



**RESEARCH ARTICLE - MATHEMATICS**

## **Dynamical Behavior of Predator-Prey Model with 'SIS Diseases in Prey Involving a Harvesting in Infected Prey**

**Khalid Khaleefah Jassim<sup>1</sup>, Haider Ghanem Sufaeh<sup>2</sup>, Mohammed Salim Ramadhan<sup>3\*</sup>**

<sup>1</sup>Department of Surveying Techniques, Institute of Technology, Middle Technical University, Baghdad, Iraq.

<sup>2</sup>Department of Civil Techniques, Institute of Technology, Middle Technical University, Baghdad, Iraq.

<sup>3\*</sup> Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq.

\* Corresponding author E-mail: [m7md\\_salim@uomustansiriyah.edu.iq](mailto:m7md_salim@uomustansiriyah.edu.iq)

| Article Info.                                                                                                                          | Abstract                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
|----------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <i>Article history:</i><br><br>Received<br>16 December 2024<br><br>Accepted<br>24 February 2025<br><br>Publishing<br>30 September 2025 | The mathematical models suggested in this article accounts for two types of predator-prey system. It was suggested that the model includes SIS (susceptible infected susceptible) disease in prey types, which is expansion by proximity between infected and susceptible species, and there is harvest in predator species. The epidemic cannot be transmitted from prey to predator during the predation process. The equations that represent the interaction between infected individuals in the prey population and susceptible individuals in the predator population are ordinary differential equations. Based on what was assumed above, any Possible balance points can be analyzed using methods of mathematical .Depending on the numerical simulations, the dynamics of the system's behavior were studied local and the dynamic behavior was studied globally using Lyabnov functions, and the variation of harvest and disease on dynamics of model's behavior was also discussed. |

This is an open-access article under the CC BY 4.0 license (<http://creativecommons.org/licenses/by/4.0/>)

*The official journal published by the College of Education at Mustansiriya University*

**Keywords:** Eco-epidemiological, mode, SIS epidemic of disease, predator and prey system, Numerical Simulation.

### **1. Introduction**

The relationship between prey and predator it will remain one of the most important and prevalent topics in mathematical ecology and ecology ratio global presence and importance. Epidemiology is concerned with studying the spread of diseases in living organisms [1], [2]. Obviously, when the infected person is still infectious and the other person is susceptible all the time, these types of diseases are called, SI (susceptible infected) diseases [3], [4]. In addition to infection don't leads to immunity so that infective become susceptible again after recovery; the diseases are called SIS diseases, see in [5], [6]. Also, once the infected organisms have an immunity of permanent after recovery from the disease, this type of disease is called SIR (susceptible infected removable) diseases. Moreover, after work of pioneering of, Kermack, Mckendrick, [7], [8] .An epidemiology system based on the classic infected-vulnerable-recovery system has emerged on the horizon and has received much research attention, Anderson and May [9], [10].

Also add to that, Majeed , and, Shawka , [11] they discussed the predator-prey system with SIS and SI in prey and diseases that can be transMay mitted in the same species through an external source or through contact. In addition to [12], [13], many researchers, scientists and engineers interested in this question have studied the dynamic system of prey and predator using different stimuli that can change the biological factor in the past

decade, example [14], [15]. Naji, Khalaf, and, Majeed [16] and similarly [17], It studies the predator-prey system, which includes the SIS in the prey. This epidemic is transmitted from the predator by the predator attacking the prey during the predation process. This epidemic is transmitted in the same type of organisms through external surrounding sources as well as through contact; Similarly, Naji, and Abdulateef in their papers [18] proposed an analysis of predator-prey model that contains SI disease that is infectious in the prey and disease that is transmitted in the same species by contact. A predator-prey model with non-local competition can produce complex dynamics, such as spatiotemporal patterns and consistently spatially heterogeneous periodic solutions [19].

Real-life examples include wolves and sheep, since sheep are somewhat infected with diseases that are contagious, while some diseases are not contagious. And also in forest life in terms of tigers and deer, as well as lions, wild buffalo, and other organisms present in the ecosystem.

In this article, a model consisting of a predator-prey system is proposed that includes SIS in prey and harvest groups in the infectious predator. Additionally, in this model, the predator species will consume Sensitive prey according to the Holling, type tow response of function and consume infected species according to a linear function of response, While an infected species predator only attacks linearly infected prey with a linear infection rate, which in turn describes the transmission of diseases between them. The number of prey consumed and the prey population density can be described as a linearly increasing function of its field with respect to the number of prey consumed by predators. The effect of Holling type-III functional response can see in [20], [21].

## 2. The Formulation of Model

This section express on a predator-prey mode involving disease in species prey Proposal for study. Let  $x(t)$  represents density of susceptible prey population at the time  $t$  and  $y(t)$  represent infected density of prey population at the time  $t$ , while  $z(t)$  denotes size the population of the predator species at the time  $t$ . Now, the assumptions represent the formulating of model

- 1) By assumption, susceptible prey is able to reproduce logistically at a constant growth rate represented by the variable ( $r > 0$ ).
- 2) Disease is transmitted in prey population by contact at infection rate  $\beta_1 > 0$ .
- 3) Once the disease subsides and infected individuals become vulnerable to reinfection by the predator, this process is characterized by a rate called the rate of recovery.  $\beta_2 > 0$ .
- 4) The predation of predator on infected and susceptible prey according to (Lotka-Volterra) responses of functional by Attack at maximum rate ( $\alpha > 0$ ), for susceptible species prey, and Attack at maximum rate ( $\gamma > 0$ ), for infected species prey. However the constants  $0 < e_i < 1$ ,  $i = 1, 2$ . represent conversion food rate from the susceptible prey and infected prey to the predator.
- 5) The decay of predator is exponentially by the death of natural rate  $d > 0$ .
- 6) Infected species prey is harvest by the rate  $e > 0$ .

Now, According to the assumptions The proposal system It can be represented by mathematically with set of differential equations in system (1).

$$\frac{dx}{dt} = rx - \alpha xz + \beta_2 y - \frac{\beta_1 xy}{1 + y}$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{\beta_1 xy}{1+y} - \beta_2 y - \gamma zy - ey \\ \frac{dz}{dt} &= e_1 \alpha xz + e_2 \gamma zy - ez\end{aligned}\tag{1}$$

Hence they are Lipschitzian. Functions and system (1) have a solution of unique for any non-negative conditions of initial. The theorem is shown the bounded of system.

**Table (1):** represents the effect of the parameters ( $e, e_1, e_2$ ) on the stability of the system.

| Parameter | Percentage of change   | Impact on system stability           |
|-----------|------------------------|--------------------------------------|
| $e$       | $0 < e \leq 0.009$     | The system is stable.                |
|           | $0.01 < e \leq 0.08$   | The system is stable.                |
|           | $0.09 < e \leq 0.1$    | The system is asymptotically stable. |
| $e_1$     | $0 < e_1 \leq 0.03$    | The system is asymptotically stable. |
|           | $0.01 < e_1 \leq 0.08$ | The system is stable.                |
|           | $0.09 < e_1 \leq 0.1$  | The system is stable.                |
| $e_2$     | $0 < e_2 \leq 0.02$    | The system is stable.                |
|           | $0.05 < e_2 \leq 0.09$ | The system is asymptotically stable. |
|           | $0.09 < e_2 \leq 0.6$  | The system is asymptotically stable. |

**Lemma (2.1):** Any solutions in system (1) which initiate in  $R_+^3$  are uniformly bounded.

**Proof:** Assume that  $w(t) = x(t) + z(t) + y(t)$  is the solution of system (1) with initial conditions

$$x(0) > 0, y(0) > 0 \text{ and } z(0) > 0$$

By Differentiating  $w$  with respect to  $t$  we get:

$$\frac{dw}{dt} \leq 2rx + \alpha xz(e_1 - 1) + \gamma yz(e_2 - 1) - (rx + dz + ey) \leq 2r - Mw$$

Where  $M = \min\{r, d, e\}$  Now, by (Sturm comparison theorem), we get:-

$$0 < w \leq \frac{2r}{M} + \left(M_{(0)} - \frac{2r}{M}\right) e^{-Mt}.$$

thus for  $t \rightarrow \infty$  we obtain  $0 < w \leq \frac{2r}{M}$  Thus, in system (1) all solutions in  $R_+^3$  are bounded uniformly.

### 3. The equilibrium points with their stabilities

There are four equilibrium points in system (1) and the conditions for the existence of any one of them are remember below

- 1- The zero point of equilibrium  $E_1 = (0,0,0)$  exists constantly .
- 2- The infected species-free point of equilibrium ,  $E_2 = (\frac{d}{e_1\alpha}, 0, \frac{r}{\alpha})$
- 3- The free of predator equilibrium point ,  $E_3 = (\hat{x}, \hat{y}, 0)$  exist provided that there is a positive root to the equations:

$$rx - \frac{\beta_1 xy}{y+1} + \beta_2 y = 0 \quad (2)$$

$$\frac{\beta_1 xy}{1+y} - (e + \beta_2)y = 0 \quad (3)$$

$$\text{from (3) we find } x = \frac{(1+y)(e + \beta_2)}{\beta_1} \quad (4)$$

By  
putting (4) in (2) we get

$$\hat{y} = \frac{r(e+\beta_2)}{\beta_1\beta_2 - (e+\beta_2)(r-\beta_1)} \quad (5)$$

By substituting the value of  $\hat{y}$  in (4) We find  $\hat{x} = \frac{(1+\hat{y})(e+\beta_2)}{\beta_1}$

Now,  $\hat{x}$  and  $\hat{y}$  are positive if the following conditions holds

$$r > \beta_1 \text{ and } \beta_1\beta_2 > (e + \beta_2)(r - \beta_1) \quad (6)$$

- 4- The positive (coexistence) equilibrium point is given by  $E_4 = (x^*, y^*, z^*)$

Where  $x^* = \frac{d-e_2\gamma y}{\alpha e_1}$  and  $z^* = \frac{\beta_1(d-e_2\gamma y) - \alpha e_1(1+y)(e+\beta_2)}{\alpha \gamma e_1(1+y)}$  while  $y^*$  is a positive root of the following equation

$$A y^2 + B y + C = 0 \quad (7)$$

Here

$$A = e_1 e_2 \gamma^2 (\beta_1 - r) - \beta_1 e_2^2 \gamma^2 - \alpha e e_1 e_2 \gamma - \alpha e_1 e_2 \gamma \beta_2 .$$

$$B = r e_1 \gamma d - r e_1 e_2 \gamma^2 - \beta_1 e_1 \alpha d + 2 \beta_1 d e_2 \gamma - \alpha \gamma e e_1 e_2 - \alpha \gamma e_1 e_2 \gamma \beta_2 + \alpha e_1 e_2 d + \alpha e_1 \beta_2 d$$

$$C = (r e_1 \gamma - \beta_1 d) d + \alpha e e_1 d .$$

Clearly equation (7) has solution of a unique positive represented by  $y^*$  if the condition  $(A < 0)$  hold with  $(C > 0)$  , now  $(A < 0)$  is holds by condition (6) Hence  $E_4$  exists uniquely in  $R_+^3$  if  $C > 0$  which is hold by the condition:

$$\beta_1 < \frac{r e_1 \gamma}{d} \quad (8)$$

Putting the value of  $y^*$  in the equations of  $x^*$  and  $z^*$  so  $x^*$  and  $z^*$  are positive if the conditions hold

$$d > e_2\gamma y^* \quad (9)$$

$$\beta_1(d - e_2\gamma y^*) > \alpha e_1(\beta_2 + e)(1 + y^*) \quad (10)$$

#### 4. Analysis of Locally Stability

For the purpose of studying the local stability of each of the above-mentioned points of equilibrium, the Jacobian matrix of system (1) was obtained at the general point  $(x, y, z)$ , written as follow

$$J(x, y, z) = \begin{bmatrix} r - \frac{\beta_1 y}{(1+y)} - \alpha z & \frac{\beta_1 xy}{(1+y)(1+y)} - \frac{\beta_1 x}{(y+1)} + \beta_2 & -\alpha x \\ \frac{\beta_1 y}{(1+y)} & \frac{\beta_1 x}{(y+1)} - \frac{\beta_1 xy}{(1+y)(1+y)} - (\gamma z + e + \beta_2) & -\gamma y \\ e_1 \alpha z & e_2 \gamma z & e_1 \alpha x + e_2 \gamma y - d \end{bmatrix} \quad (11)$$

Thus, system (1) has the following matrix near  $E_1 = (0,0,0)$

$$J(E_1) = \begin{bmatrix} r & \beta_2 & 0 \\ 0 & -(\beta_2 + e) & 0 \\ 0 & 0 & -d \end{bmatrix} \quad \text{Then, the equation of characteristic of } J(E_1) \text{ is given as}$$

$$(\lambda_1 - r)(\lambda_2 + (e + \beta_2))(\lambda_3 + d) = 0$$

The Eigen value are  $(\lambda_1 = r, \lambda_2 = -(e + \beta_2) \text{ and } \lambda_3 = -d)$ ,

Since  $\lambda_1 > 0$  and  $\lambda_2, \lambda_3 < 0$

Therefore  $E_1 = (0,0,0)$  is unstable point.

The matrix of Jacobian of system (1) around  $E_2 = (\bar{x}, \bar{0}, \bar{z})$  is written as

$$J(E_2) = (m_{ij})_{3 \times 3} \quad (12)$$

Where

$$m_{11} = 0, m_{12} = \frac{1}{e_1 \alpha} (\beta_2 e_1 \alpha - \beta_1 d), m_{13} = \frac{-d}{e_1}, m_{21} = 0,$$

$$m_{22} = \frac{1}{e_1 \alpha^2} [\alpha \beta_1 d - (e_1 \alpha \gamma r + (e + \beta_2) e_1 \alpha^2)], m_{23} = 0, m_{31} = e_1 r, m_{32} = \frac{e_1 \gamma r}{\alpha}, m_{33} = 0$$

The equation of characteristic of  $J(E_2)$  is given as

$$A_1 \lambda^2 + \lambda^3 + A_2 \lambda + A_3 = 0 \quad (13)$$

Where

$$A_1 = -m_{22} = \frac{1}{e_1 \alpha^2} [-\alpha \beta_1 d + (e_1 \alpha \gamma r + (e + \beta_2) e_1 \alpha^2)],$$

$$A_2 = m_{13} m_{31} = -dr, A_3 = -m_{13} m_{31} (m_{22} + m_{12}) = -rd \left( e + \frac{\gamma r}{\alpha} \right),$$

Now, by Roth-Hurwitz criterion  $E_2$  is asymptotically of locally stable available that

$(A_1 > 0, A_3 > 0 \text{ and } A_1 A_2 - A_3 > 0)$ , so since  $(A_3 < 0)$ , then  $E_2$  is unstable.

The matrix of Jacobian of system (1) around  $E_3 = (\hat{x}, \hat{y}, \hat{0})$  is written as

$$J(E_3) = (k_{ij})_{3 \times 3} \quad (14)$$

Where

$$k_{11} = \frac{-\beta_1 \hat{y}}{1+\hat{y}}, k_{12} = \frac{\beta_1 \hat{x} \hat{y}}{(1+\hat{y})^2} - e, k_{13} = -\alpha \hat{x}, k_{21} = \frac{\beta_1 \hat{y}}{1+\hat{y}}, k_{22} = -\frac{\beta_2 \hat{x} \hat{y}}{(1+\hat{y})^2}, k_{23} = -\gamma \hat{y}, k_{33} = k_{23} = 0, k_{33} = e_1 \alpha \hat{x} + e_2 \gamma \hat{y} - d$$

The equation of characteristic of  $J(E_3)$  is given by

$$\lambda^3 + L_1 \lambda^2 + L_2 \lambda + L_3 = 0 \quad (15)$$

Where

$$L_1 = -(k_{11} + k_{22} + k_{33}), L_2 = k_{11} k_{22} + k_{11} k_{33} + k_{22} k_{33} - k_{12} k_{21}, L_3 = k_{12} k_{21} k_{33} - k_{11} k_{22} k_{33}$$

By to Roth-Hurwiz criterion  $E_3$  is locally stable if and only if

$$L_1 > 0, L_3 > 0 \text{ and } L_1 L_2 - L_3 > 0,$$

$$\text{Now, } L_1 = -(k_{11} + k_{22} + k_{33}) = \frac{\beta_2 \hat{y}}{\hat{x}} + \frac{\beta_1 \hat{x} \hat{y}}{(1+\hat{y})^2} + d - (e_1 \alpha \hat{x} + e_2 \gamma \hat{y}),$$

$$L_1 > 0 \text{ if } d > e_1 \alpha \hat{x} + e_2 \gamma \hat{y} \quad (16)$$

$$L_3 = k_{12} k_{21} k_{33} - k_{11} k_{22} k_{33} = k_{33} (k_{12} k_{21} - k_{11} k_{22}),$$

$$L_3 = [-d + (e_1 \alpha \hat{x} + e_2 \gamma \hat{y})][(\frac{\beta_1 \hat{x} \hat{y}}{(1+\hat{y})^2} - e) \frac{\beta_1 \hat{y}}{(1+\hat{y})} + (\frac{\beta_2 \hat{y}}{\hat{x}})(-\frac{\beta_1 \hat{x} \hat{y}}{(1+\hat{y})^2})],$$

Now,  $L_3 > 0$ , if condition (16) holds with the condition

$$\frac{\beta_1 \hat{x} \hat{y}}{(\hat{y} + 1)^2} < e \quad (17)$$

$$\Delta = L_1 L_2 - L_3 \text{ since } L_1 > 0 \text{ and } L_3 > 0,$$

$$\text{Now, } \Delta = L_1 L_2 - L_3 = (L_1 + k_{33})(k_{11} k_{22} - k_{12} k_{21}) + L_1 (k_{22} k_{33} + k_{11} k_{33}),$$

The first and second term is positive if condition (17) holds with the condition

$$L_1 > -k_{33} \quad (18)$$

Thus,  $E_3 = (\hat{x}, \hat{y}, 0)$  is locally asymptotically stable.

The Jacobean matrix of system (1) around the point  $E_4 = (x^*, y^*, z^*)$

Is written as

$$J(E_4) = (a_{ij})_{3 \times 3} \quad (19)$$

Here

$$a_{11} = \frac{-\beta_2 y^*}{x^*}, a_{12} = \frac{\beta_1 x^* y^*}{(1+y^*)^2} - (\gamma z^* + e), a_{13} = -\alpha x^*,$$

$$a_{21} = \frac{\beta_1 y^*}{(1+y^*)}, a_{22} = \frac{-\beta_1 x^* y^*}{(1+y^*)^2}, a_{23} = -\gamma y^*, a_{31} = e_1 \alpha z^*, a_{32} = e_2 \gamma z^*, a_{33} = 0$$

The equation of characteristic of  $J(E_4)$  is given by

$$\lambda^3 + M_1 \lambda^2 + M_2 \lambda + M_3 = 0 \quad (20)$$

Where

$$M_1 = -(a_{11} + a_{22} + a_{33}) > 0, M_2 = a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31},$$

$$M_3 = a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32},$$

Now, Roth-Hurwitz criterion  $E_4$  is locally asymptotically provided that

$$M_1 > 0, M_2 > 0 \text{ and } \Delta = M_1 M_2 - M_3 > 0, \text{ We have } M_1 > 0,$$

$$M_3 = a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32},$$

The first, last term and third terms of  $M_3$  are positive without condition and second is positive if the conditions hold

$$\frac{\beta_1 x^* y^*}{(1+y^*)^2} > \gamma z^* + e \quad (21)$$

$$\text{Now, } \Delta = M_1 M_2 - M_3$$

$$\Delta = a_{13}(a_{31}a_{11} + a_{21}a_{32}) + a_{23}(a_{22}a_{32} + a_{12}a_{31}) + (a_{11} + a_{22})(a_{12}a_{21} - a_{11}a_{22})$$

The first term of  $\Delta$  is positive if the condition hold

$$\frac{\alpha e_1 \beta_2}{x^*} > \frac{\gamma e_2 \beta_1}{(1+y^*)} \quad (22)$$

The second and third terms are positive if the condition holds

$$\frac{\beta_1 x^* y^* e_2 \gamma}{\alpha e_1 (1+y^*)^2} < a_{12} < \frac{\beta_2 y^*}{(1+y^*)} \quad (23)$$

Thus,  $E_4 = (x^*, y^*, z^*)$  is a locally asymptotically stable .

## 5. Global Stability

These following theorems describe the global stability of all existing equilibrium points was studied above

**Theorem (5.1):** assume that the equilibrium point  $E_3 = (\hat{x}, \hat{y}, 0)$  is locally stable of asymptotically, then  $E_3$  is asymptotically of globally stable in  $R_+^3$  when this condition holds

$$\hat{\theta}_1 < \hat{\theta}_2 \quad (24)$$

Where

$$\hat{\theta}_1 = \frac{\beta_1 x \hat{y}}{(1+\hat{y})} + \frac{\beta_2 \hat{x} y}{x} + \frac{\beta_2 x \hat{y}}{\hat{x}} + \frac{\beta_1 \hat{x} y}{(1+y)} + (\alpha \hat{x} + \gamma \hat{y}) * z$$

And

$$\hat{\theta}_2 = - \left[ \frac{\beta_1 \hat{x} y}{(1+\hat{y})} + \beta_2 (y + \hat{y}) + \frac{\beta_1 x \hat{y}}{(1+y)} + dz \right]$$

**Proof:** consider the function

$$F_3(x, y, z) = \left( x - \hat{x} \ln \frac{x}{\hat{x}} - \hat{x} \right) + \left( y - \hat{y} \ln \frac{y}{\hat{y}} - \hat{y} \right) + z$$

Straightforward computation shows the derivative of  $F_3$  with respect to time t given by

$$\frac{dF_3}{dt} = \frac{\beta_1 x \hat{y}}{(\hat{y}+1)} - \frac{\beta_1 \hat{x} y}{(\hat{y}+1)} + \frac{\beta_2 \hat{x} y}{x} + \frac{\beta_2 x \hat{y}}{\hat{x}} - (\beta_2 y + \beta_2 \hat{y}) + \frac{\beta_1}{(1+y)} (\hat{x} y - x \hat{y}) + (e_1 - 1) \alpha z x + (e_2 - 1) \gamma y z - dz + (\alpha \hat{x} + \gamma \hat{y}) z.$$

Now, by Situation (24 ) and the facts of biological  $(e_1 < 1)$  with  $(e_2 < 1)$  , We get that

$$\begin{aligned} \frac{dF_3}{dt} &< \frac{\beta_1 x \hat{y}}{(1+\hat{y})} + \frac{\beta_2 x \hat{y}}{\hat{x}} + \frac{\beta_1 \hat{x} y}{(1+y)} + (\alpha \hat{x} + \gamma \hat{y}) z - \left[ \frac{\beta_1 \hat{x} y}{(1+\hat{y})} + \beta_2 (y + \hat{y}) + \frac{\beta_1 x \hat{y}}{(1+y)} + dz \right] \\ &= \hat{\theta}_1 + \hat{\theta}_2 \end{aligned}$$

Thus,  $\frac{dF_3}{dt} < 0$  . under the condition (24), clearly  $\frac{dF_3}{dt}$  is negative definite due to localy stability condition. Hence  $F_3$  is a lyapunov function with respect to the point  $E_3$  and  $E_3$  is asymptotically of globally stable.

**Theorem (5.2):** assume that the equilibrium point  $E_3 = (x^*, y^*, z^*)$  is locally stable of asymptotically, then  $E_3$  is asymptotically of globally stable in  $R_+^3$  when this condition holds

$$\theta_1^* < \theta_2^* \quad (25)$$

Where

$$\theta_1^* = \frac{\beta_1 (x y^* + x^* y)}{(1+y)} + \beta_2 \left[ \frac{x}{x^*} y^* + \frac{x^*}{x} y \right] + \alpha (x z^* + x^* z) + \gamma (y z^* + y^* z)$$



$$\theta_2^* = - \left[ \left[ \frac{x^* y \beta_1}{1 + y^*} + \frac{\beta_1 x y^*}{1 + y^*} \right] + (y\beta_2 + y^*\beta_2) + (\alpha e_1 x z^* + \alpha e_1 x^* z) + (\gamma e_2 y z^* + \gamma e_2 y^* z) \right]$$

**Proof:** consider the function

$$F_4(x, y, z) = \left( x - x^* \ln \frac{x}{x^*} - x^* \right) + \left( y - y^* - y^* \ln \frac{y}{y^*} \right) + \left( -z^* + z - z^* \ln \frac{z}{z^*} \right)$$

A straight forward computation shows the derivative of  $F_4$  with respect to  $t$  given by

$$\begin{aligned} \frac{dF_4}{dt} = & \frac{\beta_1 y x^*}{(1+y)} - \frac{\beta_1 x y^*}{(y+1)} + \frac{\beta_1 x y^*}{(1+y^*)} - \frac{\beta_1 y x^*}{(y^*+1)} + \beta_2 \frac{x^* y}{x} - \beta_2 y + \beta_2 \frac{x y^*}{x^*} - \beta_2 y^* + \alpha(x z^* + x^* z) + \\ & \alpha(e_1 - 1)(x z + x^* z^*) + \gamma(1 - e_2)(z^* y + z y^*) + \gamma(e_2 - 1)(y z + y^* z^*). \end{aligned}$$

Now, by condition (25) and the biological facts mentioned ( $e_1 < 1$ ) and ( $e_2 < 1$ )

We obtain that

$$\begin{aligned} \frac{dF_4}{dt} < & \frac{\beta_1 y x^*}{(1+y)} - \frac{\beta_1 x y^*}{(y+1)} + \frac{\beta_1 x y^*}{(y^*+1)} - \frac{\beta_1 y x^*}{(1+y^*)} + \beta_2 \frac{x^* y}{x} - \beta_2 y + \beta_2 \frac{x y^*}{x^*} - \beta_2 y^* + \alpha(x z^* + x^* z) \\ & - \alpha e_1(x z + x^* z^*) + \gamma(y z^* + y^* z) - \gamma e_2(y z^* + y^* z) = \theta_1^* + \theta_2^*. \end{aligned}$$

Thus,  $\frac{dF_4}{dt} < 0$ , by condition (25) clearly  $\frac{dF_4}{dt}$  is negative definite due to locally stability condition hence  $F_4$  is a lyapunov function with respect to  $E_4$  and the point  $E_4$  is a globally asymptotically stable.

## 6. The Numerical Simulation

This section describes the dynamics of globally of system (1) by using different groups of parameter and the sets of different of initial points which is studied by numerical simulation, this study leads to objectives which is searching the value of uneven of all Factors on the conduct of dynamics of model (1) and analytical results of assert our acquired. It is observed that, for the groups of virtual factors, the set of parameters hypothetical observed that

$$\left. \begin{aligned} r = 0.5, \alpha = 0.8, \beta_1 = 0.11, \beta_2 = 0.4, e = 0.009 \\ e_1 = 0.03, e_2 = 0.02, d = 0.002, \gamma = 0.5 \end{aligned} \right\} \quad (26)$$

As we see in the following figure:

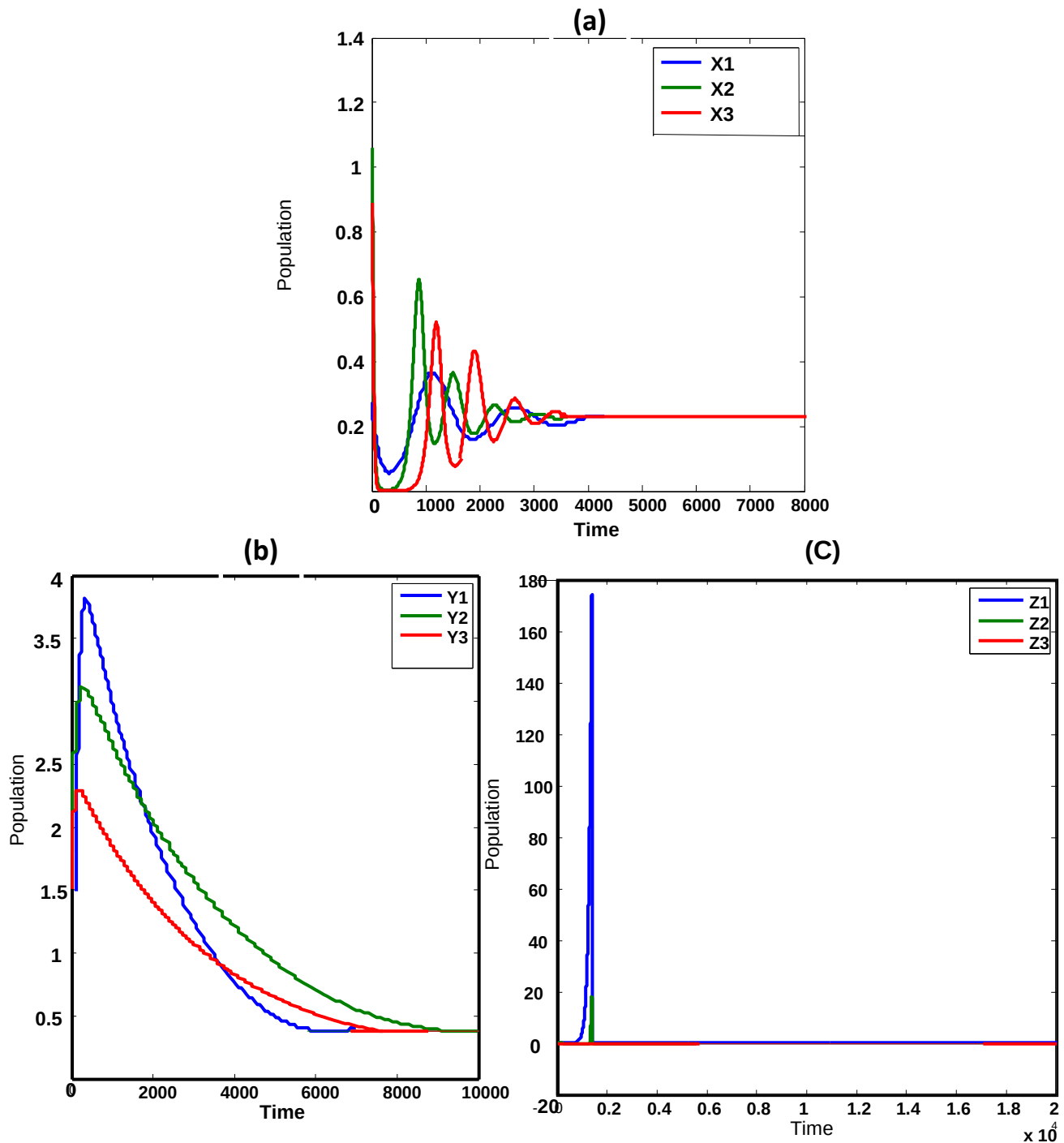


Figure (6.1) Series of time of the solution of system (1) that started from three different initial points  $(0.09, 0.2, 0.27)$ ,  $(0.1, 1, 0.3)$ , and  $(0.9, 0.2, 1)$  For data contained in (26). (a) the function of the time which is Sensitive prey tracks, (b) the function of the time which is infected prey tracks, (c) the function of the time which is tracks of predator.

Fig. (6.1) express system (1) has an asymptotically of globally stable and the solution oncoming asymptotically point  $E_4 = (0.234, 0.095, 0.021)$  asymptotically from starting three initial of different points.

Varying of growth rate  $r$  in the interval  $0.1 \leq r < 2.5$  and the rest of parameters are keeping as data in (26) observed that the system has solution(1) which is approaches as asymptotically to  $E_4$ , as show in Fig.(6.2), for typical value  $r = 0.7$ .

As we see in the following figure:

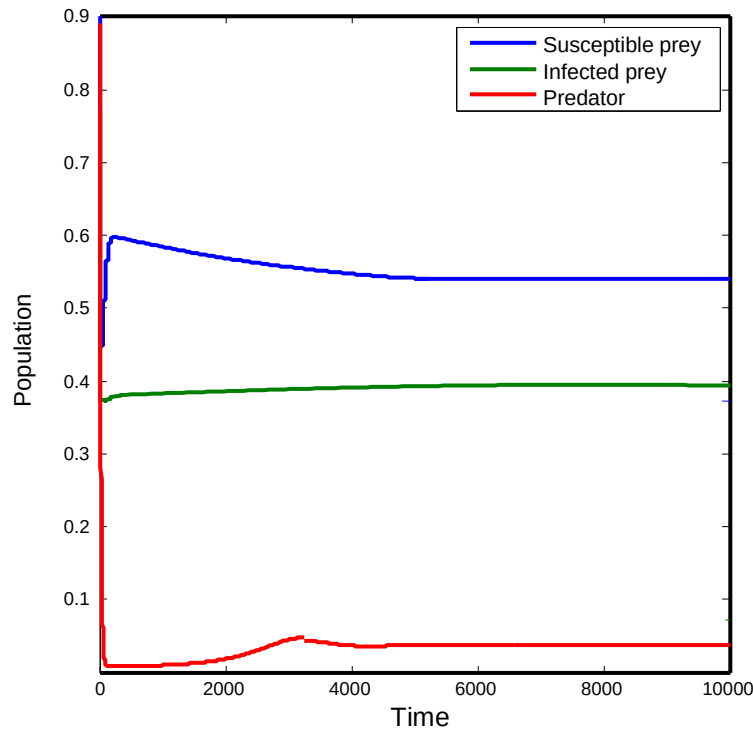


Figure (6.2) Series of time of the solution of system (1) For data contained in (26) with  $r = 0.7$ , which oncoming to  $E_4 = (0.265, 0.054, 0.082)$  in  $R_+^3$ .

Now, varying of the rate of infection of prey  $\beta_1$  and the break of Factors are keeping as data take in (26). it is notice for  $0.2 \leq \beta_1 < 0.951$  the solution (1) approaches to  $E_4$ , if we increasing the parameter for  $0.951 \leq \beta_1 < 2$ , it is observed that the system (1) asymptotically as approaches to  $E_1 = (0, 0, 0)$ , as shown in Fig(6.3), for typical value  $\beta_1 = 0.9$ .

As we see in the following figure:

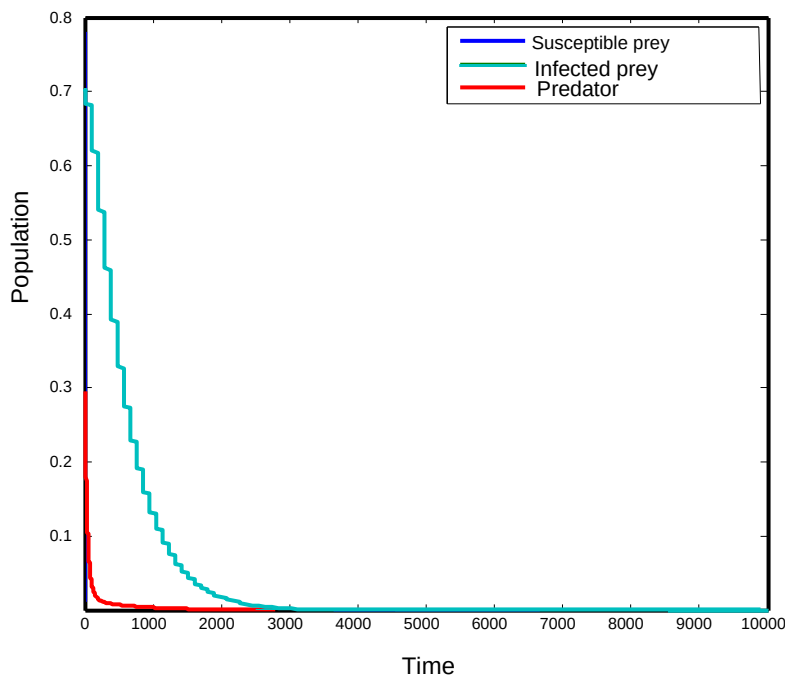


Figure.(6.3) : Series of time of the solution of system(1) For data contained in (26) for  $\beta_1 = 0.9$ , which oncoming to  $E_0 = (0, 0, 0)$  in the interior of  $R_+^3$ .

The varying the factor  $\alpha$  which represent the rate of predation on vulnerable prey, and the break of Factors are keeping as data take in (2.6), it is notice for  $0.3 \leq \alpha < 1.5$  the system (1) have a solution approaches to  $E_4$ .

Now, the varying the factor  $\gamma$  which represent the rate of predation on infected of prey, and the break of Factors are keeping as data take in (2.6), it is notice for  $0.5 \leq \gamma < 2$  the system(1) have a solution approaches to  $E_4$ .

Moreover Parameter tolerance  $d$  which represent the rate of death of predator, and the break of Factors are keeping as data take in (2.6) it is notice for  $0.09 \leq d < 1$ , the system(1) have a solution approaches to a predator free Balance point  $E_3 = (\hat{x}, \hat{y}, 0)$  as appears in Fig(3.4), For my model value  $d = 0.7$ .

**As we see in the following figure:**

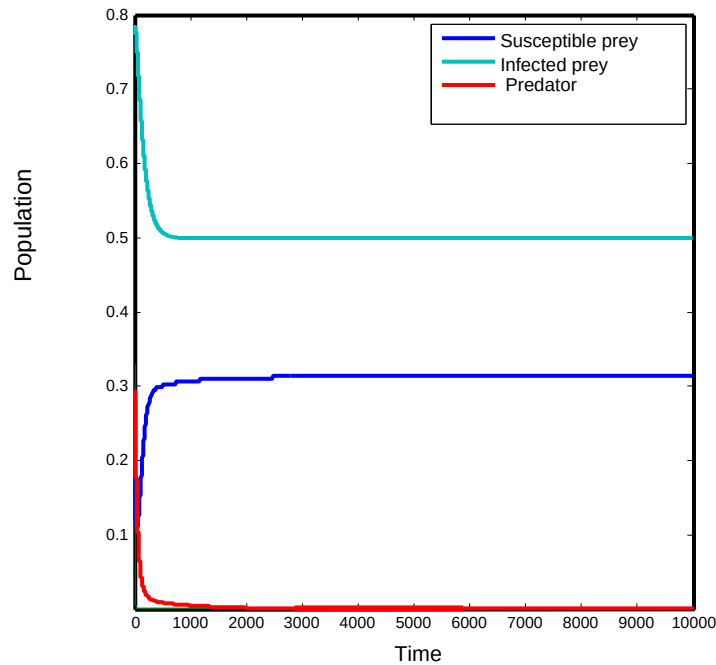
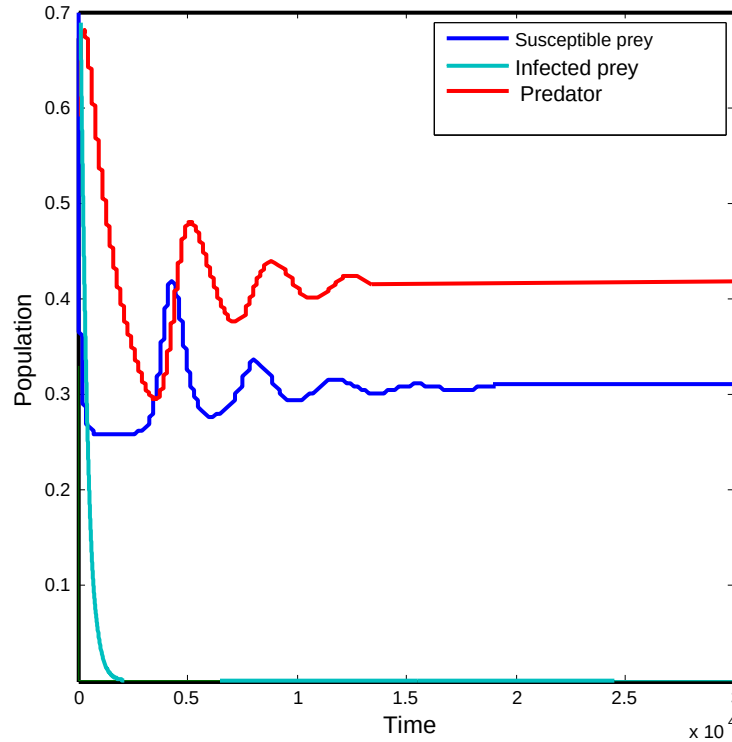


Figure (6.4): Series of time of the solution of system (1) For data contained in (2.6) for  $d = 0.7$ , which approaches to  $E_3 = (\hat{x}, \hat{y}, 0)$  in  $\mathbb{R}_+^3$ .

The change parameter  $e$ , represent the harvesting Of the injured individual, and the break of Factors are keeping as data take in (2.6) it is notice for,  $0.01 \leq e < 0.8$  the system (1) have a solution approaches to  $E_2$  as it shown in Fig.(6.5), for typical value  $e = 0.67$ .

As we see in the following figure:



figure(6.5)Series of time of the solution of system (1) for the data given in eq.(26) , for  $e = 0.67$ , which approaches to  $E_2=(x,0,z)$  in  $R_+^3$  .

The change of the parameter  $e_1$  , the rate of Food conversion from susceptible of prey to Individual of predator , and the break of Factors are keeping as data take in (2 6) it is notice for  $0.01 \leq e_1 < 0.4$  the system (1) have a solution approaches  $E_4$  .

Variance of Parameter  $e_2$  , the rate of conversion of food from infected prey individual of predator, and the break of Factors are keeping as data take in (2 6) it is notice for  $0.05 \leq e_2 < 0.6$  the system (1) have a solution approaches  $E_4$  .

Finally, varying of the rate of recovery of prey  $\beta_2$  , and the break of Factors are keeping as data take in (2 6) it is notice for  $0.02 \leq \beta_2 < 0.11$  , the system (1) have a solution approaches  $E_4$  , by increasing the parameter in the interval  $0.11 \leq \beta_2 < 2$  , the system (1) have a solution approaches  $E_2$  .

## 7. Conclusion with Discussion

The Predator-prey model, have a harvest and SIS disease in prey, proposed for studied analytical as well as numerically .By assumption the transmitted of the disease is by contact in the individual. Conditions with sufficient of the stability in system (1) by equilibrium points are obtained. The effect of all parameters are studied numerically on the dynamics behavior of the system (1) also drawn the trajectories in the typically figures which represent the solutions and according to the solutions of system (1) for the data given by (26) the conclusions obtain.

- 1- The system (1) does not have periodic dynamics for the set of data as given in eq. (26).
- 2- The varying for the values of the set parameters in eq. (2 6), system (1) have an approaches to asymptotically of globally stable point  $E_4 = (0.234, 0.095, 0.021)$  .

- 3- The effect of the parameters  $r, \alpha, \gamma, e_1, e_2$  which represent, the growth rate of a prey ,the predation rate on a susceptible individual in prey ,the rate of predation on individual of infected in prey, the rate of conversion of the food from sensitive prey to individual of predator and the rate of conversion of the food from infected prey to individual of predator, at all time and the rest of parameter keeping as data in eq.(2 6),there is not any effect on the dynamical of the behavior of system(1) and system (1)have a solution approaches to  $E_4=(x^*,y^*,z^*)$ .
- 4- The effect of the rate of individuals infection in prey  $\beta_1$  , in the interval for  $0.2 \leq \beta_1 < 0.951$  and the rest of the parameters keeping as data in eq.(2 6), system (1) have a solution approaches to  $E_4$ . However for increase the value in the interval  $0.951 \leq \beta_1 < 2$  ,the system(1) have a solution approaches to  $E_0 = (0,0,0)$  thus , the  $\beta_1 = 0.951$  is bifurcation point.
- 5- By effect the rate of mortality of predator ( $d$ ) , and the break of Factors are keeping as data take in (2 6) it is notice for  $0.09 \leq d < 1$  ,system (1) have a solution approaches to  $E_3 = (\hat{x}, \hat{y}, 0)$  .
- 6- The change of the parameter ( $e$ ) , the rate of the harvest of infected individuals in prey , and the break of Factors are keeping as data take in (2 6) it is notice for  $0.1 \leq e < 0.8$  ,system (1) have a solution approaches to  $E_3$  .
- 7- Finally, effect the rate of recovery of prey  $\beta_2$  , and the break of Factors are keeping as data take in (2 6) it is notice for ( $0.02 \leq \beta_2 < 0.11$ ) , system (1) have a solution approaches to  $E_4$  , as increase the value of the parameter ( $0.11 \leq \beta_2 < 2$ ) the solution of system(1) will approach to  $E_2=(x,0,z)$  thus , the  $\beta_2 = 0.11$  is bifurcation point.

## References

- [1] P. Panday, S. Samanta, N. Pal, and J. Chattopadhyay, "Delay induced multiple stability switch and chaos in a predator–prey model with fear effect," *Math Comput Simul*, vol. 172, pp. 134–158, 2020.
- [2] R. Yang, F. Wang, and D. Jin, "Spatially inhomogeneous bifurcating periodic solutions induced by nonlocal competition in a predator–prey system with additional food," *Math Methods Appl Sci*, vol. 45, no. 16, pp. 9967–9978, 2022.
- [3] C. Xiang, J. Huang, and H. Wang, "Bifurcations in Holling-Tanner model with generalist predator and prey refuge," *J Differ Equ*, vol. 343, pp. 495–529, 2023.
- [4] M.-G. Cojocaru, T. Migot, and A. Jaber, "Controlling infection in predator-prey systems with transmission dynamics," *Infect Dis Model*, vol. 5, pp. 1–11, 2020.
- [5] A. Al Themairi and M. A. Alqudah, "Predator-prey model of Holling-type II with harvesting and predator in disease," *Ital. J. Pure Appl. Math*, vol. 43, pp. 744–753, 2020.
- [6] M. Moustafa, M. H. Mohd, A. I. Ismail, and F. A. Abdullah, "Dynamical analysis of a fractional-order eco-epidemiological model with disease in prey population," *Adv Differ Equ*, vol. 2020, pp. 1–24, 2020.
- [7] L. J. de Vries and F. Van Langevelde, "Two different strategies of host manipulation allow parasites to persist in intermediate–definitive host systems," *J Evol Biol*, vol. 31, no. 3, pp. 393–404, 2018.

- [8] W. O. Kermack and A. G. McKendrick, "A contribution to the mathematical theory of epidemics," *Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character*, vol. 115, no. 772, pp. 700–721, 1927.
- [9] R. M. Anderson and R. M. May, "The population dynamics of microparasites and their invertebrate hosts," *Philosophical Transactions of the Royal Society of London. B, Biological Sciences*, vol. 291, no. 1054, pp. 451–524, 1981.
- [10] R. M. May, *Infectious diseases of humans: dynamics and control*. Oxford University Press, 1991.
- [11] A. A. Majeed and I. I. Shawka, "The stability analysis of eco–epidemiological system with disease, Gen," *Math. Notes*, pp. 52–72, 2016.
- [12] A. S. Abdulghafour and R. K. Naji, "The impact of refuge and harvesting on the dynamics of prey-predator system," *Sci Int (Lahore)*, vol. 30, no. 2, pp. 315–323, 2018.
- [13] H. Liu, H. Yu, C. Dai, Z. Ma, Q. Wang, and M. Zhao, "Dynamical analysis of an aquatic amensalism model with non-selective harvesting and Allee effect," *Mathematical Biosciences and Engineering*, vol. 18, no. 6, pp. 8857–8882, 2021.
- [14] X. Liu and Q. Huang, "Analysis of optimal harvesting of a predator-prey model with Holling type IV functional response," *Ecological complexity*, vol. 42, p. 100816, 2020.
- [15] S. Thota, "Prey-predator model for Awash National Park, Oromia, Ethiopia and its stability analysis with simulations," *Journal of Science and Sustainable Development*, vol. 7, no. 2, pp. 15–21, 2019.
- [16] K. Q. Khalaf, A. A. Majeed, and R. K. Naji, "The dynamics of an SIS epidemic disease with contact and external source," *Dynamics*, vol. 5, no. 4, 2015.
- [17] R. K. Naji and A. N. Mustafa, "The Dynamics of an Eco-Epidemiological Model with Nonlinear Incidence Rate," *J Appl Math*, vol. 2012, no. 1, p. 852631, 2012.
- [18] R. Naji and B. Abdulateef, "The dynamics of model with nonlinear incidence rate and saturated treatment function," *Science International*, vol. 29, no. 6, pp. 1223–1236, 2017.
- [19] H. Ghanem and A. A. Majeed, "A qualitative study of an eco-epidemiological model with (SI) epidemic disease in prey and (SIS) epidemic disease in predator involving a harvesting," *Sci. Int.(Lahore)*, vol. 30, no. 4, pp. 549–565, 2018.
- [20] M. A. Jawad, W. A. Ibrahim, and S. J. M. Al-Qaisi, "Mixed Crank-Nicolson and Galerkin Methods for Solving Nonlinear Hyperbolic Partial Differential Equation," *Mustansiriyah journal of pure and applied sciences*, vol. 2, no. 4, 2024.
- [21] A. S. Sleibi and S. Q. Hasan, "Stability of Composition Caputa–Katugampola Fractional Differential Nonlinear Control System with Delay Riemann– Katugampola," *Mustansiriyah journal of pure and applied sciences*, vol. 2, no. 4, 2024.