and explain it with an example. In section 4, we study the resize result involution graph for $PGL(2,q)$ and $PSL(2,q)$. In the last section we consider the resize result involution graph for some of sporadic groups. Various results are given about them.

## 2. Preliminaries

This section will cover some definitions and results that will required in subsequent sections. A matrix $A = (a_{ij})$ of order $n$, where the $ij$-entry is associated with the ordered pair of vertices $(x_i, x_j)$. If $\Gamma$ is a graph, then

$$a_{ij} = \begin{cases} 1 & if\ xi \text{ and } xj \text{ are adjacent in } \Gamma \\ 0 & otherwise \end{cases}$$

The matrix $A$ is the adjacency matrix of $\Gamma$ and is a symmetric $(0,1)$-matrix with zeros on the main diagonal. The spectrum of a graph $\Gamma$ is the set of numbers which are eigenvalues of $A = A(\Gamma)$, together with their multiplicities.

$$A = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ m(\lambda_1) & m(\lambda_2) & \dots & m(\lambda_n) \end{pmatrix}$$

Also, the characteristic polynomial $det(A - \lambda I)$ of $\Gamma$, and denoted by $\chi(\Gamma, A)$. Suppose that the characteristic polynomial of $\Gamma$ is $\chi(\Gamma, A) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \dots + a_0$ . In this form we know that $-a_1$ is the sum of the zeros, that is, the sum of the eigenvalues. This is also the trace of $A$ which is zero. Thus $a_1 = 0$.

**Proposition 2.1.** (Biggs 1993) The coefficients of the characteristic polynomial of a graph $\Gamma$ satisfy

1. $a_1 = 0$;
2. $a_2$ *is the number of edges of* $\Gamma$;
3. $a_3$ *is twice the number of triangles in* $\Gamma$.
4. $a_4 = 2q + \frac{1}{2}m(m+1) - \frac{1}{2}\sum_{i=1}^{n} deg(v_i)^2$ where $\Gamma$ has $m$ edges, $t$ triangles and $q$ cycles of length four.

**Theorem 2.2.** (Beineke and Wilson 2004, p.8) A graph $G$ is planar if and only if it does not contain a subgraph homeomorphic to $K3,3$ or $K5$.

**Theorem 2.3.** (Biggs 1993, p.11) A graph is bipartite if and only if its spectrum is symmetric about $0$.

**Proposition 2.4.** (Jund and Mohammed Salih 2021) Let $G$ be a finite group with $t$ involution elements. Then the number of edges in $\Gamma^{RI}(G)$ is $\frac{1}{2}(t|G| - F)$ where $F$ is the number of elements of order $4$.

**Proposition 2.5.** (Jund and Mohammed Salih 2021) Let $G$ be a finite group. Then the resize graph of $\Gamma^{RI}(G)$ is connected if and only if the result involution graph $\Gamma^{RI}(G)$ is connected.

**Corollary 2.6.** (Wilson 1972, p.28) Any simple graph with *n* vertices and more than $\frac{(n-1)(n-2)}{2}$ edges is connected.

**Lemma 2.7.** (Wilson 1972, p.12) Let $\Gamma$ be a graph with vertex set $V(\Gamma) = \{v_1, \dots, v_n\}$. Then $\sum_{i=1}^{n} deg(v_i)$ is equal to twice the number of edges.

**Definition 2.8.** (Mohammed Salih 2023) The index, $ind\, x$ is the minimum number of transpositions needed to express $x$ as a product.

**Remark 2.9.**

The conjugate elements have the same indices, that is the elements in the same class have equal indices.

## 3. Algorithm

To obtain our results, we need to take the following steps:

1. We extract all primitive permutation group G by using the GAP function AllPrimitiveGroups(DegreeOperation,n).
2. For the group G, compute the conjugacy class representatives. In particular, compute involution classes.
3. For the tuple of length 3 such that the third component lies in the involution classes. Compute the character table of $G$ if possible and remove those types which have zero structure constant. So, the first and the second components are adjacent. We collect all the adjacency vertices into a list. The structure constant can be computed by the following formula

$$n(C_1, \dots, C_k) = \frac{|C_1| \times \dots \times |C_k|}{|G|} \sum_{x_i \in Irr(G)} \frac{\chi(x_1) \dots \chi(x_k)}{\chi(1)^{k-2}}$$

(1)

With the equation (1), we compute the number of k-tuples of elements $x_i$ in the conjugacy class $C_i$ of a group such that $x_1 \dots x_k = 1$. In our case, k = 3.

4. We use the class names from the Atlas notion of finite groups. Using the YAGs package in GAP (The GAP Group: GAP – Groups 2018), draw the graph and find some properties of it.

The next example shows how to compute the edge set and draw the graph for the group $PGL(2,5)$.

**Example 3.1.** Here, we consider the projective general linear group $PGL(2,5)$.

```
gap> LoadPackage("yags");
gap> g:=PGL(2,5);
Group([ (3,6,5,4), (1,2,5)(3,4,6) ])
gap> edge:=function(g)
local ll,t,i,j,k,cc,ct,ord;
ll:=[];
cc:=List(ConjugacyClasses(g),Representative);
ord:=List(cc,Order);
ct:=CharacterTable(g);
t:=Positions(ord,2);
for i in [1..Length(cc)] do
for j in [1..Length(cc)] do
for k in t do
if i<j and ClassStructureCharTable(ct,[i,j,k])<>0 then
Add(ll,[i,j]);
fi;od;od;od;
return Set(ll);
end;
gap> e:=edge(g);;
gap> h:=GraphByEdges(e);;
gap> Draw(h);
```

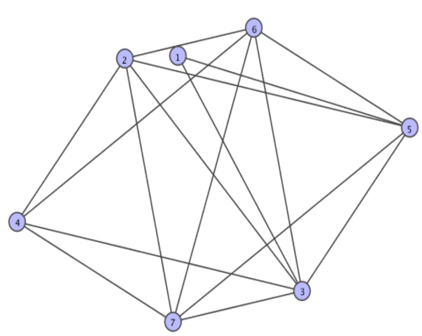


**Figure 1:** The resize result involution graph of $PGL(2,5)$

## 4 The resize result involution graph for $PGL(2,q)$ and $PSL(2,q)$

In this section, we give some results about the resize result involution graphs for $PGL(2,q)$ and $PSL(2,q)$ where $q$ is a prime power. Also, the structure of the resize result involution graphs are presented, and several graph properties are given. We mean by $C_n$ the conjugacy class with elements of order $n$.

**Lemma 4.1.** Let $q=p^n$ be a prime power. For any $C \in V(\Gamma^{RI}(PGL(2,q))$, we have

If $2|q$, then

$$deg(C)=\begin{cases} 1 & if\ C=C_1 \\ q & if\ C=C_2 \\ q-1 & otherwise \end{cases}$$

If $2\nmid q$, then

$$deg(C)=\begin{cases} 2 & if\ C=C_1 \\ q+1 & if\ C=C_2 \\ q-1 & if\ C=C_p \\ q & otherwise \end{cases}$$

Proof.

1. If $2|q$, then the group has $q+1$ conjugacy classes which are the set of vertices of $\Gamma^{RI}\ (PGL(2,q)$. The group contains only one trivial class and one involution class. The trivial class is adjacent only with the involution class. So $deg(C_1)=1$. Also, the involution class is adjacent to all other classes, thus $deg(C_2)=q$. The remaining conjugacy classes are adjacent together except with $C_1$. Hence $deg(C_i)=q-1$ if $i\neq 1,2$.

2. If $2\nmid q$, then the group has $q+2$ conjugacy classes which are the set of vertices of $\Gamma^{RI}\ (PGL(2,q))$. The group contains only one trivial class and two involution classes. These classes are adjacent. So $deg(C_1)=2$. Also, the involution class with minimum index is adjacent to all other classes, thus $deg(C_2)=q+1$. The remaining involution class is adjacent to all except the class $C_p$ which contains elements of order $p$. Hence, $deg(C_2)=q$. It has one conjugacy class $C_p$ which is adjacent to all classes except $C_1$ and $C_2$. Hence, $deg(C_p)=q-1$. The remaining other classes have degree $q$, because they are adjacent to all

vertices except the identity.

**Proposition 4.2.** The resize result involution graph of $PGL(2,q)$ has $q+2$ vertices and $\frac{q(q+1)}{2}+1$ edges if $2 \nmid q$, otherwise it has $q+1$ vertices and $\frac{q^2-q}{2}+1$ edges.

Proof. If $2 \nmid q$, then it is well known that it has $q+2$ vertices. Investigation shows that there are three vertices of degrees $2, q+1, q-1$ respectively and $q-1$ vertices of degree $q$. From Lemma 2.7, we have $2|E| = 2+q+1+q-1+q(q-1)$. Thus, the graph has $\frac{q(q+1)}{2}+1$ edges.

If $2|q$, then it is well known that it has $q+1$ vertices. Also, investigations show that there are two vertices of degrees $1, q$ respectively and $q-1$ vertices of degrees $q-1$. From Lemma 2.7, we have $2|E| = q+1+(q-1)(q-1)$. Thus, the graph has $\frac{q^2-q}{2}+1$ edges.

**Theorem 4.3.** The resize result involution graph of $PGL(2,q)$ is connected with diameter 2, raduis 1 and grith 3.

Proof. By Proposition 4.2, $|E(\Gamma^{RI}(PGL(2,q))|$ has $\frac{q(q+1)}{2}+1$ edges if $2 \nmid q$ which is greater than $\frac{(n-1)(n-2)}{2} = \frac{(q+2-1)(q+2-2)}{2}$. By Corollary 2.6, $\Gamma^{RI}(PGL(2,q)$ is connected. For the other case, we use the same argument.

From Lemma 4.1, we obtain $diam(\Gamma^{RI}(PGL(2,q)) = 2$ and $rad(\Gamma^{RI}(PGL(2,q)) = 1$.

Since $q \equiv 2 \nmid q$, then its is clear that the trivial class with involution classes produce a complete graph $K_3$. So $girth(\Gamma^{RI}(PGL(2,q))) = 3$. In the other case, the involution class with any two non-trivial conjugacy classes produce a complete graph $K_3$. Thus $girth(\Gamma^{RI}(PGL(2,q))) = 3$.

**Proposition 4.4.** $\omega(\Gamma^{RI}(PGL(2,q)) = q$.

Proof. If $2|q$, then $\Gamma^{RI}(PGL(2,q) - \{C_1\}$ is the complete induced subgraph on $q$ vertices. Therefore, the clique number of $\Gamma^{RI}(PGL(2,q)$ is $q$. If $2 \nmid q$, then $\Gamma^{RI}(PGL(2,q) - \{C_1, C_p\}$ is the complete induced subgraph on q vertices. Therefore, the clique number of $\Gamma^{RI}(PGL(2,q)$ is $q$.

**Lemma 4.5.** Let $q$ be an odd prime power with $p \neq 3$. For any $C \in V(\Gamma^{RI}(PSL(2,q))$, we have

If $q \equiv 1 \mod 4$, then either

$$deg(C) = \begin{cases} 1 & if\ C = C_1 \\ \frac{q+3}{2} & if\ C = C_2 \\ \frac{q+1}{2} & otherwise \end{cases}$$

or
$$deg(C) = \begin{cases} 1 & if\ C = C_1 \\ \frac{q+3}{2} & if\ C = C_2 \\ \frac{q-1}{2} & if\ C = C_p \\ \frac{q+1}{2} & otherwise \end{cases}$$

If $q \equiv 3 \mod 4$, then either

$$deg(C) = \begin{cases} 1 & if\ C = C_1 \\ \frac{q-1}{2} & if C = C_2\ or\ C = C_p \\ \frac{q+1}{2} & otherwise \end{cases}$$

or
$$deg(C) = \begin{cases} 1 & if\ C = C_1 \\ \frac{q-1}{2} & if\ C = C_2 \\ \frac{q-3}{2} & if\ C = C_p \\ \frac{q+1}{2} & otherwise \end{cases}$$

Proof. The proof is similar to Lemma 4.1.

**Proposition 4.6.** The resize result involution graph of $PSL(2,q)$ has $\frac{q+5}{2}$ vertices and either $\frac{q^2+4q-5}{8}$ edges or $\frac{q^2+4q-13}{8}$ edges if $q \equiv 3 \mod 4$.

Proof. It is well known that the group $PSL(2,q)$ has $\frac{q+5}{2}$ conjugacy classes, which are the set of vertices. The degrees of vertices are given in Lemma 4.5. By Lemma 2.7, either we have one trivial class, one involution class and there are $\frac{q+1}{2}$ conjugacy classes of degrees $\frac{q+1}{2}$, thus $|E(\Gamma^{RI}(PSL(2,q))| = \frac{q^2+4q-5}{8}$ or we have one trivial class, one involution class, two conjugacy classes $C_p$ of degree $\frac{q-1}{2}$ and there are $\frac{q-3}{2}$ conjugacy classes of degrees $\frac{q+1}{2}$, so $|E(\Gamma^{RI}(PSL(2,q))| = \frac{q^2+4q-13}{8}$.

**Proposition 4.7.** The resize result involution graph of $PSL(2,q)$ has $\frac{q+5}{2}$ vertices and either $\frac{q^2+4q+11}{8}$ edges or $\frac{q^2+4q+3}{8}$ edges if $q \equiv 1 \bmod 4$.

Proof. The proof is similar as Proposition 4.6.

**Theorem 4.8.** The resize result involution graph of $PSL(2,q)$ is connected with diameter $3$, radius 2 and girth $3$ if $q \equiv 3 \bmod 4$.

Proof. The connectedness follows from Proposition 4.6 and Corollary 2.6. The rest is similar to Lemma 4.3.

**Theorem 4.9.** The resize result involution graph of $PSL(2,q)$ is connected with diameter 2, radius 1 and girth 3 if $q \equiv 1 \bmod 4$.

Proof. The connectedness follows from Proposition 4.7 and Corollary 2.6. The rest is similar to Lemma 4.3.

**Proposition 4.10.** If $q \equiv 3 \bmod 4$, then $\omega(\Gamma^{RI}(PSL(2,q))$ is either $\frac{q+1}{2}$ or $\frac{q-1}{2}$.

Proof. The proof is similar as Proposition 4.4.

**Proposition 4.11.** If $q \equiv 1 \bmod 4$, then $\omega(\Gamma^{RI}(PSL(2,q))$ is either $\frac{q+1}{2}+1$ of $\frac{q+1}{2}$.

Proof. The proof is similar as Proposition 4.4.

## 5. The resize result involution graph for some sporadic groups

Here we give some results about the resize result involution graphs for some sporadic groups. Several graph properties are given. We assume that Janko group $J_1$ acts on a set of size 266, the Janko group $J_2$ acts on a set of size 100, the Held group $He$ acts on a set of size 2058 and the exceptional group $2F4(2)'$ acts on a set of size 1600.

**Theorem 5.1.** If $G$ is isomorphic to one of the groups $J_1$ or $J_2$, then $\Gamma^{RI}(G)$ is a connected graph with $diam(\Gamma^{RI}(G)) = 2$, $rad(\Gamma^{RI}(G)) = 1$ and $girth(\Gamma^{RI}(G)) = 3$.

Proof. It is well known that the Janko group $J_1$ has 15 conjugacy classes as follows:

$1A, 3A, 5A, 5B, 15A, 15B, 2A, 6A, 7A, 10A, 10B, 11A, 19A, 19B, 19C$. By Proposition 2.4, we have 92 edges which is greater than 91. So $\Gamma^{RI}(J_1)$ is connected by Corollary 2.6. The representative of the involution class $2A$ is adjacent with the other representatives of the conjugacy classes. So, the involution class has degree 14. So, it has $diam(\Gamma^{RI}(J_1)) = 2$ and $rad(\Gamma^{RI}(J_1)) = 1$. It is obvious that the conjugacy classes $3A, 5A, 2A$ form a cycle in $\Gamma^{RI}(J_1)$.

Hence $girth(\Gamma^{RI}(J_1)) = 3$.

Also, it is well known that the Janko group $J_2$ has $21$ conjugacy classes as follows:

$1A, 2A, 4A, 8A, 2B, 3A, 6A, 3B, 6B, 5A, 5B, 10A, 10B, 5C, 5D, 15A, 15B, 12A, 10C, 10D, 7A$. The proof of the rest is similar.

The adjacency matrix of $\Gamma^{RI}(J_1)$ is the following matrix:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

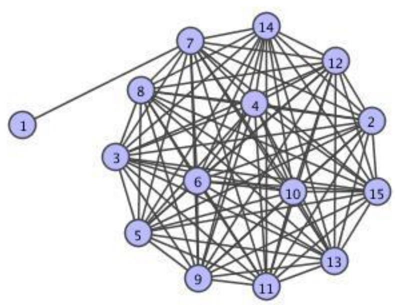


Figure 2: The resize result involution graph of $J_1$

**Lemma 5.2.** The characteristic polynomial of $\Gamma^{RI}(J_1)$is

$$\chi(\Gamma^{RI}(J_1), A_1) = x^{15} - 92x^{13} - 728x^{12} - 2925x^{11} - 7436x^{10} - 12870x^{9} - 15444x^{8} - 12441x^{7} - 5720x^{6} + 2080x^{4} + 1573x^{3} + 612x^{2} + 130x + 12$$

Proof. The proof follows from the direct computation from adjacency matrix $A_1$.

**Corollary 5.3.** The spectrum of $\Gamma^{RI}(J_1)$ is

$$M_1 = \begin{pmatrix} -1 & -1.587 & 0.581 & 13.006 \\ 12 & 1 & 1 & 1 \end{pmatrix}$$

Proof. The proof follows from Lemma 5.2.

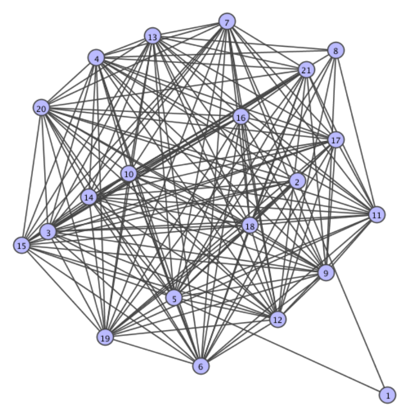


Figure 3: The resize result involution graph of $J_2$

The adjacency matrix of $\Gamma^{RI}(J_2)$ is the following matrix:

$$A_2 = \begin{pmatrix}
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&1&1&0&1&1&0&1&1&1&1&0&1&1&1&1&1&1\\
0&1&1&0&1&1&0&1&1&0&1&1&1&0&1&1&1&1&1&1&1\\
0&1&1&1&0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&0&0&0&0&1&0&1&1&0&1&1&1&1&0&0&0&0\\
1&1&1&1&1&1&0&0&0&1&0&1&0&0&0&0&0&0&0&1&1\\
0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&0&1&1&0&1&1&0&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&1&1&0&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&0&0&1&1&1&1&0&1&1&1&1&1&1&1&1\\
0&1&1&0&1&1&1&0&1&1&1&1&1&0&1&1&1&1&1&1&1\\
0&1&0&1&1&1&1&0&1&1&1&1&1&1&0&1&1&1&1&1&1\\
0&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1\\
0&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&1&1&1\\
0&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&0&1&1&1\\
0&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1&0&1&1\\
0&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&0&1\\
0&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&0
\end{pmatrix}$$

**Lemma 5.4.** The characteristic polynomial of $\Gamma^{RI}(J_2)$ is

$$\chi(\Gamma^{RI}(J_2), A_2) = x^{21} - 168x^{19} - 1636x^{18} - 7161x^{17} - 14730x^{16} + 1497x^{15} + 91040x^{14} + 247694x^{13} + 311052x^{12} + 73098x^{11} - 408036x^{10} - 744961x^{9} - 645354x^{8} - 263763x^{7} + 36000x^{6} + 109067x^{5} + 62456x^{4} + 17816x^{3} + 2424x^{2} + 96x$$

Proof. The proof follows from the direct computation from adjacency matrix $A_2$.

**Corollary 5.5.** The spectrum of $\Gamma^{RI}(J_2)$ is

$$M_2 = \begin{bmatrix} -1 & 10 \\ 0 & 1 \\ -2 & 1 \\ -3.29 & 1 \\ -2.36 & 1 \\ -2.08 & 1 \\ -0.53 & 1 \\ -1.0631 & 1 \\ 0.576 & 1 \\ 1.34 & 1 \\ 2.34 & 1 \\ 17.1 & 1 \end{bmatrix}^T$$

Proof. The proof follows from Lemma 5.4.

**Theorem 5.6.** If $G$ is isomorphic to one of the group $2F4(2)'$or $He$, then $\Gamma^{RI}(G)$ is a connected graph with $diam(\Gamma^{RI}(G)) = girth(\Gamma^{RI}(G)) = 3$ and $rad(\Gamma^{RI}(G)) = 2$.

Proof. It is clear that the group $G = 2F4(2)'$ has 22 conjugacy classes as follows: $1A, 2A, 4A, 8A, 8B, 16A, 16B, 16C, 16D, 2B,$ $3A, 4B, 6A, 12A, 12B, 4C, 8C, 8D, 5A, 10A,$ $13A, 13B$. The trivial class only adjacent with the involution classes. These classes produce a cycle of length 3. So, it has girth 3. By GAP computation, we have at least one vertex of degree $20$, that is this vertex is adjacent with all vertices except the trivial class, but it is connected with the involution classes. Therefore, $\Gamma^{RI}(G)$ is connected. Furthermore, the longest distance between any two vertices is either 2 or 3. Hence $diam(\Gamma^{RI}(G)) = 3$ and $rad(\Gamma^{RI}(G)) = 2$. The proof for the group $He$ is similar.

The adjacency matrix of $\Gamma^{RI}(2\,F4(2)')$ is the following matrix:

$$A_3 = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&1 \\ 0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&0&1&1&1&1&1&0&1&1&1&0&1&1&1&1&1&1&1&1 \\ 0&0&1&1&0&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&0&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&0&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&0&1&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1 \\ 0&0&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0 \end{pmatrix}$$

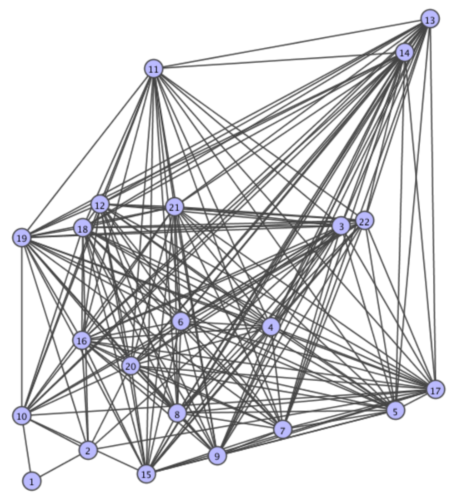


Figure 4: The resize result involution graph of $2F4(2)'$

**Lemma 5.7.** The characteristic polynomial of $\Gamma^{RI}(2F4(2)')$ is

$\chi(\Gamma^{RI}(\ 2\ \ F4(2)'), A_3) = x^{22} - 193x^{20} - 2138x^{19} - 11465x^{18} - 34752x^{17} - 47637x^{16} + 73800x^{15} + 575226x^{14} + 1649360x^{13} + 3101358x^{12} + 4245124x^{11} + 4356638x^{10} + 3335216x^{9} + 1817790x^{8} + 592776x^{7} - 4491x^{6} - 130896x^{5} - 81773x^{4} - 28410x^{3} - 6085x^{2} - 752x - 41$

Proof. The proof follows from the direct computation from adjacency matrix $A_3$.

**Corollary 5.8.** The spectrum of $\Gamma^{RI}(2F4(2)')$ is

$$M_3 = \begin{pmatrix} -1 & -3.055 & -2.510 & -0.234 & -0.4 & 2.733 & 18.618 \\ 16 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Proof. The proof follows from Lemma 5.7.

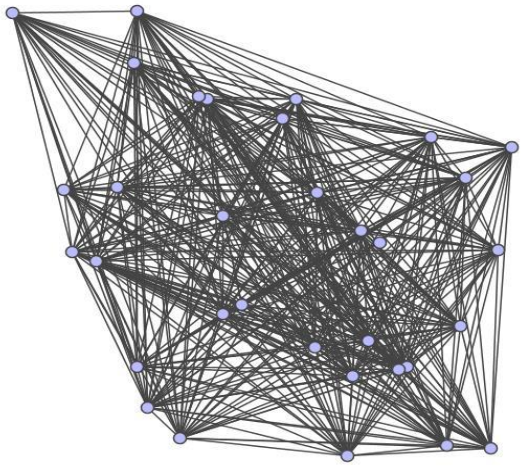

Figure 5: The resize result involution graph of $He$

The adjacency matrix of $\Gamma^{RI}(He)$ is the following matrix

$$A_4 = \begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&1&1&1&0&0&0\\
0&1&0&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1\\
0&1&1&0&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&0&1&1&1&1&0&1&1&1&1&1&0&1&1&0&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&0&1&1&1&1&1&0&1&1&1&1&0&1&1&0&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&0&1&0&0&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&0&1&1&0&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&1&1&1&1&0&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&0&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&0&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&0&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0
\end{bmatrix}$$

**Lemma 5.9.** The characteristic polynomial of $\Gamma^{RI}(He)$ is

$$\chi(\Gamma^{RI}(He), A4) = x^{33} - 451x^{31} - 7762x^{30} - 65600x^{29} - 322290x^{28} - 812286x^{27} + 574616x^{26} + 14320896x^{25} + 65369232x^{24} + 172264491x^{23} + 256011202x^{22} - 3299932x^{21} - 1177714430x^{20} - 3711418052x^{19} - 7224850396x^{18} - 10187131922x^{17} - 10685968068x^{16} - 7974564765x^{15} - 3331566798x^{14} + 819411176x^{13} + 2767370962x^{12} + 2544728738x^{11} + 1357864400x^{10} + 339653488x^{9} - 115260040x^{8} - 170899947x^{7} - 97316834x^{6} - 35567052x^{5} - 8920098x^{4} - 1502144x^{3} - 154444x^{2} - 7391x - 4.$$

Proof. The proof follows from the direct computation from adjacency matrix $A_4$.

**emark 5.10**

The Resize result involution graphs for the other maximal subgroups for $G \in \{J_1, J_2, He, 2F4(2)'\}$ which are not studied here may or may not be connected.

From Lemmas 5.2,5.4, 5.7, 5.9 and Proposition 2.1, one can compute the number of vertices, edges and triangles. Also, by Theorem 2.3, our graphs are not Bipartite. On the other hand, our graphs contain $K_{14}$, $K_{16}, K_{19}$ and $K_{26}$ as induced subgraph of $\Gamma^{RI}(J_1), \Gamma^{RI}(J_2)$, $\Gamma^{RI}(\ 2\ F4(2)')$ and $\Gamma^{RI}(He)$ respectively. So, the clique numbers are 14,16,19 and 26. Furthermore, the non-planarity follows from Theorem 2.2. So, we summarize our results in the following table.

| $\Gamma^{RI}(G)$ | $J_1$ | $J_2$ | $2F4(2)'$ | $He$ |
|---|---|---|---|---|
| # of vertices | 15 | 21 | 22 | 33 |
| # of edges | 92 | 168 | 193 | 451 |
| # of triangles | 364 | 818 | 1069 | 3881 |
| # of clique | 14 | 16 | 19 | 26 |
| Bipartite | No | No | No | No |
| Planar | No | No | No | No |

## Conclusion

In this paper, we presented the resize result involution graphs for some finite groups. Our results showed that these graphs are connected with diameter at most 3 and girth 3. Also, some graph properties are computed such as, clique number, number of edges, number of triangles and so on. This work can be further developed for other groups $PGL(n, q)$ where $q$ is a prime power. As well as finding more properties of graphs.

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