Comparing Different Estimators for Fuzzy Weibull Hazard Rate Function

م.م. أسيل نوري صالح
دائرة الدراسات والتخطيط والمتابعة - وزارة التعليم العالي

Abstract
In this paper, we compare different fuzzy estimators for hazard rate function of two parameters Weibull distribution. The two parameters (λ, θ) are estimated by maximum likelihood, moments, and percentile methods, then we apply simulation procedure through different values of (n), and fuzzy variable $(\tilde{k})$ of $(\alpha - cut\ intervals)$ and different sets of $(n = 20,40,60,80)$. The comparison has been done using statistical measure (MSE), all results shown in tables.
Keywords: Maximum Likelihood Estimator (MLE), Moments Estimator (MOM), Percentile Estimator (PEC), hazard rate.

1. Introduction

In many applications, the life times are assumed to be random variables have life time density function with crisp parameters, but in many cases, the parameters are difficult to determine, according to uncertainties and imprecision of data, this required to use fuzzy quantity, Cai et al, has been given different insight when he introduce assumptions of fuzzy state, and he use fuzzy methodology for representing failure distribution of system. The hazard rate functions have been used for several statistical analyses like economics, engineering, and time to failure of statistical distribution. Hazard rate function have been estimated using parametric and nonparametric methods, we referred to Watson and Leadbefter (1964), also Cai, K. Y. and Wen and Zhang (1991)[3] introduced fuzzy variables as a basis for a theory of fuzzy reliability in the possibility context. Also in (1991) Utkin and Grouv[20], introduced how to analyze reliability of general life time distribution under fuzzy variables. Baloui (2011), work on evaluating reliability of system using fuzzy life time distribution, and introduce methods of estimation.

2. Theoretical Aspect

Let $(t_1, t_2, \dots, t_n)$ be $i.i.d$ survival times for (n) individuals, each one of them have constant hazard rate $[\lambda(t)]$. The probability death density can be determined using mathematical induction on hazard rate function.

Many studies for failure rate (hazard rate) for a patients suffering from disease, which end with death, like cancer disease or failure of both kidneys, for these cases, the hazard rate of this type of patients is increasing function of time.

$h(t)$: Failure hazard rate function.

λ: Scale parameter.

θ: Shape parameter.

The probability death density of (n) patients suffering from dangerous disease can be determined by applying mathematical induction procedure on the hazard rate function which defined as:

$$r(t) = \frac{f(t)}{s(t)} \tag{1}$$

Were;

$$f(t, \lambda, \theta) = \lambda\theta t^{\theta-1} e^{-\lambda t^{\theta}} \tag{2}$$

Cumulative distribution function;

$$F_T(t) = 1 - e^{-\lambda t^{\theta}} \tag{3}$$

While the survival function;

$$S_T(t) = 1 - F_T(t) = e^{-\lambda t^{\theta}} \tag{4}$$

Then the hazard rate function;

$$\lambda(t) = \frac{f(t)}{S_T(t)} = \lambda\theta t^{\theta-1} \tag{5}$$

Or it may be written as equation (6) below.

When the hazard rate for a group of patients suffering from dangerous disease, and the death happen after some time ($t \geq \delta$), in this case [$\lambda(t)$] is;

$$\lambda(t) = \lambda\theta(t-\delta)^{\theta-1} \qquad t, \theta, \lambda > 0 \tag{6}$$

In this paper, we explain the methods of estimation for (θ, λ), then apply simulation to compare different estimate of fuzzy rate function.

2. Estimation Methods

2.1 Maximum Likelihood Estimators

Let $(t_1, t_2, \dots, t_n)$ be a r.v from p.d.f given in equation (2), then;

$$L = \prod_{i=1}^{n} f(t_i, \lambda, \theta) = \lambda^n \theta^n \prod_{i=1}^{n} t_i^{\theta-1} \quad e^{-\lambda \sum_{i=1}^{n} t_i^{\theta}} \tag{7}$$

$$\log L = n \log \lambda + n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log t_i - \lambda \sum_{i=1}^{n} t_i^{\theta} \tag{8}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} t_i^{\theta} = 0 \tag{9}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log t_i - \lambda \sum_{i=1}^{n} t_i^{\theta} \log t_i = 0 \tag{10}$$

$$\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} t_i^{\hat{\theta}}} \tag{11}$$

$$\hat{\theta}_{MLE} = \frac{n}{\hat{\lambda}\sum_{i=1}^{n} t_i^{\hat{\theta}} \log t_i - \sum_{i=1}^{n} \log t_i} \tag{12}$$

2.2 Derivation of r^th Moments about origin

Let;

$$\mu_r' = E(t^r) = \int_0^\infty t^r f(t,\lambda,\theta)dt \tag{13}$$

$$= \lambda\theta \int_0^\infty t^{r+\theta-1} e^{-\lambda t^\theta} dt$$

Let;

$$z = \lambda t^\theta \rightarrow \frac{z}{\lambda} = t^\theta \rightarrow t = \frac{z^{\frac{1}{\theta}}}{\lambda^{\frac{1}{\theta}}} \rightarrow dt = \frac{\frac{1}{\theta} z^{\frac{1}{\theta}-1}}{\lambda^{\frac{1}{\theta}}} dz = \frac{1}{\theta}\lambda^{-\frac{1}{\theta}} z^{\frac{1}{\theta}-1} dz$$

$$\mu_r' = \lambda^{-\frac{r}{\theta}} \int_0^\infty z^{\frac{r}{\theta}} e^{-z} dz$$

$$\mu_r' = \frac{\Gamma\left(\frac{r}{\theta}+1\right)}{\lambda^{\frac{r}{\theta}}} \tag{14}$$

When $r = 1$ we obtain the first moment;

$$\mu_1' = \frac{\Gamma\left(\frac{1}{\theta}+1\right)}{\lambda^{\frac{1}{\theta}}}$$

When $r = 2$ we obtain the second moment;

$$\mu_2' = \frac{\Gamma\left(\frac{2}{\theta}+1\right)}{\lambda^{\frac{2}{\theta}}}$$

By equating;

$$\frac{\sum_{i=1}^{n} t_i}{n} = \frac{\Gamma\left(\frac{1}{\theta}+1\right)}{\lambda^{\frac{1}{\theta}}} \tag{15}$$

$$\frac{\sum_{i=1}^{n} t_i^2}{n} = \frac{\Gamma\left(\frac{2}{\theta}+1\right)}{\lambda^{\frac{2}{\theta}}} \tag{16}$$

We obtain, ($\hat{\lambda}_{MOM}$ & $\hat{\theta}_{MOM}$)

2.3 Percentile Method

The estimators by this method obtained by minimizing the total sum square between $[F(t_i,\lambda,\theta)]$ and its non – parametric estimator like;

$$\hat{F}(t_i) = \frac{i}{n+1}$$

Let;

$$T = \sum_{i=1}^{n}\left[\left(1 - e^{-\lambda t_i^\theta}\right) - \frac{i}{n+1}\right]^2 \tag{17}$$

$$\frac{\partial T}{\partial \theta} = 2\sum_{i=1}^{n}\left[1 - e^{-\lambda t_i^\theta} - \frac{i}{n+1}\right](-1)e^{-\lambda t_i^\theta}\left(-\lambda t_i^\theta\right)\log t_i = 0$$

$$\sum_{i=1}^{n} t_i^\theta \log t_i \, e^{-\lambda t_i^\theta} - \sum_{i=1}^{n} t_i^\theta \log t_i \, e^{-2\lambda t_i^\theta} - \sum_{i=1}^{n}\left(\frac{i}{n+1}\right) t_i^\theta \log t_i \, e^{-\lambda t_i^\theta} = 0 \tag{18}$$

Solving equation (18) numerically gives ($\hat{\theta}_{PER}$).

Also from $\frac{\partial T}{\partial \lambda}$ we have;

$$\frac{\partial T}{\partial \lambda} = 2\sum_{i=1}^{n}\left[1 - e^{-\lambda t_i^\theta} - \frac{i}{n+1}\right](-1)e^{-\lambda t_i^\theta}\left(-t_i^\theta\right) = 0$$

$$\sum_{i=1}^{n}\left[1 - \frac{i}{n+1}\right] t_i^\theta e^{-\lambda t_i^\theta} - \sum_{i=1}^{n} t_i^\theta\, e^{-2\lambda t_i^\theta} = 0 \quad (19)$$

Solving equation (18) numerically gives ($\hat{\lambda}_{PEC}$).

4. Simulation

First of all we estimate the two parameters (λ, θ) by three different methods which are MLE, MOM, PEC, then using CDF to generate;

$F_T(t_i) = 1 - e^{-\lambda t_i^\theta}$

$u_i = 1 - e^{-\lambda t_i^\theta}$

$e^{-\lambda t_i^\theta} = 1 - u_i$

$-\lambda t_i^\theta = \log(1 - u_i)$

$-t_i^\theta = \left[-\frac{1}{\lambda}\log(1 - u_i)\right]^{\frac{1}{\theta}}$

Due to different sample size ($n = 20,40,60,80$), we generate different sets (θ, k, λ).

Table (1): Estimated values of ($\lambda, \theta, \tilde{k}$).

θ	0.5	0.5	0.5	2.0	2.0	2.0
k	0.3	0.6	0.8	0.3	0.6	0.8
λ	0.7	0.7	0.7	1.2	1.2	1.2

Table (1): Hazard function values for ($\lambda = 0.7, \theta = 1.5, k = 0.3$)

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_PEC	Best
20	0.5	1.16E-01	1.15E-01	0.11687	1.25E-01	9.63E-04	1.20E-03	6.90E-06	PEC
	1.0	1.38E-01	1.36E-01	0.13703	1.45E-01	0.00063	7.00E-04	2.50E-05	PEC
	1.5	1.51E-01	1.50E-01	0.14954	1.58E-01	0.00049	6.00E-04	4.03E-05	PEC
	2.0	1.61E-01	1.61E-01	0.15812	1.67E-01	0.00040	5.00E-04	5.20E-05	PEC
	2.5	1.67E-01	1.65E-01	0.16436	1.73E-01	0.00035	4.00E-04	6.06E-05	PEC
	3.0	1.72E-01	1.70E-01	0.16913	1.78E-01	0.00032	4.00E-04	6.72E-05	PEC
	3.5	1.76E-01	1.74E-01	0.17285	1.82E-01	0.00030	3.00E-04	7.22E-05	PEC
	4.0	1.80E-01	1.77E-01	0.17588	1.85E-01	0.00028	3.00E-04	7.63E-05	PEC
	4.5	1.82E-01	1.80E-01	0.17837	1.87E-01	0.00027	3.00E-04	7.95E-05	PEC

	5.0	1.84E-01	1.82E-01	0.18044	1.91E-01	0.00026	3.00E-04	8.22E-05	PEC
40	0.5	1.16E-01	1.15E-01	0.12101	1.21E-01	1.98E-06	3.16E-04	3.46E-04	MLE
	1.0	1.38E-01	1.37E-01	0.14201	1.42E-01	9.03E-06	1.94E-04	2.05E-04	MLE
	1.5	1.51E-01	1.50E-01	0.15601	1.55E-01	1.53E-05	1.38E-04	1.43E-04	MLE
	2.0	1.61E-01	1.60E-01	0.16475	1.64E-01	2.03E-05	1.08E-04	1.11E-04	MLE
	2.5	1.67E-01	1.66E-01	0.17117	1.70E-01	2.40E-05	9.12E-05	9.26E-05	MOM
	3.0	1.72E-01	1.71E-01	0.17606	1.75E-01	2.70E-05	7.96E-05	8.04E-05	MLE
	3.5	1.76E-01	1.75E-01	0.17987	1.80E-01	2.92E-05	7.16E-05	7.22E-05	MLE
	4.0	1.80E-01	1.78E-01	0.18305	1.82E-01	3.10E-05	6.60E-05	6.62E-05	MLE
	4.5	1.82E-01	1.81E-01	0.18551	1.85E-01	3.25E-05	6.16E-05	6.18E-05	MLE
	5.0	1.84E-01	1.83E-01	0.18764	1.87E-01	3.38E-05	5.83E-05	5.84E-05	MLE
60	0.5	1.16E-01	1.17E-01	0.12011	1.20E-01	7.65E-06	1.32E-04	1.94E-04	MLE
	1.0	1.38E-01	1.38E-01	0.14171	1.41E-01	4.50E-06	9.35E-05	1.23E-04	MLE
	1.5	1.51E-01	1.52E-01	0.15503	1.54E-01	2.91E-06	7.33E-05	9.05E-05	MLE
	2.0	1.61E-01	1.61E-01	0.16406	1.63E-01	2.02E-06	6.16E-05	7.27E-05	MLE
	2.5	1.67E-01	1.68E-01	0.17063	1.70E-01	1.45E-06	5.43E-05	6.20E-05	MLE
	3.0	1.72E-01	1.73E-01	0.17558	1.75E-01	1.10E-06	4.93E-05	5.50E-05	MLE
	3.5	1.76E-01	1.77E-01	0.17945	1.81E-01	8.64E-07	4.58E-05	5.02E-05	MLE
	4.0	1.80E-01	1.80E-01	0.18260	1.82E-01	6.92E-07	4.32E-05	4.66E-05	MLE
	4.5	1.82E-01	1.82E-01	0.18518	1.84E-01	5.66E-07	4.12E-05	4.40E-05	MLE
	5.0	1.84E-01	1.85E-01	0.18731	1.86E-01	4.71E-07	3.96E-05	4.21E-05	MLE
80	0.5	1.16E-01	0.11651	1.17E-01	1.18E-01	5.41E-08	1.36E-05	3.85E-05	MLE
	1.0	1.38E-01	0.13847	1.41E-01	1.41E-01	6.47E-07	5.77E-06	1.58E-05	MLE
	1.5	1.51E-01	0.15223	1.52E-01	1.52E-01	1.32E-06	2.71E-06	7.42E-06	MLE

	2.0	1.61E-01	0.16121	1.61E-01	1.61E-01	1.92E-06	1.34E-06	3.78E-06	MOM
	2.5	1.67E-01	0.16784	1.68E-01	1.68E-01	1.01E-06	6.91E-07	2.04E-06	MOM
	3.0	1.72E-01	0.17288	1.73E-01	1.73E-01	8.14E-07	3.61E-07	1.13E-06	MOM
	3.5	1.76E-01	0.17671	1.77E-01	1.77E-01	1.43E-07	1.81E-07	6.41E-07	PEC
	4.0	1.80E-01	0.18005	1.80E-01	1.80E-01	4.16E-07	8.01E-08	3.61E-07	MOM
	4.5	1.82E-01	0.18257	1.82E-01	1.82E-01	6.47E-07	3.01E-08	2.01E-07	MOM
	5.0	1.84E-01	0.18476	1.84E-01	1.87E-01	8.44E-07	1.01E-08	1.01E-07	MOM

Table (2): Hazard function values for ($\lambda = 0.7, \theta = 1.5, k = 0.6$)

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_PEC	Best
20	0.5	6.10E-01	0.6068	6.36E-01	6.41E-01	1.05E-05	8.15E-04	8.71E-04	MLE
	1.0	6.74E-01	0.6697	6.96E-01	6.97E-01	3.04E-05	4.91E-04	5.06E-04	MLE
	1.5	7.16E-01	0.7095	7.34E-01	7.34E-01	4.73E-05	3.38E-04	3.45E-04	MLE
	2.0	7.44E-01	0.7371	7.61E-01	7.61E-01	6.04E-05	2.58E-04	2.62E-04	MLE
	2.5	7.65E-01	0.7573	7.81E-01	7.81E-01	7.04E-05	2.12E-04	2.13E-04	MLE
	3.0	7.81E-01	0.7727	7.94E-01	7.94E-01	7.81E-05	1.81E-04	1.81E-04	MLE
	3.5	7.93E-01	0.7851	8.06E-01	8.06E-01	8.42E-05	1.58E-04	1.58E-04	MLE
	4.0	8.03E-01	0.7948	8.15E-01	8.15E-01	8.92E-05	1.45E-04	1.42E-04	MLE
	4.5	8.12E-01	0.8030	8.23E-01	8.23E-01	9.32E-05	1.31E-04	1.33E-04	MLE
	5.0	8.20E-01	0.8097	8.32E-01	8.32E-01	9.65E-05	1.23E-04	1.24E-04	MLE
	0.5	6.11E-01	0.6102	6.31E-01	6.33E-01	1.01E-08	3.98E-04	5.52E-04	MLE
	1.0	6.74E-01	0.6741	6.92E-01	6.93E-01	1.13E-06	2.72E-04	3.42E-04	MLE
	1.5	7.16E-01	0.7146	7.32E-01	7.31E-01	3.18E-06	2.08E-04	2.48E-04	MLE

40	2.0	7.44E-01	0.7426	7.57E-01	7.58E-01	5.10E-06	1.73E-04	1.98E-04	MLE
	2.5	7.65E-01	0.7631	7.77E-01	7.78E-01	6.68E-06	1.51E-04	1.68E-04	MLE
	3.0	7.81E-01	0.7787	7.92E-01	7.93E-01	7.97E-06	1.36E-04	1.48E-04	MLE
	3.5	7.93E-01	0.7911	8.04E-01	8.05E-01	9.02E-06	1.26E-04	1.35E-04	MLE
	4.0	8.03E-01	0.8011	8.14E-01	8.15E-01	9.87E-06	1.18E-04	1.25E-04	MLE
	4.5	8.12E-01	0.8093	8.22E-01	8.23E-01	1.05E-05	1.12E-04	1.18E-04	MLE
	5.0	8.20E-01	0.8162	8.31E-01	8.32E-01	1.13E-05	1.08E-04	1.13E-04	MLE
60	0.5	6.11E-01	0.6114	6.21E-01	6.25E-01	1.95E-06	1.32E-04	2.54E-04	MLE
	1.0	6.74E-01	0.6761	6.84E-01	6.87E-01	5.56E-07	8.94E-05	1.48E-04	MLE
	1.5	7.16E-01	0.7167	7.24E-01	7.26E-01	1.02E-07	6.83E-05	1.03E-04	MLE
	2.0	7.44E-01	0.7451	7.52E-01	7.53E-01	1.01E-09	5.62E-05	7.81E-05	MLE
	2.5	7.65E-01	0.7655	7.72E-01	7.73E-01	3.80E-08	4.86E-05	6.36E-05	MLE
	3.0	7.81E-01	0.7812	7.87E-01	7.88E-01	1.26E-07	4.35E-05	5.44E-05	MLE
	3.5	7.93E-01	0.7936	8.02E-01	8.02E-01	2.32E-07	4.01E-05	4.82E-05	MLE
	4.0	8.03E-01	0.8036	8.11E-01	8.12E-01	3.41E-07	3.72E-05	4.37E-05	MLE
	4.5	8.12E-01	0.8121	8.18E-01	8.18E-01	4.43E-07	3.52E-05	4.04E-05	MLE
	5.0	8.21E-01	0.8188	8.27E-01	8.25E-01	5.78E-07	3.36E-05	3.81E-05	MLE
80	0.5	3.11E-01	0.3111	3.15E-01	3.21E-01	8.05E-07	3.76E-05	1.46E-04	MLE
	1.0	3.74E-01	0.3758	3.82E-01	3.83E-01	4.43E-07	2.72E-05	7.97E-05	MLE
	1.5	4.16E-01	0.4171	4.22E-01	4.23E-01	2.82E-07	2.16E-05	5.18E-05	MLE
	2.0	4.44E-01	0.4453	4.48E-01	4.52E-01	1.98E-07	1.86E-05	3.10E-05	MLE
	2.5	4.65E-01	0.4661	4.71E-01	4.72E-01	1.52E-07	1.66E-05	3.01E-05	MLE
	3.0	4.83E-01	0.4821	4.85E-01	4.86E-01	1.24E-07	1.54E-05	2.53E-05	MLE
	3.5	4.93E-01	0.4944	4.97E-01	4.98E-01	1.05E-07	1.44E-05	2.23E-05	MLE
	4.0	5.03E-01	0.5045	5.07E-01	5.08E-01	9.33E-08	1.41E-05	2.01E-05	MLE
	4.5	5.12E-01	0.5131	5.15E-01	5.16E-01	8.42E-08	1.45E-05	1.83E-05	MLE

	5.0	5.21E-01	0.5201	5.22E-01	5.23E-01	7.73E-08	1.31E-05	1.73E-05	MLE

Table (3): Hazard function values for ($\lambda = 0.7, \theta = 1.5, k = 0.9$)

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_ PEC	Best
20	0.5	5.02E-01	5.28E-01	5.48E-01	3.77E-01	0.000823	0.002317	1.543E-02	RE
	1.0	5.03E-01	0.530561	0.549828	0.380805	0.000851	0.002334	0.014567	MLE
	1.5	5.03E-01	0.532232	0.551113	0.386033	0.000923	0.002423	0.013481	MLE
	2.0	5.04E-01	0.533565	0.552207	0.390687	0.000961	0.002461	0.012541	MLE
	2.5	5.03E-01	0.534782	0.553333	0.394708	0.001006	0.002526	0.011721	MLE
	3.0	5.04E-01	0.536191	0.554761	0.399112	0.001086	0.002656	0.010881	MLE
	3.5	5.06E-01	0.537703	0.556273	0.404002	0.001042	0.002586	0.010241	MLE
	4.0	5.04E-01	0.539357	0.557897	0.40943	0.001148	0.002748	0.009223	MLE
	4.5	5.08E-01	0.541387	0.55966	0.415207	0.001066	0.002594	0.008746	MLE
	5.0	5.07E-01	0.54323	0.561463	0.420597	0.001188	0.002767	0.007787	MLE
40	0.5	5.02E-01	5.18E-01	5.36E-01	4.12E-01	2.87E-04	1.18E-03	7.90E-03	MLE
	1.0	5.23E-01	5.21E-01	5.36E-01	4.14E-01	0.000295	0.001197	0.007698	MLE
	1.5	5.02E-01	5.21E-01	5.37E-01	4.16E-01	0.000310	0.001215	0.007378	MLE
	2.0	5.02E-01	5.21E-01	5.37E-01	4.18E-01	0.000316	0.001221	0.007105	MLE
	2.5	5.03E-01	5.21E-01	5.38E-01	4.20E-01	0.000327	0.001235	0.006904	MLE
	3.0	5.03E-01	5.21E-01	5.38E-01	4.21E-01	0.000341	0.001253	0.006671	MLE
	3.5	5.03E-01	5.22E-01	5.41E-01	4.23E-01	0.000351	0.001265	0.006476	MLE
	4.0	5.03E-01	5.23E-01	5.41E-01	4.24E-01	0.000370	0.001292	0.006240	MLE
	4.5	5.04E-01	5.23E-01	5.40E-01	4.26E-01	0.000370	0.001286	0.006048	MLE
	5.0	5.04E-01	5.24E-01	5.41E-01	4.28E-01	0.000394	0.001328	0.005776	MLE
	0.5	5.01E-01	4.97E-01	5.08E-01	4.13E-01	7.64E-06	5.80E-05	7.71E-03	MLE
	1.0	5.01E-01	4.97E-01	5.09E-01	4.14E-01	6.67E-06	6.00E-05	0.007569	MLE
	1.5	5.02E-01	4.98E-01	5.09E-01	4.15E-01	1.02E-05	5.05E-05	0.007603	MLE

60	2.0	5.02E-01	4.98E-01	5.09E-01	4.16E-01	8.72E-06	5.30E-05	0.007464	MLE
	2.5	5.03E-01	4.98E-01	5.10E-01	4.17E-01	1.37E-05	4.17E-05	0.007485	MLE
	3.0	5.04E-01	5.00E-01	5.10E-01	4.18E-01	2.15E-05	2.98E-05	0.007553	MLE
	3.5	5.05E-01	5.00E-01	5.10E-01	4.19E-01	1.99E-05	3.17E-05	0.007406	MLE
	4.0	5.06E-01	5.01E-01	5.11E-01	4.20E-01	2.57E-05	2.50E-05	0.007411	MLE
	4.5	5.07E-01	5.01E-01	5.11E-01	4.21E-01	3.37E-05	1.79E-05	0.007441	MLE
	5.0	5.07E-01	5.02E-01	5.12E-01	4.22E-01	3.47E-05	1.66E-05	0.007353	MLE
80	0.5	0.500051	5.09E-01	5.06E-01	4.61E-01	8.17E-05	3.31E-05	1.51E-03	MOM
	1.0	0.500171	5.09E-01	5.06E-01	4.62E-01	8.55E-05	3.57E-05	1.47E-03	MOM
	1.5	0.500466	5.10E-01	5.06E-01	4.62E-01	8.60E-05	3.62E-05	1.45E-03	MOM
	2.0	0.500597	5.10E-01	5.07E-01	4.63E-01	8.89E-05	3.84E-05	1.42E-03	MOM
	2.5	0.500667	5.10E-01	5.07E-01	4.63E-01	9.40E-05	4.22E-05	1.39E-03	MOM
	3.0	0.500831	5.11E-01	5.07E-01	4.64E-01	9.68E-05	4.42E-05	1.37E-03	MOM
	3.5	0.500841	5.11E-01	5.08E-01	4.64E-01	1.03E-04	4.84E-05	1.33E-03	MOM
	4.0	0.501007	5.11E-01	5.08E-01	4.65E-01	1.06E-04	5.10E-05	1.30E-03	MOM
	4.5	0.501131	5.12E-01	5.08E-01	4.65E-01	1.10E-04	5.35E-05	1.27E-03	MOM
	5.0	0.501331	5.12E-01	5.09E-01	4.66E-01	1.12E-04	5.51E-05	1.24E-03	MOM

Table (4): Hazard function values for $(\lambda = 0.3, \theta = 2.5, k = 0.6)$

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_ PEC	Best
20	0.5	0.501875	5.32E-01	5.59E-01	3.83E-01	8.90E-04	3.27E-03	1.40E-02	MLE
	1.0	0.503569	5.34E-01	5.61E-01	3.91E-01	9.31E-04	3.30E-03	1.26E-02	MLE
	1.5	0.504911	5.37E-01	5.63E-01	4.02E-01	1.05E-03	3.40E-03	1.07E-02	MLE
	2.0	0.508512	5.40E-01	5.65E-01	4.09E-01	9.85E-04	3.19E-03	9.91E-03	MLE
	2.5	0.508816	5.43E-01	5.67E-01	4.17E-01	1.14E-03	3.42E-03	8.36E-03	MLE
	3.0	0.511537	5.46E-01	5.70E-01	4.26E-01	1.17E-03	3.38E-03	7.30E-03	MLE
	3.5	0.514584	5.49E-01	5.72E-01	4.34E-01	1.15E-03	3.27E-03	6.56E-03	MLE

	4.0	0.517084	5.51E-01	5.74E-01	4.42E-01	1.16E-03	3.22E-03	5.70E-03	MLE
	4.5	0.517101	5.54E-01	5.76E-01	4.49E-01	1.36E-03	3.51E-03	4.63E-03	MLE
	5.0	0.523760	5.57E-01	5.79E-01	4.57E-01	1.10E-03	3.02E-03	4.51E-03	MLE
40	0.5	0.501081	5.17E-01	5.28E-01	4.29E-01	2.64E-04	7.11E-04	5.25E-03	MLE
	1.0	0.501958	5.18E-01	5.29E-01	4.31E-01	2.69E-04	7.20E-04	5.00E-03	MLE
	1.5	0.502052	5.19E-01	5.30E-01	4.34E-01	3.01E-04	7.63E-04	4.66E-03	MLE
	2.0	0.50422	5.21E-01	5.31E-01	4.37E-01	2.71E-04	7.07E-04	4.52E-03	MLE
	2.5	0.504697	5.22E-01	5.32E-01	4.39E-01	2.91E-04	7.31E-04	4.27E-03	MLE
	3.0	0.50476	5.23E-01	5.33E-01	4.42E-01	3.35E-04	7.90E-04	3.89E-03	MLE
	3.5	0.504836	5.24E-01	5.34E-01	4.45E-01	3.85E-04	8.70E-04	3.55E-03	MLE
	4.0	0.507996	5.26E-01	5.36E-01	4.48E-01	3.18E-04	7.67E-04	3.59E-03	MLE
	4.5	0.508336	5.27E-01	5.37E-01	4.51E-01	3.56E-04	8.19E-04	3.27E-03	MLE
	5.0	0.509221	5.29E-01	5.38E-01	4.54E-01	3.74E-04	8.43E-04	3.03E-03	MLE
60	0.5	0.500628	5.13E-01	5.22E-01	4.58E-01	1.43E-04	4.60E-04	1.84E-03	MLE
	1.0	0.500840	5.13E-01	5.23E-01	4.59E-01	1.58E-04	4.84E-04	1.72E-03	MLE
	1.5	0.501577	5.14E-01	5.24E-01	4.61E-01	1.61E-04	4.86E-04	1.64E-03	MLE
	2.0	0.501683	5.15E-01	5.24E-01	4.63E-01	1.77E-04	5.10E-04	1.53E-03	MLE
	2.5	0.502078	5.16E-01	5.25E-01	4.64E-01	1.89E-04	5.29E-04	1.45E-03	MLE
	3.0	0.502107	5.17E-01	5.26E-01	4.66E-01	2.12E-04	5.65E-04	1.32E-03	MLE
	3.5	0.502564	5.18E-01	5.27E-01	4.67E-01	2.24E-04	5.82E-04	1.23E-03	MLE
	4.0	0.502801	5.18E-01	5.28E-01	4.69E-01	2.43E-04	6.10E-04	1.14E-03	MLE
	4.5	0.502884	5.19E-01	5.28E-01	4.71E-01	2.69E-04	6.49E-04	1.04E-03	MLE
	5.0	0.503948	5.20E-01	5.29E-01	4.72E-01	2.67E-04	6.42E-04	1.01E-03	MLE
	0.5	0.500083	5.00E-01	5.00E-01	4.49E-01	1.54E-08	1.67E-07	2.59E-03	MLE
	1.0	0.500361	5.01E-01	5.00E-01	4.50E-01	1.26E-07	4.13E-08	2.52E-03	MLE
	1.5	0.500663	5.01E-01	5.01E-01	4.51E-01	3.53E-07	1.09E-09	2.45E-03	MLE

80	2.0	0.501127	5.02E-01	5.01E-01	4.52E-01	3.91E-07	4.29E-09	2.39E-03	MLE
	2.5	0.50195	5.02E-01	5.02E-01	4.53E-01	1.49E-07	1.01E-08	2.37E-03	MLE
	3.0	0.502231	5.03E-01	5.02E-01	4.55E-01	5.31E-07	5.62E-08	2.28E-03	MLE
	3.5	0.503876	5.04E-01	5.03E-01	4.55E-01	1.18E-07	6.38E-07	2.35E-03	MLE
	4.0	0.504618	5.04E-01	5.04E-01	4.57E-01	1.99E-07	8.39E-07	2.30E-03	MLE
	4.5	0.505347	5.05E-01	5.04E-01	4.58E-01	3.51E-07	1.11E-06	2.27E-03	MLE
	5.0	0.505474	5.05E-01	5.05E-01	4.59E-01	1.97E-08	3.88E-07	2.19E-03	MLE

Table (5): Hazard function values for $(\lambda = 0.7, \theta = 2.5, k = 0.6)$

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_ PEC	Best
20	0.5	4.69E-01	0.491671	5.18E-01	3.57E-01	5.21E-04	2.41E-03	1.25E-02	MLE
	1.0	4.69E-01	0.495357	5.21E-01	3.66E-01	6.76E-04	2.64E-03	1.06E-02	MLE
	1.5	4.74E-01	0.498891	5.24E-01	3.75E-01	6.07E-04	2.43E-03	9.86E-03	MLE
	2.0	4.86E-01	0.501971	5.26E-01	3.83E-01	2.57E-04	1.62E-03	1.05E-02	MLE
	2.5	4.87E-01	0.506153	5.29E-01	3.94E-01	3.68E-04	1.80E-03	8.70E-03	MLE
	3.0	5.01E-01	0.510174	5.33E-01	4.04E-01	8.00E-05	1.02E-03	9.50E-03	MLE
	3.5	5.01E-01	0.514397	5.36E-01	4.13E-01	1.69E-04	1.21E-03	7.82E-03	MLE
	4.0	5.03E-01	0.518132	5.40E-01	4.22E-01	2.39E-04	1.40E-03	6.44E-03	MLE
	4.5	5.04E-01	0.522615	5.44E-01	4.32E-01	3.39E-04	1.56E-03	5.20E-03	MLE
	5.0	5.22E-01	0.526951	5.47E-01	4.41E-01	2.28E-05	6.16E-04	6.55E-03	MLE
40	0.5	4.67E-01	0.478642	4.89E-01	3.85E-01	1.28E-04	4.83E-04	6.74E-03	MLE
	1.0	4.68E-01	0.480275	4.91E-01	3.89E-01	1.56E-04	5.26E-04	6.24E-03	MLE
	1.5	4.69E-01	0.482243	4.92E-01	3.93E-01	1.70E-04	5.39E-04	5.80E-03	MLE
	2.0	4.72E-01	0.484052	4.94E-01	3.97E-01	1.46E-04	4.92E-04	5.58E-03	MLE
	2.5	4.73E-01	0.485664	4.96E-01	4.01E-01	1.65E-04	5.20E-04	5.21E-03	MLE
	3.0	4.73E-01	0.487627	4.97E-01	4.05E-01	2.10E-04	5.92E-04	4.71E-03	MLE

	3.5	4.76E-01	0.489242	4.99E-01	4.08E-01	1.65E-04	5.08E-04	4.71E-03	MLE
	4.0	4.79E-01	0.491126	5.01E-01	4.11E-01	1.43E-04	4.62E-04	4.60E-03	MLE
	4.5	4.82E-01	0.492622	5.02E-01	4.15E-01	1.21E-04	4.17E-04	4.51E-03	MLE
	5.0	4.83E-01	0.494359	5.04E-01	4.18E-01	1.36E-04	4.35E-04	4.19E-03	MLE
60	0.5	4.70E-01	0.469355	4.71E-01	4.16E-01	1.96E-07	2.19E-06	2.87E-03	MLE
	1.0	4.71E-01	0.470655	4.73E-01	4.18E-01	3.07E-07	1.78E-06	2.79E-03	MLE
	1.5	4.72E-01	0.471682	4.74E-01	4.20E-01	2.68E-07	1.88E-06	2.68E-03	MLE
	2.0	4.73E-01	0.473078	4.75E-01	4.23E-01	2.14E-07	5.60E-06	2.50E-03	MLE
	2.5	4.74E-01	0.474161	4.76E-01	4.24E-01	8.72E-08	4.85E-06	2.46E-03	MLE
	3.0	4.74E-01	0.475423	4.77E-01	4.26E-01	1.88E-06	1.08E-05	2.28E-03	MLE
	3.5	4.77E-01	0.476468	4.78E-01	4.28E-01	3.38E-09	3.60E-06	2.36E-03	MLE
	4.0	4.77E-01	0.47758	4.80E-01	4.30E-01	1.12E-06	1.02E-05	2.16E-03	MLE
	4.5	4.77E-01	0.478931	4.81E-01	4.32E-01	3.84E-06	1.79E-05	2.00E-03	MLE
	5.0	4.79E-01	0.480154	4.82E-01	4.34E-01	2.13E-06	1.31E-05	1.97E-03	MLE
80	0.5	4.67E-01	0.474738	4.87E-01	4.38E-01	5.32E-05	3.75E-04	8.86E-04	MLE
	1.0	4.68E-01	0.475592	4.88E-01	4.39E-01	6.52E-05	4.02E-04	8.10E-04	MLE
	1.5	4.69E-01	0.476454	4.88E-01	4.40E-01	4.97E-05	3.58E-04	8.36E-04	MLE
	2.0	4.70E-01	0.477258	4.89E-01	4.42E-01	5.72E-05	3.75E-04	7.85E-04	MLE
	2.5	4.71E-01	0.478261	4.90E-01	4.43E-01	5.98E-05	3.78E-04	7.41E-04	MLE
	3.0	4.72E-01	0.479214	4.91E-01	4.45E-01	5.60E-05	3.64E-04	7.31E-04	MLE
	3.5	4.72E-01	0.480092	4.92E-01	4.46E-01	6.24E-05	3.78E-04	6.88E-04	MLE
	4.0	4.72E-01	0.480972	4.92E-01	4.47E-01	7.42E-05	4.00E-04	6.27E-04	MLE
	4.5	4.76E-01	0.481891	4.93E-01	4.49E-01	3.44E-05	2.95E-04	7.58E-04	MLE
	5.0	4.76E-01	0.482766	4.94E-01	4.50E-01	4.35E-05	3.17E-04	7.02E-04	MLE

Table (6): Hazard function values for ($\lambda = 0.7, \theta = 2.5, k = 0.9$)

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_ PEC	Best
20	0.5	5.02E-01	0.534465	5.60E-01	3.97E-01	1.03E-03	3.32E-03	1.12E-02	MLE
	1.0	5.03E-01	0.536992	5.62E-01	4.05E-01	1.15E-03	3.48E-03	9.71E-03	MLE
	1.5	5.05E-01	0.539653	5.64E-01	4.13E-01	1.22E-03	3.55E-03	8.48E-03	MLE
	2.0	5.07E-01	0.542337	5.66E-01	4.20E-01	1.22E-03	3.49E-03	7.59E-03	MLE
	2.5	5.09E-01	0.544743	5.69E-01	4.27E-01	1.31E-03	3.60E-03	6.60E-03	MLE
	3.0	5.11E-01	0.547595	5.71E-01	4.34E-01	1.35E-03	3.59E-03	5.83E-03	MLE
	3.5	5.15E-01	0.550793	5.73E-01	4.43E-01	1.29E-03	3.41E-03	5.10E-03	MLE
	4.0	5.15E-01	0.553348	5.75E-01	4.50E-01	1.47E-03	3.64E-03	4.18E-03	MLE
	4.5	5.16E-01	0.556316	5.78E-01	4.58E-01	1.61E-03	3.80E-03	3.40E-03	MLE
	5.0	5.23E-01	0.560071	5.81E-01	4.67E-01	1.38E-03	3.34E-03	3.11E-03	MLE
40	0.5	5.02E-01	0.519484	5.28E-01	4.12E-01	3.22E-04	7.14E-04	8.06E-03	MLE
	1.0	5.05E-01	0.520622	5.29E-01	4.15E-01	2.49E-04	5.94E-04	8.14E-03	MLE
	1.5	5.07E-01	0.522141	5.30E-01	4.19E-01	2.31E-04	5.53E-04	7.79E-03	MLE
	2.0	5.07E-01	0.523316	5.32E-01	4.21E-01	2.62E-04	5.99E-04	7.35E-03	MLE
	2.5	5.08E-01	0.524483	5.33E-01	4.24E-01	2.74E-04	6.18E-04	6.99E-03	MLE
	3.0	5.09E-01	0.525902	5.34E-01	4.29E-01	3.03E-04	6.53E-04	6.40E-03	MLE
	3.5	5.10E-01	0.527188	5.35E-01	4.32E-01	3.02E-04	6.56E-04	6.04E-03	MLE
	4.0	5.10E-01	0.528382	5.36E-01	4.35E-01	3.41E-04	7.06E-04	5.58E-03	MLE
	4.5	5.12E-01	0.529807	5.38E-01	4.39E-01	3.23E-04	6.88E-04	5.29E-03	MLE
	5.0	5.13E-01	0.531177	5.39E-01	4.42E-01	3.40E-04	7.06E-04	4.99E-03	MLE
	0.5	5.00E-01	0.512966	5.24E-01	4.40E-01	1.64E-04	5.48E-04	3.59E-03	MLE
	1.0	5.00E-01	0.513796	5.24E-01	4.42E-01	1.83E-04	5.78E-04	3.39E-03	MLE
	1.5	5.00E-01	0.514391	5.25E-01	4.43E-01	1.98E-04	6.00E-04	3.26E-03	MLE
	2.0	5.00E-01	0.515208	5.26E-01	4.45E-01	2.21E-04	6.34E-04	3.06E-03	MLE

60	2.5	5.01E-01	0.516013	5.26E-01	4.47E-01	2.30E-04	6.43E-04	2.90E-03	MLE
	3.0	5.02E-01	0.516954	5.27E-01	4.49E-01	2.36E-04	6.45E-04	2.73E-03	MLE
	3.5	5.02E-01	0.517695	5.28E-01	4.51E-01	2.56E-04	6.78E-04	2.60E-03	MLE
	4.0	5.02E-01	0.518343	5.28E-01	4.52E-01	2.68E-04	6.97E-04	2.48E-03	MLE
	4.5	5.02E-01	0.519195	5.29E-01	4.54E-01	2.96E-04	7.37E-04	2.32E-03	MLE
	5.0	5.03E-01	0.520158	5.30E-01	4.56E-01	3.10E-04	7.56E-04	2.16E-03	MLE
80	0.5	5.00E-01	0.50456	5.10E-01	4.56E-01	2.09E-05	9.40E-05	1.90E-03	MLE
	1.0	5.01E-01	0.50512	5.10E-01	4.57E-01	2.03E-05	9.17E-05	1.88E-03	MLE
	1.5	5.01E-01	0.50583	5.11E-01	4.59E-01	2.63E-05	1.03E-04	1.78E-03	MLE
	2.0	5.01E-01	0.506438	5.11E-01	4.60E-01	3.21E-05	1.13E-04	1.70E-03	MLE
	2.5	5.03E-01	0.507033	5.12E-01	4.60E-01	1.96E-05	8.76E-05	1.78E-03	MLE
	3.0	5.03E-01	0.507524	5.12E-01	4.61E-01	1.75E-05	8.30E-05	1.75E-03	MLE
	3.5	5.04E-01	0.508228	5.13E-01	4.63E-01	1.65E-05	7.98E-05	1.73E-03	MLE
	4.0	5.04E-01	0.508784	5.14E-01	4.64E-01	1.89E-05	8.48E-05	1.66E-03	MLE
	4.5	5.05E-01	0.509438	5.14E-01	4.65E-01	1.82E-05	8.20E-05	1.64E-03	MLE
	5.0	5.05E-01	0.510082	5.15E-01	4.66E-01	2.30E-05	9.01E-05	1.56E-03	MLE

Table (7): Hazard function values for $(\lambda = 1.5, \theta = 1.5, k = 0.3)$

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_ PEC	
20	0.5	4.24E-01	0.434824	4.55E-01	3.05E-01	1.27E-04	9.90E-04	1.41E-02	MLE
	1.0	4.33E-01	0.442932	4.62E-01	3.22E-01	9.06E-05	7.96E-04	1.25E-02	MLE
	1.5	4.35E-01	0.450692	4.68E-01	3.38E-01	2.40E-04	1.09E-03	9.37E-03	MLE
	2.0	4.50E-01	0.457632	4.75E-01	3.53E-01	5.43E-05	5.98E-04	9.37E-03	MLE
	2.5	4.53E-01	0.464586	4.81E-01	3.68E-01	1.44E-04	8.02E-04	7.18E-03	MLE
	3.0	4.65E-01	0.472913	4.88E-01	3.84E-01	5.72E-05	5.14E-04	6.66E-03	MLE
	3.5	4.66E-01	0.480402	4.95E-01	3.98E-01	2.16E-04	8.44E-04	4.53E-03	MLE
	4.0	4.75E-01	0.488561	5.02E-01	4.14E-01	1.83E-04	7.52E-04	3.73E-03	MLE

	4.5	4.78E-01	0.495583	5.08E-01	4.26E-01	2.93E-04	8.97E-04	2.79E-03	MLE
	5.0	4.89E-01	0.502298	5.15E-01	4.38E-01	1.76E-04	6.80E-04	2.57E-03	MLE
40	0.5	7.24E-01	0.734824	7.55E-01	6.05E-01	3.00E-01	3.01E-01	3.14E-01	MLE
	1.0	7.33E-01	0.742932	7.62E-01	6.22E-01	3.00E-01	3.01E-01	3.12E-01	MLE
	1.5	7.35E-01	0.750694	7.68E-01	6.38E-01	3.00E-01	3.01E-01	3.09E-01	MLE
	2.0	7.50E-01	0.757635	7.75E-01	6.53E-01	3.00E-01	3.01E-01	3.09E-01	MLE
	2.5	7.53E-01	0.764586	7.81E-01	6.68E-01	3.00E-01	3.01E-01	3.07E-01	MLE
	3.0	7.65E-01	0.772915	7.88E-01	6.84E-01	3.00E-01	3.01E-01	3.07E-01	MLE
	3.5	7.66E-01	0.780402	7.95E-01	6.98E-01	3.00E-01	3.01E-01	3.05E-01	MLE
	4.0	7.75E-01	0.788562	8.02E-01	7.14E-01	3.00E-01	3.01E-01	3.04E-01	MLE
	4.5	7.78E-01	0.795583	8.08E-01	7.26E-01	3.00E-01	3.01E-01	3.03E-01	MLE
	5.0	7.89E-01	0.802298	8.15E-01	7.38E-01	3.00E-01	3.01E-01	3.03E-01	MLE
60	0.5	4.22E-01	0.427531	4.34E-01	3.64E-01	2.99E-05	1.46E-04	3.34E-03	MLE
	1.0	4.24E-01	0.430528	4.37E-01	3.69E-01	3.83E-05	1.61E-04	3.03E-03	MLE
	1.5	4.25E-01	0.433081	4.40E-01	3.74E-01	6.31E-05	2.12E-04	2.65E-03	MLE
	2.0	4.27E-01	0.435867	4.42E-01	3.78E-01	7.13E-05	2.24E-04	2.47E-03	MLE
	2.5	4.28E-01	0.43853	4.45E-01	3.82E-01	1.13E-04	2.89E-04	2.10E-03	MLE
	3.0	4.28E-01	0.441676	4.48E-01	3.87E-01	1.79E-04	3.78E-04	1.74E-03	MLE
	3.5	4.30E-01	0.444329	4.50E-01	3.91E-01	2.19E-04	4.22E-04	1.51E-03	MLE
	4.0	4.31E-01	0.446837	4.52E-01	3.95E-01	2.42E-04	4.40E-04	1.35E-03	MLE
	4.5	4.33E-01	0.449553	4.55E-01	3.99E-01	2.69E-04	4.76E-04	1.19E-03	MLE
	5.0	4.34E-01	0.451959	4.57E-01	4.02E-01	3.38E-04	5.63E-04	9.80E-04	MLE
	0.5	4.22E-01	0.420042	4.25E-01	3.66E-01	5.52E-06	9.18E-06	3.24E-03	MLE
	1.0	4.27E-01	0.422097	4.27E-01	3.69E-01	2.70E-05	9.89E-09	3.45E-03	MOM

80	1.5	4.30E-01	0.424291	4.30E-01	3.72E-01	3.79E-05	9.02E-07	3.46E-03	MOM
	2.0	4.33E-01	0.426116	4.31E-01	3.75E-01	4.23E-05	1.82E-06	3.38E-03	MOM
	2.5	4.34E-01	0.428208	4.33E-01	3.78E-01	2.93E-05	1.63E-07	3.13E-03	MOM
	3.0	4.34E-01	0.430311	4.35E-01	3.81E-01	1.70E-05	6.00E-07	2.91E-03	MOM
	3.5	4.35E-01	0.432182	4.37E-01	3.83E-01	1.06E-05	2.47E-06	2.77E-03	MOM
	4.0	4.37E-01	0.434000	4.39E-01	3.85E-01	7.65E-06	3.85E-06	2.63E-03	MOM
	4.5	4.37E-01	0.436083	4.41E-01	3.89E-01	1.48E-06	1.21E-05	2.34E-03	MOM
	5.0	4.42E-01	0.438072	4.43E-01	3.92E-01	1.73E-05	2.24E-07	2.52E-03	MOM

Table (8): Hazard function values for ($\lambda = 1.5, \theta = 1.5, k = 0.6$)

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_PEC	Best
20	0.5	6.18E-01	0.580313	5.93E-01	4.56E-01	1.42E-03	6.22E-04	2.63E-02	MOM
	1.0	6.36E-01	0.620064	6.29E-01	5.06E-01	2.57E-04	4.38E-05	1.68E-02	MOM
	1.5	6.55E-01	0.655173	6.66E-01	5.51E-01	1.37E-09	1.26E-04	1.09E-02	MLE
	2.0	6.74E-01	0.689786	7.02E-01	5.98E-01	2.62E-04	8.21E-04	5.76E-03	MLE
	2.5	6.76E-01	0.717027	7.30E-01	6.32E-01	1.69E-03	2.87E-03	1.94E-03	MLE
	3.0	8.17E-01	0.745636	7.57E-01	6.67E-01	5.11E-03	3.58E-03	2.24E-02	MOM
	3.5	8.70E-01	0.774795	7.87E-01	7.04E-01	9.00E-03	6.84E-03	2.74E-02	MOM
	4.0	8.71E-01	0.802218	8.15E-01	7.38E-01	4.68E-03	3.12E-03	1.75E-02	MOM
	4.5	8.95E-01	0.826306	8.40E-01	7.69E-01	4.66E-03	2.93E-03	1.58E-02	MOM
	5.0	9.21E-01	0.857497	8.71E-01	8.06E-01	4.08E-03	2.54E-03	1.33E-02	MOM
40	0.5	5.34E-01	0.549087	5.67E-01	4.44E-01	2.40E-04	1.10E-03	7.95E-03	MOM
	1.0	6.24E-01	0.573491	5.90E-01	4.75E-01	2.56E-03	1.18E-03	2.22E-02	MOM
	1.5	6.78E-01	0.592894	6.08E-01	5.00E-01	7.31E-03	4.89E-03	3.20E-02	MOM
	2.0	6.98E-01	0.611368	6.26E-01	5.23E-01	7.45E-03	5.09E-03	3.07E-02	MOM
	2.5	7.06E-01	0.629325	6.44E-01	5.45E-01	5.95E-03	3.90E-03	2.62E-02	MOM
	3.0	7.11E-01	0.650765	6.65E-01	5.71E-01	3.64E-03	2.16E-03	1.96E-02	MOM

	3.5	7.18E-01	0.667923	6.81E-01	5.92E-01	2.53E-03	1.35E-03	1.59E-02	MOM
	4.0	7.25E-01	0.683448	6.97E-01	6.12E-01	1.71E-03	7.63E-04	1.28E-02	MOM
	4.5	7.43E-01	0.701323	7.16E-01	6.34E-01	1.76E-03	7.67E-04	1.20E-02	MOM
	5.0	7.87E-01	0.714354	7.29E-01	6.50E-01	5.34E-03	3.46E-03	1.89E-02	MOM
60	0.5	5.00E-01	0.531697	5.21E-01	4.34E-01	9.79E-04	4.05E-04	4.44E-03	MOM
	1.0	5.01E-01	0.550005	5.39E-01	4.56E-01	2.42E-03	1.43E-03	2.01E-03	MOM
	1.5	5.01E-01	0.567225	5.56E-01	4.77E-01	4.35E-03	2.98E-03	5.75E-04	PEC
	2.0	5.07E-01	0.582071	5.70E-01	4.95E-01	5.64E-03	4.02E-03	1.50E-04	PEC
	2.5	5.17E-01	0.594547	5.82E-01	5.09E-01	5.95E-03	4.22E-03	6.97E-05	PEC
	3.0	5.53E-01	0.608287	5.96E-01	5.25E-01	3.02E-03	1.84E-03	7.91E-04	PEC
	3.5	5.57E-01	0.620217	6.09E-01	5.40E-01	4.03E-03	2.69E-03	2.90E-04	PEC
	4.0	5.61E-01	0.631796	6.21E-01	5.54E-01	5.03E-03	3.57E-03	5.40E-05	PEC
	4.5	5.65E-01	0.642846	6.32E-01	5.67E-01	6.09E-03	4.53E-03	5.31E-06	PEC
	5.0	5.71E-01	0.654035	6.44E-01	5.81E-01	6.94E-03	5.39E-03	1.11E-04	PEC
80	0.5	5.11E-01	0.521611	5.29E-01	4.10E-01	1.21E-04	3.35E-04	1.02E-02	MLE
	1.0	5.27E-01	0.534105	5.41E-01	4.26E-01	4.43E-05	1.86E-04	1.04E-02	MLE
	1.5	5.30E-01	0.546974	5.53E-01	4.41E-01	2.96E-04	5.62E-04	7.80E-03	MLE
	2.0	5.34E-01	0.560554	5.67E-01	4.58E-01	7.24E-04	1.09E-03	5.73E-03	MLE
	2.5	5.52E-01	0.572061	5.78E-01	4.72E-01	4.19E-04	6.87E-04	6.30E-03	MLE
	3.0	5.62E-01	0.582338	5.88E-01	4.85E-01	4.29E-04	6.71E-04	5.94E-03	MLE
	3.5	5.67E-01	0.593375	5.99E-01	4.99E-01	6.74E-04	9.68E-04	4.66E-03	MLE
	4.0	5.75E-01	0.603742	6.09E-01	5.13E-01	8.43E-04	1.18E-03	3.84E-03	MOM
	4.5	5.76E-01	0.614613	6.20E-01	5.26E-01	1.48E-03	1.89E-03	2.54E-03	MLE
	5.0	5.96E-01	0.624546	6.30E-01	5.38E-01	8.33E-04	1.17E-03	3.27E-03	MLE

Table (9): Hazard function values for $(\lambda = 1.5, \theta = 1.5, k = 0.9)$

n	t_i	Real	mle	mom	PEC	Mse_mle	Mse_mom	Mse_ PEC	Best
20	0.5	5.39E-01	0.616776	6.30E-01	5.13E-01	6.10E-03	8.37E-03	6.37E-04	PEC
	1.0	5.63E-01	0.676876	6.88E-01	5.89E-01	1.30E-02	1.56E-02	6.96E-04	PEC
	1.5	5.94E-01	0.723962	7.37E-01	6.48E-01	1.69E-02	2.04E-02	2.97E-03	PEC
	2.0	6.92E-01	0.773883	7.87E-01	7.09E-01	6.68E-03	9.05E-03	2.96E-04	PEC
	2.5	7.23E-01	0.820065	8.35E-01	7.66E-01	9.35E-03	1.26E-02	1.79E-03	PEC
	3.0	7.28E-01	0.855701	8.72E-01	8.09E-01	1.62E-02	2.06E-02	6.54E-03	PEC
	3.5	8.07E-01	0.888683	9.05E-01	8.47E-01	6.72E-03	9.70E-03	1.66E-03	PEC
	4.0	8.32E-01	0.921431	9.40E-01	8.88E-01	8.02E-03	1.17E-02	3.19E-03	PEC
	4.5	9.57E-01	0.950983	9.70E-01	9.24E-01	3.37E-05	1.81E-04	1.08E-03	PEC
	5.0	9.74E-01	0.978661	1.000000	9.59E-01	1.78E-05	6.29E-04	2.47E-04	PEC
40	0.5	5.28E-01	0.540542	5.62E-01	4.49E-01	1.73E-04	1.22E-03	6.11E-03	MLE
	1.0	5.42E-01	0.580546	6.00E-01	4.98E-01	1.47E-03	3.30E-03	1.99E-03	MLE
	1.5	6.20E-01	0.610725	6.28E-01	5.34E-01	7.86E-05	6.79E-05	7.34E-03	MOM
	2.0	6.41E-01	0.636782	6.55E-01	5.66E-01	1.90E-05	1.87E-04	5.66E-03	MLE
	2.5	7.13E-01	0.663252	6.80E-01	5.96E-01	2.49E-03	1.11E-03	1.36E-02	MOM
	3.0	7.20E-01	0.690193	7.05E-01	6.27E-01	8.99E-04	2.23E-04	8.58E-03	MOM
	3.5	7.21E-01	0.712644	7.27E-01	6.54E-01	7.57E-05	2.86E-05	4.47E-03	MOM
	4.0	7.50E-01	0.736833	7.51E-01	6.83E-01	1.59E-04	1.74E-06	4.38E-03	MOM
	4.5	7.69E-01	0.758748	7.73E-01	7.10E-01	1.03E-04	1.48E-05	3.51E-03	MOM
	5.0	7.88E-01	0.778932	7.92E-01	7.33E-01	8.29E-05	1.55E-05	3.01E-03	MOM
	0.5	5.08E-01	0.534478	5.34E-01	4.39E-01	7.00E-04	6.91E-04	4.75E-03	MOM
	1.0	6.48E-01	0.559492	5.58E-01	4.66E-01	7.86E-03	8.07E-03	3.32E-02	MLE
	1.5	6.55E-01	0.583662	5.82E-01	4.95E-01	5.12E-03	5.33E-03	2.58E-02	MLE
	2.0	6.82E-01	0.60514	6.05E-01	5.23E-01	5.87E-03	5.96E-03	2.53E-02	MLE

60	2.5	6.90E-01	0.624021	6.23E-01	5.45E-01	4.40E-03	4.49E-03	2.11E-02	MLE
	3.0	7.01E-01	0.639252	6.39E-01	5.62E-01	3.80E-03	3.89E-03	1.92E-02	MLE
	3.5	7.08E-01	0.654223	6.53E-01	5.80E-01	2.92E-03	3.03E-03	1.65E-02	MOM
	4.0	7.20E-01	0.672561	6.71E-01	6.02E-01	2.23E-03	2.34E-03	1.39E-02	MLE
	4.5	7.22E-01	0.691232	6.91E-01	6.25E-01	9.46E-04	9.69E-04	9.31E-03	PEC
	5.0	7.25E-01	0.704734	7.05E-01	6.43E-01	4.28E-04	4.23E-04	6.86E-03	MOM
80	0.5	5.21E-01	0.532211	5.38E-01	4.53E-01	1.33E-04	3.06E-04	4.62E-03	MLE
	1.0	5.22E-01	0.554251	5.60E-01	4.78E-01	1.05E-03	1.44E-03	1.91E-03	MLE
	1.5	5.41E-01	0.574675	5.80E-01	5.01E-01	1.11E-03	1.46E-03	1.61E-03	MLE
	2.0	5.64E-01	0.592742	5.98E-01	5.23E-01	8.11E-04	1.14E-03	1.69E-03	MOM
	2.5	5.75E-01	0.609438	6.14E-01	5.43E-01	1.21E-03	1.57E-03	9.73E-04	PEC
	3.0	5.93E-01	0.626451	6.32E-01	5.64E-01	1.15E-03	1.54E-03	8.20E-04	PEC
	3.5	6.18E-01	0.640548	6.46E-01	5.81E-01	5.10E-04	7.80E-04	1.39E-03	MLE
	4.0	6.45E-01	0.654754	6.60E-01	5.98E-01	8.81E-05	2.24E-04	2.27E-03	MLE
	4.5	6.53E-01	0.66751	6.74E-01	6.13E-01	2.15E-04	4.23E-04	1.57E-03	MLE
	5.0	6.55E-01	0.679749	6.86E-01	6.27E-01	6.12E-04	9.43E-04	7.76E-04	MLE

Conclusion

The given values of (t_i) are measured in months, and due to simulation results, the first best estimator of fuzzy hazard function is (MLE), the second best one is (PEC) and finally the third one is MOM, these results depend on estimated parameters also on (t_i) values as well as vague factor $(\tilde{k})$ for moderate values of $(\tilde{k} = 0.3, 0.6)$, the percentage of best estimators are proportional between three estimators (MOM, PEC, MLE), for this reasons the production operation must be always under consideration and observation to make $(\tilde{k}_i)$ as small as possible.

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