Mustansiriyah Journal of Pure and Applied Sciences, Print ISSN: 2957-9910. Online ISSN: 2957-9929, Vol. 3, No. 4, 2025, Pages 189-195
DOI: 10.47831/mjpas.v3i4.412

MJPAS

MUSTANSIRIYAH JOURNAL OF PURE AND APPLIED SCIENCES

Journal homepage: https://mjpas.uomustansiriyah.edu.iq/index.php/mjpas

RESEARCH ARTICLE - MATHEMATICS

Investigating the Traits of a New Quasi Paranormal Operator Created by Fuzzy Soft Hilbert Description

Salim Dawood Mohsen[1], Mohammed Jasim Mohammed[*]

[*,1]Mustansiriyah University, College of Education, Department of mathematics, Baghdad, Iraq

* *Corresponding author: E-mail: mr.mohammed.jassim@gmail.com*

Article Info.

Article history:

Received
12 April 2025

Accepted
18 May 2025

Publishing
30 September 2025

Abstract

This paper introduced a new type of fuzzy soft quasi paranormal operator (QFSPO) in Hilbert space (HS) and examined some of its main properties. To improve the operator's efficacy, it was compared to other operators in certain conditions. The fact that this type is interesting has been supported by very important theories for example, the possibility of applying the proposed operator condition to the inverse if it exists, and other theories.

The official journal published by the College of Education at Mustansiriyah University

Keywords: fuzzy theory (FT), fuzzy soft (FS), quasi fuzzy soft -paranormal operators (QFSPO), fuzzy soft Hilbert space, FS-Norm.

1. Introduction

Uncertainty and ambiguity abound in our lives, leading to major and complex challenges in all areas of economic, environmental, and other life. In 1965, Zadeh discovered his theory, which changed many mathematical concepts. It is the fuzzy theory (FT), which is considered a generalization of classical set [1]. Since then, the world has begun to rely on this innovative method for data analysis and accuracy in decision-making. Since then, the fuzzy set (FS) has solved all the problems that classical logic cannot solve, and it has become of interest to scientists and researchers because it is more general than the classical logic What is acquired under the work of the fuzzy theory is more expressive power and more comprehensive, and in addition, it is more systematic and easy to provide solutions with lower costs for many problems. [2-5]. While researchers are still preoccupied with the fuzzy theory (FT) and the events that it caused in terms of tremendous development, the scientist Molodtsov appeared with new development ideas in 1999, when he first presented his theory, which was known as the soft theory (ST) [6]. This development aims to deal with data in a more accurate way. It is an interesting topic, as the scientist Maji and others introduced a new concept, which is merging the two concepts, i.e. merging the fuzzy theory (FT) with the soft theory (ST), so that we have a new term that has revolutionized mathematics under the name fuzzy soft theory (FST) [7-8]. This is a remarkable achievement in itself, as it covers a wide area and scope, has a wonderful vision of the future, free from impurities, and pays great attention to every minute detail that produces the best results [9]. Research has continued on this topic, and many researchers have delved into these topics [10-14]. This development had to be included in the topics of functional analysis because of its extreme importance, as it is considered the cornerstone of all branches of pure mathematics, there is no doubt that the theory of operators [TO] is considered the most important pillar in this science, and it was necessary for it to have its share in the subject of fuzzy mathematics, and this is what we will work on in this research, Where they started studying fuzzy soft metric (FSMS), fuzzy soft norm space(FSNS), etc.[15-23]. the soft fuzzy point was created to develop research in this field [24-25]. In 2020, Freud and a group of other researchers presented a theory that is considered one of the most important current theories, cited in many solid studies. This theory is the theory of fuzzy soft operators (FSO), through studying fuzzy soft linear operators (FSLO), with a discussion of a group of important advantages and characteristics. In addition, many applications were proposed by a significant number of researchers [26-30].

2. Preliminaries

This section outlines the crucial principles relevant to the study of $\eta -$ quasi fuzzy soft paranormal operator (briefly "$\eta - QFSPO$") within fuzzy soft Hilbert space (FS-HS).

Definition 2.1[28]: Assume that U is the universal set, The fuzzy set $\tilde{A}$ in U is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x): x \in U\}$, where $\mu_{\tilde{A}}: U \to [0,1]$ is said the membership function.

Definition 2.2 [8]: Assume that U is the universal set, with D as a set of parameters,$P(U)$ the power set of U,and $\tilde{A} \subseteq D$,let H be a function from $\tilde{A}$ to $P(U)$, $H_{\tilde{A}} = \{H(e) \in P(U): e \in \tilde{A}\}$. $H_{\tilde{A}}$ or the pair is $(H, \tilde{A})$ said to be soft set on U with regards $\tilde{A}$

Definition 2.3 [28]: Let $H: \tilde{A} \to I^U$ is mapping and fuzzy soft collection is $\{e \in A, H_{\tilde{A}}(e) \in I^U\}$ then $H_{\tilde{A}}$ is said to be fuzzy soft set on U and symbolized by FS.

Definition 2.4 [10]: The fuzzy soft point (FS-Point) on U, It is a special case of FSS, If $x \in U$ and e$\in$ A, such that:

$$\mu_{H(e)}(e) = \begin{cases} \alpha \text{ if } x = x_o \; and \; e = e_0 \in \tilde{A} \\ 0 \; if \; u \in U - \{x_0\} \; or \; e \in A - \{e_0\} \; where \; \alpha \in (0,1] \end{cases}$$

Definition 2.5 [27]: let $\widetilde{U}$ fuzzy soft vector, and a function $\widetilde{\|\,.\,\|}: \widetilde{U} \to \tilde{R}^{+}(\tilde{A})$ is said FS-Norm on $\widetilde{U}$, if $\widetilde{\|\,.\,\|}$ satisfies the following:

(a) $\|\widetilde{\tilde{x}_{\mu_{H(e)}}}\| \geq \tilde{0} \; \forall \tilde{x}_{\mu_{H(e)}} \in \widetilde{U}$, and $\|\widetilde{\tilde{x}_{\mu_{H(e)}}}\| = \tilde{0} \leftrightarrow \tilde{x}_{\mu_{H(e)}} = \tilde{0}$

(b) $\|\tilde{r} \cdot \widetilde{\tilde{x}_{\mu_{H(e)}}}\| = |\tilde{r}| \|\widetilde{\tilde{x}_{\mu_{H(e)}}}\|$, $\forall \tilde{x}_{\mu_{H(e)}} \in \widetilde{U}, \forall \tilde{r} \in \tilde{R}^{+}(\tilde{A})$.

(c) $\|\widetilde{\tilde{x}_{\mu_{1_{H(e_1)}}} + \tilde{y}_{\mu_{2_{H(e_2)}}}}\| \leq \|\widetilde{\tilde{x}_{\mu_{1_{H(e_1)}}}}\| + \|\widetilde{\tilde{y}_{\mu_{2_{H(e_2)}}}}\|, \forall \; \tilde{x}_{\mu_{1_{H(e_1)}}}, \tilde{y}_{\mu_{2_{H(e_2)}}} \in \widetilde{U}$.

$\widetilde{U}$ with FS-Norm $\widetilde{\|\,.\,\|}$ is called FS-Normed space (FSN-Space).

Definition 2.6 [27]: The mapping $\widetilde{\langle .,. \rangle} : \widetilde{U} \times \widetilde{U} \to \left(\tilde{C}(A) or \; \tilde{R}(A)\right)$ is said to be FS-inner product over FS-vector space $\widetilde{U}$, Provided that the following conditions are satisfied:

(a) $\langle \widetilde{\tilde{x}_{\mu_{1_{H(e_1)}}}, \tilde{x}_{\mu_{1_{H(e_1)}}}} \rangle \geq \tilde{0}$ and $\langle \widetilde{\tilde{x}_{\mu_{1_{H(e_1)}}}, \tilde{x}_{\mu_{1_{H(e_1)}}}} \rangle = \tilde{0} \Leftrightarrow \tilde{x}_{\mu_{1_{H(e_1)}}} = \tilde{0}$

(b) $\langle \widetilde{\tilde{x}_{\mu_{1_{H(e_1)}}}, \tilde{y}_{\mu_{2_{\mathcal{G}(e_2)}}}} \rangle = \overline{\langle \widetilde{\tilde{y}_{\mu_{2_{H(e_2)}}}, \tilde{x}_{\mu_{1_{H(e_1)}}}} \rangle}$

(c) $\langle \widetilde{\tilde{\alpha} \tilde{x}_{\mu_{1_{H(e_1)}}}, \tilde{y}_{\mu_{2_{H(e_2)}}}} \rangle = \tilde{\alpha} \langle \widetilde{\tilde{x}_{\mu_{1_{H(e_1)}}}, \tilde{y}_{\mu_{2_{H(e_2)}}}} \rangle$, for all $\tilde{\alpha} \in \tilde{C}(A)$.

(d) $\langle \widetilde{\tilde{x}_{\mu_{1_{H(e_1)}}} + \tilde{y}_{\mu_{2_{H(e_2)}}}, \tilde{z}_{\mu_{3_{H(e_3)}}}} \rangle = \langle \widetilde{\tilde{x}_{\mu_{1_{H(e_1)}}}, \tilde{z}_{\mu_{3_{H(e_3)}}}} \rangle + \langle \widetilde{\tilde{y}_{\mu_{2_{H(e_2)}}}, \tilde{z}_{\mu_{3_{H(e_3)}}}} \rangle$

$\forall \tilde{x}_{\mu_{1H(e_1)}}, \tilde{y}_{\mu_{2H(e_2)}}, \tilde{z}_{\mu_{3H(e_3)}} \in \tilde{U}$

$\tilde{U}$ with FS-inner product $\widetilde{\langle .,. \rangle}$ said to FS-inner product space (FSI), and denoted by $(\tilde{U}, \widetilde{\langle .,. \rangle})$.

Definition 2.7: [27]: Let $\tilde{H}$ be FS-inner product space and it is called fuzzy soft Hilbert space if $\tilde{H}$ is complete space and symbolize by FSH-space.

Definition 2.8: [30]: Assume that $\tilde{H}$ be FSHS and $\tilde{Ŧ}: \tilde{H} \to \tilde{H}$is FS-operator thus $\tilde{Ŧ}$ is said to be fuzzy soft linear operator (FSL-operator) if:

a) $\tilde{Ŧ}(\widetilde{\tilde{x}_{\mu 1H(e1)} + \tilde{y}_{\mu 2H(e2)}}) = \tilde{Ŧ}(\widetilde{\tilde{x}_{\mu 1H(e1)}}) + \tilde{Ŧ}(\widetilde{\tilde{y}_{\mu 2H(e2)}})$, For all $\tilde{x}_{\mu 1H(e1)}, \tilde{y}_{\mu 2H(e2)} \in \tilde{H}$

b) $\tilde{Ŧ}(\widetilde{\lambda \tilde{x}_{\mu 1H(e1)}}) = \lambda \tilde{Ŧ}(\widetilde{\tilde{x}_{\mu 1H(e1)}})$, For all $\tilde{x}_{\mu 1H(e1)} \in \tilde{H}$ and $\lambda \in \tilde{R}^{+}(\tilde{A})$.

Definition 2.9: [30]: Assume that $\tilde{H}$ is FSH-space and $\tilde{Ŧ}$ belongs to $\tilde{B}(\tilde{H})$, then FS-adjoint operator $\tilde{Ŧ}^{*}$is defined as:$\langle \widetilde{\tilde{Ŧ}\tilde{x}_{\mu 1H(e1)}, \tilde{y}_{\mu 2H(e2)}} \rangle \cong \langle \widetilde{\tilde{x}_{\mu 1H(e1)}, \tilde{Ŧ}^{*} \tilde{y}_{\mu 2H(e2)}} \rangle \; \forall \, \tilde{x}_{\mu 1H(e1)}, \tilde{y}_{\mu 2H(e2)} \tilde{\in} \tilde{H}$, where $\tilde{B}(\tilde{H})$ The collection of all fuzzy soft bounded linear operator in Hilbert space.

Definition 2.10: [21]: Assume that $\tilde{H}$ be FSH-space and $\tilde{Ŧ}$ belongs to $\tilde{B}(\tilde{H})$, then $\tilde{Ŧ}$ is called fuzzy soft normal operator (FS-Normal) if $\tilde{Ŧ}\tilde{Ŧ}^{*} \cong \tilde{Ŧ}^{*}\tilde{Ŧ}$

Definition 2.11: [22]: Let $\tilde{Ŧ}$ FS-operator of $\tilde{H}$. Then $\tilde{Ŧ}$ is called FS-self adjoint operator if $\tilde{Ŧ} \cong \tilde{Ŧ}^{*}$.

Definition 2.12:[28]: Assume that $\tilde{H}$ is FSH-space and $\tilde{Ŧ}$ belongs to $\tilde{B}(\tilde{H})$, then $\tilde{Ŧ}$ is said to be FS-isometry operator if achieve $\langle \widetilde{\tilde{Ŧ}\, \tilde{x}_{\mu 1H(e1)}, \tilde{Ŧ}\, \tilde{y}_{\mu 2H(e2)}} \rangle \cong \langle \widetilde{\tilde{x}_{\mu 1H(e1)}, \tilde{y}_{\mu 2H(e2)}} \rangle \; \forall \, \tilde{x}_{\mu 1H(e1)}, \tilde{y}_{\mu 2H(e2)} \tilde{\in} \tilde{H}$,

Definition 2.13 [27]: AFS-operator Fuzzy $\tilde{I}$ it is called soft identity operator $\tilde{I} : \tilde{H} \to \tilde{H}$if $\tilde{I}(\tilde{x}_{\mu 1H(e1)}) = \tilde{x}_{\mu 1H(e1)}, \forall \tilde{x}_{\mu 1H(e1)} \tilde{\in} \tilde{H}$
Definition 2.14 [30]: Assume that $\tilde{H}$ is FSH-space and $\tilde{Ŧ}$ belongs to $\tilde{B}(\tilde{H})$, then $\tilde{Ŧ}$ is said FS-unitary operator if achieve $\tilde{Ŧ}\tilde{Ŧ}^{*} = \tilde{I} = \tilde{Ŧ}^{*}\tilde{Ŧ}$.

Definition 2.15 [30]: Assume that $\tilde{H}$ is FSH-space with $\tilde{Ŧ}$ belongs to $\tilde{B}(\tilde{H})$, then $\tilde{Ŧ}$ be FS-paranormal operator if achieved : $\|\widetilde{Ŧ^{2}\tilde{x}_{\mu H(e)}}\| \|\widetilde{\tilde{x}_{\mu H(e)}}\| \tilde{\geq} \|\widetilde{\tilde{Ŧ}\tilde{x}_{\mu H(e)}}\|^{2} \forall \, \tilde{x}_{\mu H(e)} \tilde{\in} \tilde{H}$ So: $\|\widetilde{\tilde{Ŧ}^{2}\tilde{x}_{\mu H(e)}}\| \tilde{\geq} \|\widetilde{\tilde{Ŧ}\tilde{x}_{\mu H(e)}}\|^{2}$ for all $\tilde{x}_{\mu H(e)}$ belongs in $\tilde{H}$.

We know that $\tilde{H}$ is FS-Hilbert space with $\tilde{Ŧ}$ belongs to $\tilde{B}(\tilde{H})$, hence $\tilde{Ŧ}$ is called FS-Paranormal operator and symbolized by (FSPO).

3. Results and discussion

In this section, we present an important definition within the theory operators and discuss the theoretical aspects and properties related to this term within the framework of the fuzzy soft theory in Hilbert space.

Definition 3.1: Let $\tilde{H}$ is FSH-space with $\tilde{Ŧ}$ belongs in $\tilde{B}(\tilde{H})$, An operator $\tilde{Ŧ}$ is said η −quasi fuzzy soft paranormal operator, for anon negative integer η,(briefly "$\eta -$ QFSPO")if it achieves $\left\|\widetilde{\tilde{Ŧ}^{\eta+1}\tilde{x}_{\mu H(e)}}\right\|^{2} \tilde{\leq} \|\widetilde{\tilde{Ŧ}^{\eta+2}\tilde{x}_{\mu H(e)}}\| \|\widetilde{\tilde{Ŧ}^{\eta}\tilde{x}_{\mu H(e)}}\|$ for all $\tilde{x}_{\mu H(e)} \tilde{\in} \tilde{H}$.

Remark 3.2: From the above definition, the following fact: for $= 1$, 1-fuzzy soft quasi paranormal operator, obviously gives the class of quasi fuzzy soft paranormal operator for any $\tilde{x}_{\mu H(e)} \tilde{\in} \tilde{H}$.

Theorem 3.3: The $\eta -$ quasi fuzzy soft paranormal operator is closed under scalar multiplication.

Proof: Assume that $\tilde{Ŧ}$ belongs in $\tilde{B}(\tilde{H})$ is η -QFSPO and let ξ be any complex scalar.

For all $\tilde{x}_{\mu H(e)} \tilde{\in} \tilde{H}$ we have

$$\left\|(\xi\tilde{Ŧ})^{\widetilde{\eta+1}}\tilde{x}_{\mu H(e)}\right\|^2 \cong |\xi|^{2\eta+2}\left\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\right\|^2 \lesssim |\xi|^{2\eta+2}(\|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\|\|\tilde{Ŧ}^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|)$$

$$\cong \|(\xi\tilde{Ŧ})^{\widetilde{\eta+2}}\tilde{x}_{\mu H(e)}\|\|(\xi\tilde{Ŧ})^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|$$, then, $\xi\tilde{Ŧ}$ is also η -QFSPO.

Theorem 3.4: If $\tilde{Ŧ}$ belongs in $\tilde{B}(\tilde{H})$, is invertible η -QFSPO, then $\tilde{Ŧ}^{-1}$ is also η -QFSPO.

Proof: Since $\tilde{Ŧ}$ is a η -QFSPO, then $\left\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\right\|^2 \lesssim \|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\|\|\tilde{Ŧ}^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|$ for all $\tilde{x}_{\mu H(e)} \tilde{\in} \tilde{H}$.

Then: $\dfrac{\left\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\right\|^2}{\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\|} \lesssim \dfrac{\|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\|\|\tilde{Ŧ}^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\|}$

$$\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\| \lesssim \frac{\|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\|\|\tilde{Ŧ}^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\|}$$

$$\frac{\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\|} \lesssim \frac{\|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\|\|\tilde{Ŧ}^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\|\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\|}$$

$\dfrac{\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\|} \lesssim \dfrac{\|\tilde{Ŧ}^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\|}$, Now replacing $\tilde{x}_{\mu H(e)}$ by $\tilde{Ŧ}^{-\eta-2}\,\tilde{x}_{\mu H(e)}$, we have

$$\frac{\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{Ŧ}^{-2\eta-2}}\,\tilde{x}_{\mu H(e)}\|}{\|\tilde{Ŧ}^{\eta+2}\widetilde{\tilde{Ŧ}^{-2\eta-2}}\,\tilde{x}_{\mu H(e)}\|} \lesssim \frac{\|\tilde{Ŧ}^{\eta}\widetilde{\tilde{Ŧ}^{-2\eta-2}\,\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{\eta+1}\widetilde{\tilde{Ŧ}^{-2\eta-2}}\,\tilde{x}_{\mu H(e)}\|}$$

$$\frac{\left\|\tilde{Ŧ}^{-\eta-1}\,\widetilde{\tilde{x}_{\mu H(e)}}\right\|}{\left\|\tilde{Ŧ}^{-\eta}\,\widetilde{\tilde{x}_{\mu H(e)}}\right\|} \lesssim \frac{\left\|\tilde{Ŧ}^{-\eta-2}\,\widetilde{\tilde{x}_{\mu H(e)}}\right\|}{\left\|\tilde{Ŧ}^{-\eta-1}\,\widetilde{\tilde{x}_{\mu H(e)}}\right\|}$$

$$\frac{\|\tilde{Ŧ}^{-\eta-1}\,\widetilde{\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{-\eta}\,\widetilde{\tilde{x}_{\mu H(e)}}\|}.\|\tilde{Ŧ}^{-\eta-1}\,\widetilde{\tilde{x}_{\mu H(e)}}\| \lesssim \frac{\|\tilde{Ŧ}^{-\eta-2}\,\widetilde{\tilde{x}_{\mu H(e)}}\|}{\|\tilde{Ŧ}^{-\eta-1}\,\widetilde{\tilde{x}_{\mu H(e)}}\|}.\|\tilde{Ŧ}^{-\eta-1}\,\widetilde{\tilde{x}_{\mu H(e)}}\|$$

$$\frac{\|\tilde{Ŧ}^{-\eta-1}\,\widetilde{\tilde{x}_{\mu H(e)}}\|^2}{\|\tilde{Ŧ}^{-\eta}\,\widetilde{\tilde{x}_{\mu H(e)}}\|}\|\tilde{Ŧ}^{-\eta}\,\widetilde{\tilde{x}_{\mu H(e)}}\| \lesssim \|\tilde{Ŧ}^{-\eta-2}\,\widetilde{\tilde{x}_{\mu H(e)}}\|\|\tilde{Ŧ}^{-\eta}\,\widetilde{\tilde{x}_{\mu H(e)}}\|$$

$\|\tilde{Ŧ}^{-\eta-1}\,\widetilde{\tilde{x}_{\mu H(e)}}\|^2 \lesssim \|\tilde{Ŧ}^{-\eta-2}\,\widetilde{\tilde{x}_{\mu H(e)}}\|\|\tilde{Ŧ}^{-\eta}\,\widetilde{\tilde{x}_{\mu H(e)}}\|$ for all $\tilde{x}_{\mu H(e)} \tilde{\in} \tilde{H}$. This shows that $\tilde{Ŧ}^{-1}$ η -QFSPO.

Theorem 3.5: Assume $\tilde{Ŧ}$ belongs to $\tilde{B}(\tilde{H})$ is η -QFSPO, If $\tilde{ʄ}$ is unitary equivalent to operator $\tilde{Ŧ}$, then $\tilde{ʄ}$ is a η -QFSPO.

Proof: Let $\tilde{Ŧ}$ belongs to $\tilde{B}(\tilde{H})$ is η -QFSPO,

given that operator $\tilde{ʄ}$ is unitary equivalent to operator $\tilde{Ŧ}$, there exists an unitary operator U so that $\tilde{ʄ} = U^*\tilde{Ŧ}\,U$,and $\tilde{Ŧ}$ is η -QFSPO then:

$\tilde{Ŧ}^{*\eta+1}\tilde{Ŧ}^{\eta+1} - 2\lambda\tilde{Ŧ}^{*\eta+2}\tilde{Ŧ}^{\eta+2} + \lambda^2\tilde{Ŧ}^{*\eta}\tilde{Ŧ}^{\eta} \gtrsim 0$ for all non-negative λ.

Hence, $\tilde{ʄ}^{*\eta+2}\tilde{ʄ}^{\eta+2} - 2\lambda\tilde{ʄ}^{*\eta+1}\tilde{ʄ}^{\eta+1} + \lambda^2\tilde{ʄ}^{*\eta}\tilde{ʄ}^{\eta} \gtrsim 0, \forall\lambda > 0$

$= \left(U^*\tilde{Ŧ}U\right)^{*\eta+2}\left(U^*\tilde{Ŧ}U\right)^{\eta+2} - 2\lambda\left(U^*\tilde{Ŧ}U\right)^{*\eta+1}\left(U^*\tilde{Ŧ}U\right)^{\eta+1} + \lambda^2\left(U^*\tilde{Ŧ}U\right)^{*\eta}(U^*\tilde{Ŧ}\,U)^{\eta} \gtrsim 0,$

$= U^*(\tilde{Ŧ}^{*\eta+1}\tilde{Ŧ}^{\eta+1} - 2\lambda\tilde{Ŧ}^{*\eta+2}\tilde{Ŧ}^{\eta+2} + \lambda^2\tilde{Ŧ}^{*\eta}\tilde{Ŧ}^{\eta})U \gtrsim 0, \forall\lambda > 0.$ So $\tilde{ʄ}$ is η -QFSPO.

Theorem 3.6: Assume $\widetilde{Ŧ}$ belongs in $\widetilde{B}(\widetilde{H})$ is η -QFSPO, if $\widetilde{Ŧ}$ an isometric operator with $\mathcal{S}$, then $\widetilde{Ŧ}\mathcal{S}$ is η -QFSPO.

Proof: Assume that $\mathcal{S}$ an isometric operator and let be $M = \widetilde{Ŧ}\mathcal{S}$. Given that $\widetilde{Ŧ}$ an isometric operator with $\mathcal{S}$ so that $\widetilde{Ŧ}\mathcal{S} = \mathcal{S}\widetilde{Ŧ}$, $\widetilde{Ŧ}\mathcal{S}^* = \mathcal{S}^*\widetilde{Ŧ}$ and $\mathcal{S}^*\mathcal{S} = I$.

Now, $M^{*\eta+1}M^{\eta+1} - 2\lambda M^{*\eta+2}M^{\eta+2} + \lambda^2 M^{*\eta}M^{\eta} \gtrapprox 0,$

$= (\widetilde{Ŧ}\mathcal{S})^{*\eta+1}(\widetilde{Ŧ}\mathcal{S})^{\eta+1} - 2\lambda(\widetilde{Ŧ}\mathcal{S})^{*\eta+2}(\widetilde{Ŧ}\mathcal{S})^{\eta+2} + \lambda^2(\widetilde{Ŧ}\mathcal{S})^{*\eta}(\widetilde{Ŧ}\mathcal{S})^{\eta} \gtrapprox 0,$

$= \widetilde{Ŧ}^{*3+t}\widetilde{Ŧ}^{3+t} - 2\lambda\widetilde{Ŧ}^{*2+t}\widetilde{Ŧ}^{2+t} + \lambda^2\widetilde{Ŧ}^{*1+t}\widetilde{Ŧ}^{1+t} \gtrapprox 0$, So $\widetilde{Ŧ}\mathcal{S}$ is η -QFSPO.

Theorem 3.7: Assume $\widetilde{Ŧ}$ belongs in $\widetilde{B}(\widetilde{H})$ be η -QFSPO. If $\mathcal{A}$ is a closed $\widetilde{Ŧ}$ invariant subset of $\widetilde{H}$, then, the restriction $\widetilde{Ŧ}_{/\mathcal{A}}$ is a η -QFSPO.

Proof: Let $\widetilde{Ŧ}$ belongs in $\widetilde{B}(\widetilde{H})$ is η -QFSPO, $\|\widetilde{(\widetilde{Ŧ}_{/\mathcal{A}})^{\eta+1}\tilde{x}_{\mu H(e)}}\|^2 \cong \left\|\widetilde{Ŧ}^{\eta+1}\widetilde{\tilde{x}_{\mu H(e)}}\right\|$

$$\lessapprox \left\|\widetilde{Ŧ}^{\eta+2}\widetilde{\tilde{x}_{\mu H(e)}}\right\| \left\|\widetilde{Ŧ}^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\right\|$$

$$\cong \left\|\widetilde{(\widetilde{Ŧ}_{/\mathcal{A}})^{\eta+2}}\tilde{x}_{\mu H(e)}\right\| \left\|\widetilde{(\widetilde{Ŧ}_{/\mathcal{A}})^{\eta}\tilde{x}_{\mu H(e)}}\right\|,$$

This implies that $(\widetilde{Ŧ}_{/\mathcal{A}})$ is η -QFSPO.

Theorem 3.8: Assume $\widetilde{Ŧ}_1 , \widetilde{Ŧ}_2$ belongs in $\widetilde{B}(\widetilde{H})$ are $\eta -$ QFSPO, and $\widetilde{Ŧ}_1 \, \widetilde{Ŧ}_2 = \widetilde{Ŧ}_2 \, \widetilde{Ŧ}_1$ then the product $\widetilde{Ŧ}_1 . \widetilde{Ŧ}_2$ is η -QFSPO.

Proof: Since $\widetilde{Ŧ}_1 , \widetilde{Ŧ}_2$ commute, we have

$(\widetilde{Ŧ}_1 . \widetilde{Ŧ}_2)^{\eta+2} = \widetilde{Ŧ}_1{}^{\eta+2} . \widetilde{Ŧ}_2{}^{\eta+2}$, $(\widetilde{Ŧ}_1 . \widetilde{Ŧ}_2)^{\eta+1} = \widetilde{Ŧ}_1{}^{\eta+1} . \widetilde{Ŧ}_2{}^{\eta+1}$ and $(\widetilde{Ŧ}_1 . \widetilde{Ŧ}_2)^{\eta} = \widetilde{Ŧ}_1{}^{\eta} . \widetilde{Ŧ}_2{}^{\eta}$

By the property of the norm

$\left\|(\widetilde{Ŧ}_1 \; \widetilde{Ŧ}_2)\widetilde{^{\eta+1}\tilde{x}}_{\mu H(e)}\right\| \cong \left\|\widetilde{Ŧ}_1{}^{\eta+1} \; \widetilde{Ŧ}_2{}^{\widetilde{\eta+1}}\tilde{x}_{\mu H(e)}\right\| \lessapprox \left\|\widetilde{\widetilde{Ŧ}_1{}^{\eta+1}}\right\| \left\|\widetilde{Ŧ}_2{}^{\widetilde{\eta+1}}\tilde{x}_{\mu H(e)}\right\|$

Squaring both sides $\left\|(\widetilde{Ŧ}_1 . \widetilde{Ŧ}_2)\widetilde{^{\eta+1}\tilde{x}}_{\mu H(e)}\right\|^2 \lessapprox \left\|\widetilde{\widetilde{Ŧ}_1{}^{\eta+1}}\right\|^2 \left\|\widetilde{Ŧ}_2{}^{\widetilde{\eta+1}}\tilde{x}_{\mu H(e)}\right\|^2$

By hypothesis, each operator$(\widetilde{Ŧ}_1 , \widetilde{Ŧ}_1)$ satisfies η -QFSPO.

$\left\|(\widetilde{Ŧ}_1)\widetilde{^{\eta+1}\tilde{x}}_{\mu H(e)}\right\|^2 \lessapprox \|(\widetilde{Ŧ}_1)\widetilde{^{\eta+2}\tilde{x}}_{\mu H(e)}\| \|(\widetilde{Ŧ}_1)^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|$ for all $\tilde{x}_{\mu H(e)} \widetilde{\in} \widetilde{H}$.

$\left\|(\widetilde{Ŧ}_2)\widetilde{^{\eta+1}\tilde{x}}_{\mu H(e)}\right\|^2 \lessapprox \|(\widetilde{Ŧ}_2)\widetilde{^{\eta+2}\tilde{x}}_{\mu H(e)}\| \|(\widetilde{Ŧ}_2)^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|$ for all $\tilde{x}_{\mu H(e)} \widetilde{\in} \widetilde{H}$.

Multiplying these two inequalities yields:

$\left\|(\widetilde{Ŧ}_1)\widetilde{^{\eta+1}\tilde{x}}_{\mu H(e)}\right\|^2 \left\|(\widetilde{Ŧ}_2)\widetilde{^{\eta+1}\tilde{x}}_{\mu H(e)}\right\|^2 \lessapprox \left\|\widetilde{(\widetilde{Ŧ}_1)^{\eta+2}}\right\| \left\|(\widetilde{Ŧ}_2)\widetilde{^{\eta+2}\tilde{x}}_{\mu H(e)}\right\| \left\|(\widetilde{Ŧ}_1)^{\eta}\right\| \|(\widetilde{Ŧ}_2)^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|$

Under the additional assumption that the norm is multiplicative for these powers i.e.

$\left\|(\widetilde{Ŧ}_1 \, \widetilde{Ŧ}_2\widetilde{)^{\eta+2}}\tilde{x}_{\mu H(e)}\right\| \|((\widetilde{Ŧ}_1 \, \widetilde{Ŧ}_2)^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\| \cong \left\|\widetilde{(\widetilde{Ŧ}_1)^{\eta+2}}\right\| \left\|(\widetilde{Ŧ}_2)\widetilde{^{\eta+2}\tilde{x}}_{\mu H(e)}\right\| \left\|(\widetilde{Ŧ}_1)^{\eta}\right\| \|(\widetilde{Ŧ}_2)^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|,$

By combining the above inequalities, we get:

$\left\|(\widetilde{Ŧ}_1 . \widetilde{Ŧ}_2)\widetilde{^{\eta+1}\tilde{x}}_{\mu H(e)}\right\|^2 \lessapprox \left\|\widetilde{(\widetilde{Ŧ}_1)^{\eta+1}}\right\|^2 \left\|(\widetilde{Ŧ}_2)\widetilde{^{\eta+1}\tilde{x}}_{\mu H(e)}\right\|^2$

$$\lessapprox \left\|\widetilde{(\widetilde{Ŧ}_1)^{\eta+2}}\right\| \left\|(\widetilde{Ŧ}_2)\widetilde{^{\eta+2}\tilde{x}}_{\mu H(e)}\right\| \left\|(\widetilde{Ŧ}_1)^{\eta}\right\| \|(\widetilde{Ŧ}_2)^{\eta}\widetilde{\tilde{x}_{\mu H(e)}}\|$$

$$\cong \|(\widetilde{T}_1 . \widetilde{T}_2 \widetilde{)^{\eta+2}} \tilde{x}_{\mu H(e)}\| \left\|\left(\widetilde{T}_1 . \widetilde{T}_2\right)^{\eta} \tilde{x}_{\mu H(e)}\right\|,$$

Thus, $\left\|(\widetilde{T}_1 . \widetilde{T}_2 \widetilde{)^{\eta+1} \tilde{x}_{\mu H(e)}}\right\|^2 \lesseqq \|(\widetilde{T}_1 . \widetilde{T}_2 \widetilde{)^{\eta+2}} \tilde{x}_{\mu H(e)}\| \left\|\left(\widetilde{T}_1 . \widetilde{T}_2\right)^{\eta} \tilde{x}_{\mu H(e)}\right\|$, then $\widetilde{T}_1 . \widetilde{T}_2$ is η −QFSPO.

4. Conclusions

In this paper, we introduce (η -QFSPO) in a fuzzy soft Hilbert space (FS-HS). We also provide useful observations and properties that may pave the way for future work related to this class of operators. This class can also solve some types of differential equations When certain conditions mentioned in this research are met

Acknowledgement

We would like to thank Al-Mustansiriya University for its continuous support, especially the College of Education. In addition, we extend our gratitude to the reviewers for their valuable comments.

References

[1] L. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] M. Riaz and M. R. Hashmi, “Linear diophantine fuzzy set and its applications towards multi-attribute decision-making problems,” *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 4, pp. 5417–5439, 2019.

[3] M. Riaz and S. T. Tehrim, “Cubic bipolar fuzzy ordered weighted geometric aggregation operators and their application using internal and external cubic bipolar fuzzy data,” *Computational and Applied Mathematics*, vol. 38, no. 87, pp. 1–25, 2019.

[4] M. Riaz and S. T. Tehrim, “Multi-attribute group decision making based on cubic bipolar fuzzy information using averaging aggregation operators,” *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 2, pp. 2473–2494, 2019.

[5] M. R. Hashmi, M. Riaz, and F. Smarandache, “M-polar neutrosophic topology with applications to multi-criteria decision making in medical diagnosis and clustering analysis,” *International Journal of Fuzzy Systems*, vol. 22, no. 1, pp. 273–292, 2020.

[6] D. Molodtsov, "Soft set theory-First results," *Computers and Mathematics with Applications*, vol. 37, no. 4–5, pp. 19–31, 1999.

[7] P. K. Maji, A. R. Roy, and R. Biswas, “An application of soft sets in a decision making problem,” *Computers and Mathematics with Applications*, vol. 44, pp. 1077–1083, 2002.

[8] P. K. Maji, R. Biswas, and A. R. Roy, “Soft set theory,” *Computers and Mathematics with Applications*, vol. 45, pp. 555–562, 2003.

[9] S. Das and S. K. Samanta, “Soft Metric,” *Annals of Fuzzy Mathematics and Informatics*, vol. 6, no. 1, pp. 77–94, 2013.

[10] B. S. Reddy et al., “Soft n-normed linear spaces: generalizations and extensions from soft normed spaces,” *Communications in Mathematics and Applications*, vol. 15, no. 1, pp. 265–277, 2024.

[11] A. J. Edan, “Some results on co fuzzy metric space,” *Mustansiriyah Journal of Pure and Applied Sciences*, vol. 3, no. 2, pp. 88–95, 2025.

[12] S. D. Mohsen, “Novel Results of K Quasi (ℷ - M)-hyponormal Operator,” *Iraqi Journal of Science*, vol. 65, no. 1, pp. 374–380, 2024.

[13] S. D. Mohsen and Y. H. Thiyab, “Some Characteristics of Completeness Property In Fuzzy Soft b Metric Space,” *Journal of Applied Science and Engineering*, vol. 27, no. 3, pp. 2227–2232, 2023.

[14] M. I. Yazar et al., "Results on soft Hilbert spaces," *TWMS Journal of Applied and Engineering Mathematics*, vol. 9, no. 1, pp. 159–164, 2019.

[15] S. D. Mohsen and A. M. Khalaf, “Further characteristics of the (k, n, N)–quasinormal operators,” *AIP Conference Proceedings*, vol. 3282, no. 1, 2025.

[16] S. D. Mohsen, "On (n, D) quasi operators," *Iraqi Journal for Computer Science and Mathematics*, vol. 5, no. 1, pp. 175–180, 2024.
[17] A. M. A. El-latif et al., "Strictly wider class of soft sets via supra soft δ-closure operator," *Journal of Analysis and Applications*, vol. 22, no. 47, pp. 1–13, 2024.
[18] S. D. Mohsen and H. K. Mousa, "Another Results Related of Fuzzy Soft Quasi Normal Operator in Fuzzy Soft Hilbert Space," *Journal of Physics: Conference Series*, vol. 232, no. 1, p. 012050, 2022.
[19] S. D. Mohsen, "Solvability of (λ, μ) Commuting Operator Equations for Bounded Generalization of Hyponormal Operators," *Iraqi Journal of Science*, vol. 63, no. 9, pp. 3854–3860, 2022.
[20] N. Faried et al., "Fuzzy soft inner product spaces," *Applied Mathematics and Information Sciences*, vol. 14, no. 4, pp. 709–720, 2020.
[21] F. Nashat et al., "On Fuzzy Soft Linear Operators in Fuzzy Soft Hilbert Spaces," *Abstract and Applied Analysis*, vol. 2020, pp. 1–13, 2020.
[22] S. D. Mohsen, "On (k,m) - n −Paranormal Operators," *Iraqi Journal of Science*, vol. 64, no. 6, pp. 3087–3092, 2023.
[23] F. Nashat et al., "Fuzzy soft Hilbert spaces," *Journal of Mathematics and Computer Science*, vol. 22, no. 2, pp. 142–157, 2021.
[24] S. D. Mohsen, "Some generalizations of fuzzy soft (k^*– Â)-quasinormal operators in fuzzy soft Hilbert spaces," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 6, pp. 1133–1143, 2023.
[25] S. D. Mohsen, "New generalizations for M-hyponormal operators," *Iraqi Journal of Science*, vol. 64, no. 12, pp. 6477–6482, 2023.
[26] S. D. Mohsen and H. K. Mousa, "On fuzzy soft m-hyponormal operator," *Journal of Education for Pure Science-University of Thi-Qar*, vol. 12, no. 2, pp. 119–129, 2022.
[27] F. Nashat et al., "Fuzzy soft Hermitian operators," *Advances in Mathematics: Scientific Journal*, vol. 9, no. 1, pp. 73–82, 2020.
[28] P. Rajarajeswari and J. Vanitha, "Some special operators on multi interval valued fuzzy soft matrix," *IOSR Journal of Mathematics*, vol. 16, no. 4, pp. 48–54, 2020.
[29] A. Radharamani et al., "Fuzzy Unitary Operator in Fuzzy Hilbert Space and its Properties," *International Journal of Research and Analytical Reviews*, vol. 5, no. 4, pp. 258–261, 2018.
[30] A. Radharamani and T. Nagajothi, "Fuzzy Soft Paranormal Operator in Fuzzy Soft Hilbert Space," *Communications on Applied Nonlinear Analysis*, vol. 31, no. 2, pp. 129–141, 2024.