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Mixed Triple Optimal Control Vector Problem for Nonlinear Triple Elliptic System

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Abstract

This paper studies the mixed triple optimal control vector problem (MTOCVP) controlled by nonlinear system of triple elliptic PDEs (or for short TNES) under appropriate hypotheses. The existence theorem of a unique triple state vector solution (TSVS) for the weak formulation resulting from the TNES is demonstrated by employing the Browder theorem, with fixed mixed triple control vector (MTCV). The Lipschitz operator between the MTCV and its conforming TSVS is demonstrated continuous. The existence theorem of a mixed triple optimal control vector (MTOCV) controlling by the TNES is stated and demonstrated, under suitable assumptions.

Keywords: Mixed Triple Optimal Control Vector, Triple Nonlinear Elliptic System, Browder theorem, Lipschitz Continuous Operator.

مسألة السيطرة الامثلية المتجهه الثلاثية المختلطة لنظام اهليجي غير خطي

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الخلاصة

يتناول هذا البحث دراسة مسألة السيطرة الامثلية المتجهه الثلاثية المختلطة مسيطرة بنظام اهليجي ثلاثي غير خطي بوجود شروط ملائمة. تم برهان مبرهنة وجود متجه حل ثلاثي وحيد للصيغة الضعيفة الناتجة من النظام الاهليجي الثلاثي الغير خطي باستخدام مبرهنة بروادر مع كون متجه السيطرة الثلاثية المختلطة ثابتاً. تم برهان استمرارية مؤثر ليبشيتز بين متجه السيطرة الثلاثية المختلطة ومتجه الحل الثلاثي للحالة المقابل له. تم برهان مبرهنة وجود سيطرة امثلية متجهه مختلطة والمسيطرة بالنظام الاهليجي الثلاثي الغير خطي بوجود شروط مناسبة.

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1. Introduction

The subject of control theory has many implementations in real life. Mainly in the fields of science and engineering, like that in elective power [1], economy [2], medicine [3], robot science [4], conditions for the weather [5], removal mission for space debris[6], chemistry[7], biology [8], management [9], aircraft [10] and many other branches of the field of sciences. Optimal control problems (OCPs) towards the field of mathematical science are usually described by ordinary differential equations (ODEs) [11], [12] and partial differential equations (PDEs) [13], [15] in general. Through the past few years, many researchers concerned their focus on studding OCPs dominating by “single” PDEs of hyperbolic, parabolic and elliptic kinds [16]-[18], and by couple PDEs of theses the three kinds[19]- [21], whilst other investigators focused their studies about mixed OCPs controlling by couple PDEs of the kinds of parabolic and elliptic resp. [22] &[24]. On the other hand, the researchers [25]- [27] focused their attentions about studding OCPs dominating by triple PDEs of the three kinds of hyperbolic, parabolic and elliptic PDEs.

All of these previous studies motivated us to search for a new type of the OCPs, that is the MOCVP controlling with TNES, because this type of problem has not been addressed yet, and on the other hand, this type of study give a green light for the investigators about studding the numerical solution of this type of problems. Of course, such types of new study “on proposing mathematical model” requires generalization the governing PDEs (the TNES) including of the boundary conditions, and the objective function (OF) needs to be generalized.

The paper begins with giving a description about the new proposed problem, the MOCVP. At first the weak formulation (WF) of the TNES is formulated. Then the existence and uniqueness theorem of the TSVS of the WF of the TNES for a given MTCV is developed and demonstrated through utilizing the theorem of Browder [20] under appropriate hypotheses. The continuity of the Lipschitz operator (LIPO) between the MTCV and its conforming TSVS has also demonstrated. Finally, and under appropriate hypotheses, the existence theorem of a MTOCV relating with TNES is developed and demonstrated.

2. Problem Depiction

Consider Γ be a boundary of the bounded domain $\Omega \subset \mathbb{R}^2$. The MTOCVP is described by the following TNES:

$$A_1 y_1 + a_{10}(x)y_1 - b(x)y_2 - c(x)y_3 + f_1(x, \vec{y}, u_1) = f_2(x, u_1) \quad (1)$$

$$A_2 y_2 + a_{20}(x)y_2 + b(x)y_1 + d(x)y_3 + h_1(x, \vec{y}) = h_2(x) \quad (2)$$

$$A_3 y_3 + a_{30}(x)y_3 + c(x)y_1 - d(x)y_2 + k_1(x, \vec{y}, u_3) = k_2(x, u_3) \quad (3)$$

$$\sum_{i,j=1}^2 a_{1ij} \frac{\partial y_1}{\partial n} = 0, \quad \text{in } \Gamma \quad (4)$$

$$\sum_{i,j=1}^2 a_{2ij} \frac{\partial y_2}{\partial n} = u_2, \quad \text{in } \Gamma \quad (5)$$

$$\sum_{i,j=1}^2 a_{3ij} \frac{\partial y_3}{\partial n} = 0, \quad \text{in } \Gamma \quad (6)$$

where $A_\iota y_\iota = -\sum_{i,j=1}^2 \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial y_\iota}{\partial x_i})$, $a_{ij}(x), a_{10}(x), b(x), c(x), d(x) \in C^\infty(\Omega)$, for $\iota = 1, 2, 3$, $\vec{u} = (u_1, u_2, u_3) \in L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)$ is the MTCV, $\vec{y} = (y_1, y_2, y_3) = (y_1(x), y_2(x), y_3(x)) \in (H^1(\Omega))^3$ is its TSVS, $(f_1, h_1, k_1) = (f_1(x, h_1(x, \vec{y}), k_1(x, \vec{y}, u_3))) \in (L^2(\Omega))^3$ and $(f_2, h_2, k_2) = (f_2(x, u_1), h_2(x), k_2(x, u_3)) \in (L^2(\Omega))^3$ are a vector of given functions for all $x = (x_1, x_2) \in \Omega$.

The set of the MTCV is $\vec{W} = \{\vec{u} \in L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega) | \vec{u} \in \vec{U} \subset \mathbb{R}^3 \text{ a.e. in } \Omega \times \Gamma \times \Omega\}$, where $\vec{U} = U_1 \times U_2 \times U_3$ has the convexity property.

The objective function is represented as:

$$G_o(\vec{u}) = \iint_{\Omega} [g_{01}(x, \vec{y}, u_1)] dx_1 dx_2 + \int_{\Gamma} [g_{02}(x, u_2)] d\gamma + \iint_{\Omega} [g_{03}(x, \vec{y}, u_3)] dx_1 dx_2 \quad (7)$$

The MTOCVP is to minimize (7) subject to $\vec{u} \in \vec{W}$.

Let $\vec{V} = V \times V \times V = H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$. Let (v, v) and $(\vec{v}, \vec{v})_{(H^1(\Omega))^3}$ ($\|v\|_{H^1(\Omega)}$ and $\|\vec{v}\|_{(H^1(\Omega))^3} = \sum_{i=1}^3 \|v_i\|_{H^1(\Omega)}$) are represent the inner product (the norm) in $H^1(\Omega)$ and in $\vec{V}$.

2.1. Weak formulation of the TSVS:

The WF of ((1)-(6)) is

$$a_1(y_1, v_1) + (a_0 y_1, v_1)_{\Omega} - (b y_2, v_1)_{\Omega} - (c y_3, v_1)_{\Omega} + (f_1(\vec{y}, u_1), v_1)_{\Omega} = (f_2(u_1), v_1)_{\Omega}, \forall v_1 \in V_1 \quad (8)$$

$$a_2(y_2, v_2) + (a_0 y_2, v_2)_{\Omega} + (b y_1, v_2)_{\Omega} + (d y_3, v_2)_{\Omega} + (h_1(\vec{y}), v_2)_{\Omega} = (h_2, v_2)_{\Omega} + (u_2, v_2)_{\Gamma}, \forall v_2 \in V_2 \quad (9)$$

$$a_3(y_3, v_3) + (a_0 y_3, v_3)_{\Omega} + (c y_1, v_3)_{\Omega} - (d y_2, v_3)_{\Omega} + (k_1(\vec{y}, u_3), v_3)_{\Omega} = (k_2(x, u_3), v_3)_{\Omega}, \forall v_3 \in V_3 \quad (10)$$

where $(v, v)_{\Omega}$ ($(v, v)_{\Gamma}$) refer to the inner product in $L^2(\Omega)$ ($L^2(\Gamma)$) resp.

Adding ((8)-(10)) to get

$$a(\vec{y}, \vec{v}) + (f_1(\vec{y}, u_1), v_1)_{\Omega} + (h_1(\vec{y}), v_2)_{\Omega} + (k_1(\vec{y}, u_3), v_3)_{\Omega} = (f_2(u_1), v_1)_{\Omega} + (h_2, v_2)_{\Omega} + (u_2, v_2)_{\Gamma} + (k_2(x, u_3), v_3)_{\Omega} \quad (11)$$

where $a(\vec{y}, \vec{v}) = a_1(y_1, v_1) + (a_0 y_1, v_1)_{\Omega} - (b y_2, v_1)_{\Omega} - (c y_3, v_1)_{\Omega} + a_2(y_2, v_2) + (a_0 y_2, v_2)_{\Omega} + (b y_1, v_2)_{\Omega} + (d y_3, v_2)_{\Omega} + a_3(y_3, v_3) + (a_0 y_3, v_3)_{\Omega} + (c y_1, v_3)_{\Omega} - (d y_2, v_3)_{\Omega}$

with $a_i(y_i, v_i) = \sum_{i,j=1}^2 a_{ij} \frac{\partial y_i}{\partial x_i} \frac{\partial v_i}{\partial x_j}$, $i = 1, 2, 3$.

The following assumptions are useful in the study of the existence solution for the TSVS.

Assumptions (ASSU)1:

a) $a(\vec{y}, \vec{y}) \geq c \|\vec{y}\|_{(H^1(\Omega))^3}^2 > 0$, $\forall \vec{y} \in \vec{V}$.

b) $|a(\vec{y}, \vec{v})| \leq \ell_1 \|\vec{y}\|_{(H^1(\Omega))^3} \|\vec{v}\|_{(H^1(\Omega))^3}$, $\forall \vec{y} \in \vec{V}$, $\ell_1 > 0$

c) $f_1(x, \vec{y}, u_1)$, $h_1(x, \vec{y})$, $k_1(x, \vec{y}, u_3)$ are of type of Caratheodory (TC) on $\Omega \times \mathbb{R}^2 \times U_1$, $\Omega \times \mathbb{R}^2$ and $\Omega \times \mathbb{R}^2 \times U_3$ resp. and the following conditions for $\phi_1, \phi_2, \phi_3 \in L^2(\Omega)$ and $\bar{c}_1, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_3 \geq 0$ are held

$$|f_1(x, \vec{y}, u_1)| \leq \phi_1(x) + \bar{c}_1 |\vec{y}| + \bar{c}_1 |u_1|, |h_1(x, \vec{y})| \leq \phi_2(x) + \bar{c}_2 |\vec{y}|$$

$$|k_1(x, \vec{y}, u_3)| \leq \phi_3(x) + \bar{c}_3 |\vec{y}| + \bar{c}_3 |u_3|$$

d) $f_1(x, \vec{y}, u_1)$, $h_1(x, \vec{y})$, $k_1(x, \vec{y}, u_3)$ are monotonic (MC) for any $x \in \Omega$ w.r.t.

$(\vec{y}, u_1)$, $\vec{y}$, and $(\vec{y}, u_3)$ resp., with $f_1(x, 0, u_1) = 0$, $\forall (x, u_1) \in \Omega \times U_1$

$h_1(x, 0) = 0$, $\forall x \in \Omega$, $k_1(x, 0, u_3) = 0$, $\forall (x, u_3) \in \Omega \times U_3$.

e) $f_2(x, u_1)$, $h_2(x)$, $k_2(x, u_3)$ are of CT on $\Omega \times U_1$, Ω and $\Omega \times U_3$ resp. and the following conditions for $\phi_4, \phi_5, \phi_6 \in L^2(\Omega)$ and $\bar{c}_1, \bar{c}_3 \geq 0$ are satisfied

$$|f_2(x, u_1)| \leq \phi_4(x) + \bar{c}_1 |u_1|, \forall (x, u_1) \in \Omega \times U_1$$

$$|h_2(x)| \leq \phi_5(x), \quad \forall x \in \Omega$$

$$|k_2(x, u_3)| \leq \phi_6(x) + \bar{c}_3 |u_3|, \quad \forall (x, u_3) \in \Omega \times U_3$$

Main Results

3. Existence of a unique TSVS

The demonstration of existence of the unique TSVS for the WF (11) is demonstrated in the next theorem.

Theorem 3.1: In addition to ASSU1, if f_1 is monotone strictly. Then for each given MTCV $\vec{u} \in \vec{W}$, then $\vec{y} \in \vec{V}$ is the unique TSVS of WF (11).

Proof: Let $\bar{A}: \vec{V} \longrightarrow \vec{V}^*$, then the WF (11) can be expressed as

$$\langle \bar{A}(\vec{y}), \vec{V} \rangle = (\vec{F}(\vec{u}), \vec{V}) \quad (12)$$

where $\langle \bar{A}(\vec{y}), \vec{V} \rangle = a(\vec{y}, \vec{v}) + (f_1(\vec{y}, u_1), v_1)_\Omega + (h_1(\vec{y}), v_2)_\Omega + (k_1(\vec{y}, u_3), v_3)_\Omega$,

and $(\vec{F}(\vec{u}), \vec{V}) = (f_2(u_1), v_1)_\Omega + (h_2, v_2)_\Omega + (u_2, v_2)_\Gamma + (k_2(u_3), v_3)_\Omega$.

Then

- $\bar{A}$ is coercive (from ASSU1 (a, d)).
- The mapping $\vec{y} \longrightarrow \langle \bar{A}(\vec{y}), \vec{V} \rangle$ is continuous w.r.t. $\vec{y}$ (from ASSU1(b, c) with applying Proposition (3.1)).
- $\bar{A}$ is monotone strictly w.r.t. $\vec{y}$ (from ASSU1(a, d) with utilizing portion (i) above).

At the end by applying the Border's theorem, once get that the TSVS $\vec{y} \in \vec{V}$ of (12) is unique.

4. Existence of the MTOCV

In this section, the existence of a MTOCV is studied under appropriate hypotheses. The following lemmas and assumptions are necessary in this study.

Lemma 4.1: With ASSU1, if the functions $f_1, f_2, (k_1, k_2)$ are LIP w.r.t. $u_1(u_3)$, h_1 is LIP w.r.t. $\vec{y}$, h_2 is bounded. Thus the LIP operator $\vec{u} \longrightarrow \vec{y}_{\vec{u}}$ from $\vec{W} \in L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)$ satisfies the continuity property, i.e. $\|\vec{\Delta y}\|_{(L^2(\Omega))^3} \leq L \|\vec{\Delta u}\|_{L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)}$, with $L > 0$.

Proof: Let $\vec{u}, \vec{u} \in \vec{W}$ are two given MTOCV, then from Theorem 3.1 $\vec{y}$ and $\vec{y}$ represent the TSVS of (11), subtracting the WF (for $\vec{y}$) from that for $\vec{y}$, setting $\vec{\Delta y} = \vec{y} - \vec{y}$, $\vec{\Delta u} = \vec{u} - \vec{u}$, with $\vec{v} = \vec{\Delta y}$, then collecting the resulting WF to secure

$$\begin{aligned} & a_1(\Delta y_1, \Delta y_1) + (a_0 \Delta y_1, \Delta y_1)_\Omega + a_2(\Delta y_2, \Delta y_2) + (a_0 \Delta y_2, \Delta y_2)_\Omega + a_3(\Delta y_3, \Delta y_3) + \\ & (a_0 \Delta y_3, \Delta y_3)_\Omega + (f_1(\vec{y} + \vec{\Delta y}, u_1 + \Delta u_1) - f_1(\vec{y}, u_1), \Delta y_1)_\Omega + (h_1(\vec{y} + \vec{\Delta y}) - \\ & h_1(\vec{y}), \Delta y_2)_\Omega + (k_1(\vec{y} + \vec{\Delta y}, u_3 + \Delta u_3) - k_1(\vec{y}, u_3), \Delta y_3)_\Omega = (f_2(u_1 + \Delta u_1) - \\ & f_2(u_1), \Delta y_1)_\Omega + (\Delta u_2, \Delta y_2)_\Gamma + (k_2(u_3 + \Delta u_3) - k_2(u_3), \Delta y_3)_\Omega, \quad (13) \end{aligned}$$

Taking the absolute value of (13), it yields

$$\begin{aligned} c \|\vec{\Delta y}\|_{(H^1(\Omega))^3}^2 & \leq \left| (f_1(\vec{y} + \vec{\Delta y}, u_1 + \Delta u_1) - f_1(\vec{y}, u_1), \Delta y_1)_\Omega \right| + \\ & \left| (h_1(\vec{y} + \vec{\Delta y}) - h_1(\vec{y}), \Delta y_2)_\Omega \right| + \left| (f_2(u_1 + \Delta u_1) - f_2(u_1), \Delta y_1)_\Omega \right| + \\ & \left| (k_1(\vec{y} + \vec{\Delta y}, u_3 + \Delta u_3) - k_1(\vec{y}, u_3), \Delta y_3)_\Omega \right| + \left| (\Delta u_2, \Delta y_2)_\Gamma \right| + \\ & \left| (k_2(u_3 + \Delta u_3) - k_2(u_3), \Delta y_3)_\Omega \right| \quad (14) \end{aligned}$$

After utilizing ASSU1 (a, d), the LIP property, and the trace operator in (14), it becomes

$$\begin{aligned} c \|\vec{\Delta y}\|_{(H^1(\Omega))^3}^2 & \leq 2c_1 \|\vec{\Delta u}\|_{L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)} \|\vec{\Delta y}\|_{(H^1(\Omega))^3} \\ \implies \|\vec{\Delta y}\|_{(H^1(\Omega))^3} & \leq c_2 \|\vec{\Delta u}\|_{L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)}, \quad \text{with } c_2 = \frac{2c_1}{c} \end{aligned}$$

Therefore

$$\|\vec{\Delta y}\|_{(L^2(\Omega))^3} \leq L \|\vec{\Delta u}\|_{L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)}, \quad \text{with } L = cc_2.$$

ASSU2: Assume that g_{01}, g_{02}, g_{03} are of TC on $\Omega \times R^3 \times U_1$, $\Omega \times R^3 \times U_2$ and $\Omega \times R^3 \times U_3$ resp. and the following are held

$$|g_{0\tau}(x, \vec{y}, u_\tau)| \leq \psi_{0\tau}(x) + c_{0\tau} \vec{y}^2 + \bar{c}_{0\tau} u_\tau^2, \text{ where } \psi_{0\tau} \in L^1(\Gamma), \text{ and } \bar{c}_{0\tau} \geq 0, \text{ for } \tau = 1, 3, \\ |g_{02}(x, u_2)| \leq \psi_{02}(x) + \bar{c}_{02} u_2^2, \text{ where } \psi_{02} \in L^1(\Gamma), \text{ and } \bar{c}_{02} \geq 0.$$

Lemma 4.2 : With ASSU2, the functional $\vec{u} \longrightarrow G_0(\vec{u})$ which is defined on $L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)$ is continuous.

Proof: By employing Proposition 3.1 in [22] and ASSU2, the functional $\int_\Gamma g_{02}(x, u_2) dx$ and $\iint_\Omega g_{0\tau}(x, \vec{y}, u_\tau) dx$ (for $\tau = 1, 3$) are continuous on $(L^2(\Gamma))^3$ and $(L^2(\Omega))^3$, which give the continuity of

$$G_0(\vec{u}) = \iint_\Omega g_{01}(x, \vec{y}, u_1) dx + \int_\Gamma g_{02}(x, u_2) d\gamma + \iint_\Omega g_{03}(x, \vec{y}, u_3) dx \\ \text{on } L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega).$$

Theorem 4.1: Assume that $\vec{W} \neq \emptyset$, f_1, k_1 are independent functions of u_1 and u_3 resp. f_2, k_2 are linear w.r.t. u_1 and u_3 resp., and they satisfy (with $\phi_\tau \in L^2(\Omega)$, $\bar{c}_\tau \geq 0$, $\tau = 1, 2, 3$): $|f_1(x, \vec{y})| \leq \phi_1(x) + \bar{c}_1 |\vec{y}|$, $|h_1(x, \vec{y})| \leq \phi_2(x) + \bar{c}_2 |\vec{y}|$, and $|k_1(x, \vec{y})| \leq \phi_3(x) + \bar{c}_3 |\vec{y}|$. If G_0 is coercive, and $g_{0\tau}$ (for $\tau = 1, 2, 3$) satisfy the convexity property w.r.t u_τ , then there is a MTOCV.

Proof: From the condition $\vec{W} \neq \emptyset$, then there is $\vec{w} \in \vec{W}$, and a minimum sequence (seq.) $\{\vec{u}_n\} = \{(u_{1n}, u_{2n}, u_{3n})\} \in \vec{W}$, s.t. $\lim_{n \rightarrow \infty} G_0(\vec{u}_n) = \inf_{\vec{w} \in \vec{W}} G_0(\vec{w})$.

But from the theorem of Egorov [23] $\vec{W}$ is closed, but $\vec{W}$ satisfy the convexity and bounded property since $\vec{U}$ satisfy them, hence $\vec{W}$ is compact weakly, then there is a subsequence (subseq) of $\{\vec{u}_n\}$, let be again $\{\vec{u}_n\}$ which weakly converges to $\vec{u} \in \vec{W}$, i.e. $\vec{u}_n \longrightarrow \vec{u}$ weakly in $L^2(\Omega) \times L^2(\Gamma) \times L^2(\Omega)$. Then from Theorem 3.1, conforming to the seq. TMCV $\{\vec{u}_n\}$ there is a seq. of TSVS $\{\vec{y}_n\}$ with $\|\vec{y}_n\|_{(H^1(\Omega))^3}$ is bounded for all n of the seq. of the WF :

$$a_1(y_{1n}, v_1) + (a_0 y_{1n}, v_1)_\Omega - (b y_{2n}, v_1)_\Omega - (c y_{3n}, v_1)_\Omega + a_2(y_{2n}, v_2) + (a_0 y_{2n}, v_2)_\Omega + \\ (b y_{1n}, v_2)_\Omega + (d y_{3n}, v_2)_\Omega + a_3(y_{3n}, v_3) + (a_0 y_{3n}, v_3)_\Omega + (c y_{1n}, v_3)_\Omega - (d y_{2n}, v_3)_\Omega + \\ (f_1(\vec{y}_n), v_1)_\Omega + (h_1(\vec{y}_n), v_2)_\Omega + (k_1(\vec{y}_n), v_3)_\Omega = (f_2(x) u_{1n}, v_1)_\Omega + (h_2(x), v_2)_\Omega + \\ (u_{2n}, v_2)_\Gamma + (k_2(x) u_{3n}, v_3)_\Omega \quad (15)$$

Hence, there is a subseq of $\{\vec{y}_n\}$ (from Alaoglu's theorem) let be one more time $\{\vec{y}_n\}$ s.t. $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\vec{V}$.

To prove that (15) converges to:

$$a_1(y_1, v_1) + (a_0 y_1, v_1)_\Omega - (b y_2, v_1)_\Omega - (c y_3, v_1)_\Omega + a_2(y_2, v_2) + (a_0 y_2, v_2)_\Omega + \\ (b y_1, v_2)_\Omega + (d y_3, v_2)_\Omega + a_3(y_3, v_3) + (a_0 y_3, v_3)_\Omega + (c y_1, v_3)_\Omega - (d y_2, v_3)_\Omega + \\ (f_1(\vec{y}), v_1)_\Omega + (h_1(\vec{y}), v_2)_\Omega + (k_1(\vec{y}), v_3)_\Omega = (f_2(x) u_1, v_1)_\Omega + (h_2(x), v_2)_\Omega + (u_2, v_2)_\Gamma + \\ (k_2(x) u_3, v_3)_\Omega \quad (16)$$

Let $v_\tau \in C(\bar{\Omega})$, for $\tau = 1, 2, 3$, since $y_{1n} \longrightarrow y_1$ weakly in V_1 , and hence in $L^2(\Omega) \forall \tau = 1, 2, 3$. Now subtracting (16) from (15), then taking the absolute value for the resulting equality, with utilizing the inequality of Cauchy-Schwarz on the L.H.S. of the resulting equality, it yields

$$|a_1(y_{1n} - y_1, v_1) + (a_0(y_{1n} - y_1), v_1)_\Omega - (b(y_{2n} - y_1), v_1)_\Omega - (c(y_{3n} - y_3), v_1)_\Omega + \\ a_2(y_{2n} - y_2, v_2) + (a_0(y_{2n} - y_2), v_2)_\Omega + (b(y_{1n} - y_1), v_2)_\Omega + (d(y_{3n} - y_3), v_2)_\Omega + \\ a_3(y_{3n} - y_3, v_3) + (a_0(y_{3n} - y_3), v_3)_\Omega + (c(y_{1n} - y_1), v_3)_\Omega - (d(y_{2n} - y_2), v_3)_\Omega| \\ \leq c_1 \|y_{1n} - y_1\|_{H^1(\Omega)} \|v_1\|_{H^1(\Omega)} + c_2 \|y_{1n} - y_1\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} - c_3 \|y_{2n} - \\ y_1\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} - c_4 \|y_{3n} - y_3\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} + c_5 \|y_{2n} - y_2\|_{H^1(\Omega)} \|v_2\|_{H^1(\Omega)} + \\ c_6 \|y_{2n} - y_2\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)} + c_7 \|y_{1n} - y_1\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)} + c_8 \|y_{3n} - y_3\|_{L^2(\Omega)} \|v_2\|_{L^2(\Omega)} + \\ c_9 \|y_{3n} - y_3\|_{H^1(\Omega)} \|v_3\|_{H^1(\Omega)} + c_{10} \|y_{3n} - y_3\|_{L^2(\Omega)} \|v_3\|_{L^2(\Omega)} - c_{11} \|y_{1n} - \\ y_1\|_{L^2(\Omega)} \|v_3\|_{L^2(\Omega)} + c_{12} \|y_{2n} - y_2\|_{L^2(\Omega)} \|v_3\|_{L^2(\Omega)} \quad (17)$$

From ASSU2 and Proposition 3.1 in [22], the continuity of the functionals $\iint_{\Omega} f_1(x, \vec{y}_n) v_1 dx$, $\iint_{\Omega} h_1(x, \vec{y}_n) v_2 dx$, $\iint_{\Omega} k_1(x, \vec{y}_n) v_3 dx$ are produced w.r.t. $\vec{y}_n$. But $\vec{y}_n \longrightarrow \vec{y}$ weakly in $(L^2(\Omega))^3$, then by applying the compactness theorem [22] to secure $\vec{y}_n \longrightarrow \vec{y}$ strongly in $(L^2(\Omega))^3$, hence

$$(f_1(\vec{y}_n), v_1)_{\Omega} + (h_1(\vec{y}_n), v_2)_{\Omega} + (k_1(\vec{y}_n), v_3)_{\Omega} \longrightarrow (f_1(\vec{y}), v_1)_{\Omega} + (h_1(\vec{y}), v_2)_{\Omega} + (k_1(\vec{y}), v_3)_{\Omega} \quad (18)$$

beside this, since $u_{\tau n} \longrightarrow u_{\tau}$ weakly in $L^2(\Omega) \forall \tau = 1, 3$ and $u_{2n} \longrightarrow u_2$ weakly in $L^2(\Gamma)$ then $(f_2(x)(u_{1n} - u_1), v_1)_{\Omega} + (u_{2n} - u_2, v_2)_{\Gamma} + (k_2(x)(u_{3n} - u_3), v_3)_{\Omega} \longrightarrow 0 \quad (19)$

The convergent of (15) to (16) it secures from (18) and (19). This convergent holds $\forall \vec{v} \in \vec{V}$ (from the density of $(c(\bar{\Omega}))^3$ in $\vec{V}$) which implies the limit point $\vec{y} = \vec{y}_{\vec{u}}$ is TSVS of (16).

By Lemma 4.2 and ASSU2 on $g_{0\tau}(x, \vec{y}, u_{\tau})$ give that the integral $\iint_{\Omega} g_{0\tau}(x, \vec{y}, u_{\tau}) dx$ (for $\tau = 1, 3$) is weakly lower semicontinuous w.r.t. u_{τ} , i.e.

$$\begin{aligned} \iint_{\Omega} g_{0\tau}(x, \vec{y}, u_{\tau}) dx &\leq \lim_{n \rightarrow \infty} \iint_{\Omega} g_{0\tau}(x, \vec{y}, u_{\tau}) dx \\ &= \lim_{n \rightarrow \infty} \iint_{\Omega} [g_{0\tau}(x, \vec{y}, u_{1n}) - g_{0\tau}(x, \vec{y}_n, u_{1n})] dx + \\ &\quad \lim_{n \rightarrow \infty} \iint_{\Omega} g_{0\tau}(x, \vec{y}_n, u_{1n}) dx \\ &= \lim_{n \rightarrow \infty} \iint_{\Omega} g_{0\tau}(x, \vec{y}_n, u_{1n}) dx, \text{ for } \tau = 1, 3 \end{aligned}$$

and,

$$\int_{\Gamma} g_{02}(x, u_2) d\gamma \leq \lim_{n \rightarrow \infty} \int_{\Gamma} g_{02}(x, u_2) d\gamma$$

Hence,

$$\begin{aligned} \sum_{\tau=1, \tau \neq 2}^3 \iint_{\Omega} g_{0\tau}(x, \vec{y}, u_{\tau}) dx + \int_{\Gamma} g_{02}(x, u_2) d\gamma &\leq \lim_{n \rightarrow \infty} \sum_{\tau=1, \tau \neq 2}^3 \iint_{\Omega} g_{0\tau}(x, \vec{y}_n, u_{\tau n}) dx \\ \text{i.e. } G_0(\vec{u}) &\leq \lim_{n \rightarrow \infty} G_0(\vec{u}_n) = \lim_{n \rightarrow \infty} G_0(\vec{u}_n) = \inf_{\vec{w} \in \vec{W}} G_0(\vec{w}). \end{aligned}$$

Hence $\vec{u}$ is a MTOCV.

Example: Let $\Omega = (0,1) \times (0,1)$, and consider the TNES ((1)-(3)), with

$$f_1(x, \vec{y}, u_1) = \varphi_1(x) + \cos(y_1) + \sin(y_2) - \cos(\bar{y}_1) - \sin(\bar{y}_2) + (2 + u_1 - \bar{u}_1)y_1,$$

$$h_1(x, \vec{y}) = \varphi_2(x) + \cos(y_2) + \sin(y_3) + y_1 - \cos(\bar{y}_2) - \bar{y}_1 - \sin(\bar{y}_3)$$

$$k_1(x, \vec{y}, u_3) = \varphi_3(x) + 0.5|y_3|y_3 + 0.5|y_1|y_1 + y_2 + (3 + u_3 - \bar{u}_3)y_3$$

$$f_2(x, u_1) = 2\bar{y}_1 + u_1 - \bar{u}_1, \quad h_2(x) = \varphi_3(x, t), \quad k_2(x, u_3) = 0.5|\bar{y}_1|\bar{y}_1 + \bar{y}_2 + 0.5|\bar{y}_3|\bar{y}_3 + 3\bar{y}_3 + u_3 - \bar{u}_3.$$

where $\varphi_i(x)$ are given functions, $\forall i = 1, 2, 3, 4$.

The OF is

$$G_0(\vec{u}) = \iint_{\Omega} [(y_1 - \bar{y}_1)^2 + (u_1 - \bar{u}_1)^2] dx_1 dx_2 + \int_{\Gamma} [(u_2 - \bar{u}_2)^2] d\gamma + \iint_{\Omega} [(y_3 - \bar{y}_3)^2 + (u_3 - \bar{u}_3)^2] dx_1 dx_2$$

$$\text{with } U_1 = [-1, 1], U_2 = \{-1, 1\}, U_3 = [-1, 1].$$

First, since the functions f_1, h_1, k_1, f_2, h_2 , and k_1 satisfy ASSU 1, then for $\vec{u} \in \vec{W}$ be any given MTCV, then from Theorem 3.1, the WF (11), has a unique TSVS $\vec{y} \in \vec{V}$.

Second, since, $g_{02}(x, t, u_2) = (u_2 - \bar{u}_2)^2$, $g_{0i}(x, t, y_i, u_i) = (y_i - \bar{y}_i)^2 + (u_i - \bar{u}_i)^2$, for each $i = 1, 3$ satisfy ASSU 2, if the functions $f_1(x, t, \vec{y}), k_1(x, \vec{y})$ which the defined above are independent on u_1, u_3 resp., G_0 is coercive, and satisfy the convexity property w.r.t $\vec{u}$, then by theorem 4.1, there is a MTOCV $\vec{u} \in \vec{W}$ for the MTOCVP.

Conclusions

During our study for the proposed problem “MTOCVP” controlling by the triple nonlinear PDES of elliptic types, it was concluding that; the existence theorem for a triple vector “state” solutions for the weak formulation of the for the TNES with given mixed triple

control vector was stated and proved successfully through utilizing the Browder's theorem. The continuity of Lipchitz operator between the MTCV and its conforming TSVS was demonstrated. The theory of the existence of MTOCV associated with the TNES was developed and demonstrated, under appropriate hypotheses.

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