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Homotopy Perturbation-Laplace Method for Solving Random Ordinary Differential Equations

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Abstract:
In this paper, the homotopy perturbation method will be used in connection with Laplace transformation method to give a hybrid approach as a modification of the homotopy perturbation method to find the approximate solutions of random ordinary differential equations. The approximate solution is proved also to converge to the exact solution, in which the analysis of the proof is based on mean square convergence of the sequence of a random process. The proposed hybrid approach is effectively used to find the exact solution for the considered examples, which are simulated and solved using two generations of Brownian motion with a total length of signal processing, namely 500 and 1000 generations.

Keywords: Homotopy perturbation method, Laplace transformation, Random ordinary differential equations, Stochastic Process, Brownian motion.

طريقة الاضطراب الهوموتوبي– لابلاس لحل المعادلات التفاضلية الاعتيادية العشوائية

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الخلاصة:
في هذا البحث، تم إستخدام طريقة الهوموتوبي المضطربة مع تحويلات لابلاس لإعطاء نهج هجين مطّور لطريقة هموتوبي المضطربة لإيجاد الحلول التقريبية للمعادلات التفاضلية الإعتيادية العشوائية. تم إثبات ان الحلول التقريبية متقاربة إلى الحل الدقيق، حيث تعتمد طريقة البرهان على متوسط مربعات التقارب لمتسلسلات العمليات العشوائية. تم استخدام النهج الهجين المقترح بشكل فعال لإيجاد الحل الدقيق للأمثلة قيد الدراسة، والتي تم محاكاتها وحلها باستخدام جيلين من الحركة البراونية مع إجمالي من التوليدات العشوائية وهما 500 و 1000 جيل.

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1. Introduction

The random ordinary differential equation (RODE) is defined as differential equations including random elements in their vector field. There are many researchers whose wrote a lot of articles interested in random differential equations that have emerged in recent years, such as [1] and [2]. To solve the RODE which sometimes does not close to the exact solution or is difficult to evaluate, then it is important and essential to derive and use analytical approximate or numerical methods to find the solution for such equations with its results close to the exact solution as possible [3].

In the previous mathematical fields of science and engineering, RODE's were studied with the vector field including random variables instead of depending on stochastic processes parts [4], [5]. Such RODE's are still of major relevance in the field of uncertainty qualification community, which are a specific instance of stochastic models explored by Xiaoying Han and Peter E. Kloden, and will therefore not be discussed separately [6], [7] and [8]. In many real-life applications such as engineering, natural sciences, biology (the population growth problem), physics and in chemistry (the problems that include the rate of change that count on the interaction of the basic particles), etc., which may contained the stochastic and/or random process, that are going to produce models as RODE's, so the solution of the differential equation that is evaluated experimentally are indeed not predictable, [9].

The homotopy perturbation method (HPM) is a semi-analytical methodology for solving a variety of linear or nonlinear mathematical problems. J. He suggested the HPM in 1999 [10], and he developed a novel perturbation method facility using the homotopy approach. It does not need small parameters in the equations, also can easily exclude the limitations of the traditional perturbation techniques. Indeed, the small parameter assumption is still depended on all known perturbation methods. So, these small parameters for several nonlinear problems need numerous techniques for evaluating or estimating. Such parameters are supersensitive and then any small swap in them will impact the results. A suitable alternative of small parameters gives us perfect results.

The aim of this paper is to find the approximate solution of RODE's using a hybrid approach between Laplace transformation method in connection with the HPM and for comparison purpose then we demonstrate the convergence of the obtained approximate solution to the exact analytical solution. The hybrid approach is introduced by combining the Laplace transformation method with the HPM, which will be abbreviated as LHPM, for the sake of evaluating a highly efficient closed form of the solution. Two illustrative examples, for linear and nonlinear RODEs, are given and simulated two simulated Brownian motion in order to illustrate the applicability of the proposed approach of this article. Finally, some conclusions gathered from this work are summarized.

2. Preliminaries

In this section, we will give some basic concepts, which are necessary to understand this work. Stochastic process $x(t, \omega)$ which is a family of random variables that is denoted in this work by $x_t(\omega)$ (or briefly x_t) of two variables t and ω, where $t\in[t_0, T] \subset [0,\infty)$, $T\in\mathbb{R}$, $\omega\in\Omega$ on the probability space (Ω, A, P), which assuming real values and as a function is P-measurable with respect to ω for each fixed t. The independent variable t is assumed to represent the time increment, while $x_t(.)$ represents a random variable on the above probability space Ω, and $x_.(\omega)$ is considered a sample path or trajectory of the stochastic process, [11].

Definition 2.1: [12] [A stochastic process W_t, $t \geq 0$, is said to be a Brownian motion or Wiener process, if":
1. "$p(\{\omega \in \Omega \mid W_0(\omega) = 0\}) = 1$, i.e., $p(W_0 = 0) = 1$".
2. "For $0 < t_0 < t_1 < \ldots < t_n$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1} \ldots$, $W_{t_n} - W_{t_{n-1}}$ are independent".
3. "For an arbitrary t and $h > 0$, $W_{t+h} - W_t$ has a normal distribution with mean 0 and variance h".

In stochastic calculus, convergence of sequence of random variables may be defined using different approaches and among them is given in the next definition which will be used in this paper.

Definition 2.2: [13] "A sequence of random variables $\{x_{t_n}(\omega)\}$, $n \in \mathbb{N}$, such that $E(x_{t_n}^2(\omega)) < \infty$, for all $n \in \mathbb{N}$ is said to be converges in the mean square to $x(\omega)$ if $\lim_{n\to\infty} \left|x_{t_n}(\omega) - x_t(\omega)\right|^2 = 0$".

Now, consider the probability space (Ω, A, P) and let $W_t : [0,T]\times\Omega \longrightarrow \mathbb{R}^m$ be an $\mathbb{R}^m$-valued stochastic process with continuous sample paths. Also, let $g : \mathbb{R}^d\times\mathbb{R}^m \longrightarrow \mathbb{R}^d$ be a continuous function, then RODEs may be defined as [11]:

$$\frac{dx_t(\omega)}{dt} = g(x_t(\omega), W_t(\omega)),\ x_t(\omega) \in \mathbb{R}^d,\ t \in [0,T]. \tag{1}$$

To investigate the existence and uniqueness theorem, after a noise sample path is fixed, the RODE (1) will be interpreted as an ODE with random variables included. Because the noise modifies the vector field over time, it will resemble a non-autonomous ODE. If the vector field function g in the RODE (1) is continuous in both of its variables and the sample path of the noise process W_t are continuous too, then the vector filed function $g(x_t(\omega), W_t(\omega))$ of the related non-autonomous ODE for each fixed ω is continuous in both of its variables. Hence, conventional existence and uniqueness theorem of ODEs may be extended and generalized to the RODEs and in this case is stated without proof. Among the classical assumptions used in this theorem is to suppose that the vector field g given in Equation (1) is at least continuous in both of its variables and the sample paths of the noise W_t are continuous. Fixing a sample path, i.e., ω with $g(x_t(\omega), W_t(\omega))$ then, the solution of the initial value problem (1) is a continuously differentiable function $x_t(\omega) : [t_0, T] \longrightarrow \mathbb{R}^d$ with $x_{t_0}(\omega) = x_{t_0}$, such that Equation (1) is satisfied when integrating both sides of this equation gives the integral equation [11]:

$$x_t(\omega) = x_{t_0} + \int_{t_0}^{t} g(x_s(\omega), W_s(\omega))\, ds,\ t \in [t_0, T]. \tag{2}$$

Hence, a solution of the Equation (1) is a solution of the integral Equation (2). The converse also holds, whenever the solution $x_t(\omega)$ is differentiable.

3. Application of the HPM for solving RODEs

Ji Huan He presented the standard HPM in 1999 as a strong tool for solving a wide range of linear and nonlinear problems. The HPM is viewed as a hybrid of the conventional perturbation approach and homotopy analysis (which has a basis in topology), but it is not limited to very small parameters as standard perturbation methods are. For example, the HPM technique takes only a few iterations to achieve extremely precise results, rather than minimal parameters or a linearization approach [10], [14].

To see how the HPM approach works, consider the following broad nonlinear problem in operator form:

$$A(U) - f(t) = 0,\ \ t \in D, \tag{3}$$

with the following boundary conditions:

$$B\left(u,\frac{\delta u}{\delta t}\right)=0,\ t\in\partial D, \tag{4}$$

where A is a generic differential operator, B is a boundary operator, $f(t)$ is a known analytical function, and δD is the domain's boundary. Operator A may be decomposed into two operators L and N, where L is linear and N is nonlinear, such that the Equation (3) can be modified and rewritten as:

$$L(U)+N(U)-f(t)=0. \tag{5}$$

In general, a homotopy function $u(t,p): D\times[0,1]\longrightarrow\mathbb{R}$ can be constructed to satisfy the homotopy Equation [10]:

$$H(u,p)=(1-p)[L(u)-L(U_0)]+p[L(u)+N(u)-f(t)]=0, p\in[0,1], t\in D \tag{6}$$

or equivalently:

$$H(u,p)=L(u)-L(U_0)+p[L(U_0)+N(u)-f(t)]=0, p\in[0,1], t\in D \tag{7}$$

where $p\in[0,1]$ is a homotopy parameter, and u_0 is the first approximation for the solution of Equation (5) that satisfies the boundary conditions. Also, from Equation (6), one can have:

$$H(u,0)=L(u)-L(U_0)=0,$$
$$H(u,1)=L(u)+N(u)-f(t)=0.$$

Assuming that solution for Equation (6) after the equating the like powers of p can be written as a power series of p as:

$$u(t,p)=\sum_{i=0}^{\infty}p^i u_i(t)=u_0+pu_1+p^2u_2+\cdots. \tag{8}$$

Substituting back Equation (8) into Equation (7) and equating similar powers of p terms produces values for the sequence u_0, u_1, u_2, …, and when $p\longrightarrow 1$, it yields in the approximate solution for Equation (3) in the form:

$$U=\lim_{p\to 1}u(t,p)=\sum_{i=0}^{\infty}p^i u_i(t)=u_0+u_1+u_2+\cdots. \tag{9}$$

Now, the HPM will be used to solve the RODE's (1), But firstly, rewrite Equation (1) in the operator form as:

$$L(x_t(\omega))+N(x_t(\omega))-g(t,x_t(\omega),W_t(\omega))=0 \tag{10}$$

and by letting $L(x_t(\omega))=\frac{dx_t(\omega)}{dt}$, $N(x_t(\omega))=g(t,x_t(\omega),W_t(\omega))$, then the following homotopy may be constructed $u_t(\omega,p): D\times\Omega\times[0,1]\longrightarrow\mathbb{R}$, which satisfies:

$$H(u,p)=\frac{du_t(\omega,p)}{dt}-\frac{dx_{t_0}(\omega)}{dt}+p\left[\frac{dx_{t_0}(\omega)}{dt}-g(t,u_t(\omega,p),W_t(\omega))\right]=0 \tag{11}$$

where $p\in[0,1]$, $\frac{dx_{t_0}(\omega)}{dt}$ is the derivative of the initial approximation for the solution of Equation (1). From (11) it follows that:

$$H(u,0)=\frac{du_t(\omega,p)}{dt}-\frac{dx_{t_0}(\omega)}{dt}=0,$$
$$H(u,1)=\frac{du_t(\omega,p)}{dt}-g(t,u_t(\omega,p),W_t(\omega))=0,$$

and the variation practicability of p from 0 to 1 is just like that of changing of $u_t(\omega,p)$ from $x_{t_0}(\omega)$ to $x_t(\omega)$. Therefore:

$$\frac{du_t(\omega,0)}{dt}-\frac{dx_{t_0}(\omega)}{dt}\cong\frac{du_t(\omega,1)}{dt}-g(t,u_t(\omega,1),W_t(\omega))$$

And

$$x_{t_0}(\omega)\cong x_t(\omega),\ t\in D.$$

Next, we assume that the solution Equation (11) can express as:

$$u_t(\omega,p)=\sum_{i=0}^{\infty}p^i u_{i_t}(\omega). \tag{12}$$

Therefore, the approximate solution of Equation (1) is defined upon taking the limit as $p \longrightarrow 1$, i.e.,

$$x_t(\omega) = \lim_{p\to 1} u_t(\omega, p) = \sum_{i=0}^{\infty} u_{i_t}(\omega). \tag{13}$$

By substituting the approximated solution (12) in Equation (11), one can get:

$$\sum_{i=0}^{\infty} p^i \frac{du_{i_t}(\omega)}{dt} - \frac{dx_{t_0}(\omega)}{dt} + p\left[\frac{dx_{t_0}(\omega)}{dt} - g\left(\mathrm{t}, \sum_{i=0}^{\infty} p^i\, u_{i_t}(\omega), W_t(\omega)\right)\right] = 0.$$

Now, equating the terms with identical powers of p, we get:

$$p^0:\ \frac{du_{0_t}(\omega)}{dt} = \frac{dx_{t_0}(\omega)}{dt}$$

$$p^1:\ \frac{du_{1_t}(\omega)}{dt} = g(t, u_{0_t}(\omega), W_t(\omega)) - \frac{dx_{t_0}(\omega)}{dt}$$

$$p^j:\ \frac{du_{j_t}(\omega)}{dt} = g(t, u_{(j-1)_t}(\omega), W_t(\omega)),\ \text{for all } j = 2,3,\dots.$$

Consequently, by applying the first integral operator to the above differential equations in order to calculate $u_{0_t}, u_{1_t}, u_{2_t}, \dots$, implying:

$$u_{0_t}(\omega) = x_{t_0}(\omega)$$

$$u_{1_t}(\omega) = \int_{t_0}^{t} g(t, u_{0_s}(\omega), W_s(\omega))\ ds - x_{t_0}(\omega)$$

$$u_{j_t}(\omega) = \int_{t_0}^{t} g\left(t, u_{(j-1)_s}(\omega), W_s(\omega)\right) ds,\ \text{ for all } j = 2,3,\dots.$$

Then, by using Equation (12), the approximate solution of RODEs (1) utilizing HPM is:

$$x_t(\omega) = \lim_{p\to 1} u_t(\omega, p)$$

$$= \lim_{p\to 1}\left[u_{0_t}(\omega) + p u_{1_t}(\omega) + p^2 u_{2_t}(\omega) + \dots\right] = \sum_{i=0}^{\infty} u_{i_t}(\omega). \tag{14}$$

4. Convergence Analysis

In this section, the convergence of the solution series of (12) will be presented in connection with the mean square convergence.

We start with the following which is known in kind of literatures as Cauchy Schwartz and triangle inequalities.

Lemma 4.1: [15] Suppose that *X* and *Y* are random variables, then:

(i) $E(|XY|) \le \sqrt{E(|X|^2)}\ \sqrt{E(|Y|^2)}$.

(ii) $\sqrt{E(|X \pm Y|^2)} \le \sqrt{E(|X|^2)} + \sqrt{E(|Y|^2)}$.

In addition, inequality (i) is known as the Cauchy- Schwartz inequality for expectation, and the inequality (ii) is known as the triangle inequality relate also to expectation.

Theorem 4.2: Suppose that $A \subset \mathbb{R}$ is a Banach space with a norm $\|.\| = \sqrt{E|.|^2}$ over which the sequence $u_{i_t}(\omega)$ of (14) is defined. Assume also that the initial approximation $u_{0_t}(\omega)$ remains inside the ball of the solution $u_t(\omega)$. Taking $r \in \mathbb{R}$ to be a constant, then the following statements hold:

(a) If $\left\|v_{(k+1)_t}(\omega)\right\| \le r\ \left\|v_{k_t}(\omega)\right\|$ for all *k*, given some $0 < r < 1$, then the series solution given by Equation (12) is absolutely converges when *p = 1* to the series given in Equation (14) over the domain in which *t* is defined.

(b) If the series solution defined in Equation (14) is convergent, then it converges to the exact solution of the nonlinear problem (10).

Proof: (a) The proof is evidently based on the ratio test of the power series p. However, to be able to provide an estimate of the HPM's truncation error, we will briefly present the entire proof here.

Suppose that $S_{n_t}(\omega)$ is a sequence of partial sums of the series (14), we need to show that $S_{n_t}(\omega)$ is a Cauchy sequence in the Banach space A. For this objective, consider

$$\begin{aligned}\left\|S_{(n+1)_t}(\omega)-S_{n_t}(\omega)\right\| &= \left\|u_{(n+1)_t}(\omega)\right\| \\ &\le r\ \left\|u_{n_t}(\omega)\right\|\ \le r^2\left\|u_{(n-1)_t}(\omega)\right\| \le r^3\left\|u_{(n-2)_t}(\omega)\right\| \\ &\le\ \dots\ \le\ r^{n+1}\left\|u_{0_t}(\omega)\right\|. \end{aligned} \qquad (15)$$

It should be noted that, according to inequality (15), all of the approximations obtained from homotopy (11) will fall within the ball containing $u_t(\omega)$. For every $m,n\in\mathbb{N}, n>m$, and using (15) and the triangle inequality repeatedly, we have:

$$\begin{aligned}\left\|S_{n_t}(\omega)-S_{m_t}(\omega)\right\| &= \sqrt{E\left|S_{n_t}(\omega)-S_{m_t}(\omega)\right|^2} \\ &= \sqrt{E\left|\begin{matrix}\left(S_{n_t}(\omega)-S_{(n-1)_t}(\omega)\right)+\left(S_{(n-1)_t}(\omega)-S_{(n-2)_t}(\omega)\right)\\ +\ldots+\left(S_{(m+1)_t}(\omega)-S_{m_t}(\omega)\right)\end{matrix}\right|^2} \\ &= \sqrt{E\left|u_{n_t}(\omega)+u_{(n-1)_t}(\omega)+\ldots+u_{(m+1)_t}(\omega)\right|^2} \\ &= \sqrt{E\left|\sum_{i=m+1}^{n} u_{i_t}(\omega)\right|^2} \\ &\le \sum_{i=m+1}^{n}\sqrt{E\left|u_{i_t}(\omega)\right|^2}\ \text{(by Lemma 4.1)} \\ &= \sqrt{E\left|u_{(m+1)_t}(\omega)\right|^2}+\sqrt{E\left|u_{(m+2)_t}(\omega)\right|^2}+\ldots+\sqrt{E\left|u_{n_t}(\omega)\right|^2} \\ &= \left\|u_{(m+1)_t}(\omega)\right\|+\left\|u_{(m+2)_t}(\omega)\right\|+\ldots+\left\|u_{n_t}(\omega)\right\| \\ &\le\ r^{m+1}\left\|u_{0_t}(\omega)\right\|+r^{m+2}\left\|u_{0_t}(\omega)\right\|+\ldots+r^{n}\left\|u_{0_t}(\omega)\right\| \\ &= \left(r^{m+1}+r^{m+2}+\ldots+r^{n}\right)\left\|u_{0_t}(\omega)\right\| \\ &= r^{m+1}\left(1+r+r^2+\ldots+r^{n-m-1}\right)\left\|u_{0_t}(\omega)\right\| \\ &\le r^{m+1}\left(1+r+r^2+\ldots\right)\left\|u_{0_t}(\omega)\right\| \\ &= \left(\frac{r^{m+1}}{1-r}\right)\left\|u_{0_t}(\omega)\right\|. \end{aligned} \qquad (16)$$

Since $0<r<1$, then we get from inequality (16)

$$\lim_{m,n\to\infty}\left\|S_{n_t}(\omega)-S_{m_t}(\omega)\right\| = 0. \qquad (17)$$

Therefore, $S_{n_t}(\omega)$ is a Cauchy sequence in the Banach space A, which implies that the series solution (14) is convergent.

(b) Since, by hypothesis, the approximate series solution (14) is converge, then $\lim_{n\to\infty}\ u_{n_t}(\omega)=0$, and further producing the homotopy series coefficients $u_{i_t}(\omega)$ of Equation (12).

Now, from Equation (6), we get after substituting $u_t(\omega,1)$ instead of u:

$$0 = H(u_t(\omega,1),1) = (1-1)\left[L\left(u_t(\omega,1)\right)-L(U_0)\right]+1\times\left[L\left(u_t(\omega,1)\right)+N\left(u_t(\omega,1)\right)-f(t)\right].$$

Thus, $L\left(u_t(\omega,1)\right)+N\left(u_t(\omega,1)\right)-f(t)=0$, i.e., $L\left(u_t(\omega,1)\right)+N\left(u_t(\omega,1)\right)=f(t)$, which means that $u_t(\omega,1)=\sum_{i=0}^{\infty}u_{i_t}(\omega)$ satisfied the original RODE (3), so $u_t(\omega,1)=\sum_{i=0}^{\infty}u_{i_t}(\omega)$ is the exact solution of the problem. This the complete the proof. □

5. Application of Laplace-HPM

In this section, an improved approach will be introduced which is a hybrid between the Laplace transformation method and the HPM (abbreviated as LHPM), this method may give a closed form and accurate results.

To introduce this approach, apply first Laplace transformation to both sides of the homotopy Equation (11) getting:

$$\mathrm{L}\left\{\frac{du_t(\omega,p)}{dt}-\frac{dx_{t_0}(\omega)}{dt}+p\ \left[\frac{dx_{t_0}(\omega)}{dt}-g(t,u_t(\omega,p),W_t(\omega))\right]\right\}=0$$

and so

$$\mathrm{L}\left\{\frac{du_t(\omega,p)}{dt}\right\}=\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}\right\}-p\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}-g\big(t,u_t(\omega,p),W_t(\omega)\big)\right\},$$

using the differential properties of the Laplace transformation we get:

$$s\mathrm{L}\{u_t(\omega,p)\}-u(0)=\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}\right\}-p\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}-g(t,u_t(\omega,p),W_t(\omega))\right\},$$

and so:

$$\mathrm{L}\{u_t(\omega,p)\}=\frac{1}{s}\left\{u(0)+\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}\right\}\right\}-p\left\{\frac{1}{s}\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}-g\big(t,u_t(\omega,p),W_t(\omega)\big)\right\}\right\}. \quad (18)$$

Utilizing the inverse of the Laplace transform for both sides of Equation (18), getting:

$$u_t(\omega,p)=\mathrm{L}^{-1}\left\{\frac{1}{s}\left\{u(0)+\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}\right\}\right\}\right\}-p\mathrm{L}^{-1}\left\{\frac{1}{s}\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}g\big(t,u_t(\omega,p),W_t(\omega)\big)\right\}\right\}. \quad (19)$$

Assuming that the solutions of (10) may be written as a power series of p, as provided in Equation (12). Then substituting the Equation (12) into Equation (19), we get:

$$\sum_{i=0}^{\infty}p^i\,u_{i_t}(\omega)=\mathrm{L}^{-1}\left\{\frac{1}{s}\left\{u(0)+\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}\right\}\right\}\right\}-$$
$$p\mathrm{L}^{-1}\left\{\frac{1}{s}\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}-g\big(t,\textstyle\sum_{i=0}^{\infty}p^i\,u_{i_t}(\omega),W_t(\omega)\big)\right\}\right\}.$$

Comparing coefficients of p with the same power leads to:

$$p^0:\ u_{0_t}(\omega)=\mathrm{L}^{-1}\left\{\frac{1}{s}\left\{u(0)+\mathrm{L}\left\{\frac{dx_{t_0}(\omega)}{dt}\right\}\right\}\right\}$$
$$p^1:\ u_{1_t}(\omega)=\mathrm{L}^{-1}\left\{\frac{1}{s}\mathrm{L}\left\{g(t,u_{0_t}(\omega),W_t(\omega))-\frac{dx_{t_0}(\omega)}{dt}\right\}\right\}$$
$$p^j:\ u_{j_t}(\omega)=\mathrm{L}^{-1}\left\{\frac{1}{s}\mathrm{L}\left\{g\left(t,u_{(j-1)_t}(\omega),W_t(\omega)\right)\right\}\right\},\ \text{for all } j=2,3,\dots .$$

Assuming that the initial approximation has the form $u(0)=x_{t_0}(\omega)=\alpha_0$, $u'(0)=\alpha_1$, …, $u^{(n-1)}(0)=\alpha_{n-1}$: therefore, the approximate solution may be obtained as follows:

$$x_t(\omega)=\lim_{p\to1}u_t(\omega,p)$$
$$=\lim_{p\to1}\big(u_{0_t}(\omega)+pu_{1_t}(\omega)+p^2u_{2_t}(\omega)+\dots\big)=\textstyle\sum_{i=0}^{\infty}u_{i_t}(\omega).$$

6. Illustrative Examples and Numerical Simulation:

In this section, two examples will be simulated and solved using the previously suggested HPM and LHPM, but first, it is crucial to highlight that the production of distinct discretized Brownian motions within the unit interval [0,1] will be explored. Figure 1 illustrates these generations, which have total numbers N = 500 and 1000.

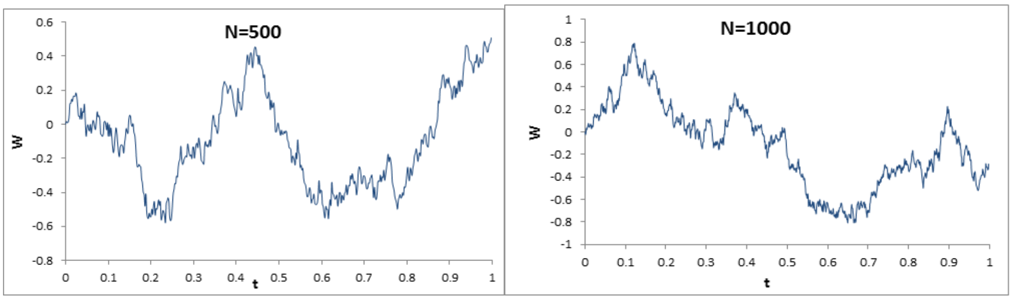

Figure 1: Discretized Brownian path with 500 and 1000 generations.

Example 6.1: Consider the problem of solving the linear RODE [16]:

$$\frac{dx_t(\omega)}{dt} = -x_t(\omega) + \sin(W_t(\omega)),\ x_{t_0}(\omega) = x_{0_t}(\omega) = 1,\ t \in [0,1], \tag{20}$$

with the exact solution for comparison purposes as it is given in [7] by:

$$x_t(\omega) = e^{-t} + e^{-t}\int_0^t e^s \sin\big(W_s(\omega)\big)\, ds.$$

First, consider the HPM and take the initial guess approximation to be $x_{0_t}(\omega) = 1$, and by assuming that $L(x) = \frac{dx_t(\omega)}{dt} + x_t(\omega)$, $N(x) = 0$, $g(t, x_t(\omega,p), W_t(\omega)) = \sin(W_t(\omega))$. Hence, define the homotopy function *H(u,p)* as:

$$H(u,p) = \frac{du_t(\omega,p)}{dt} + u_t(\omega,p) - \frac{dx_{0_t}(\omega)}{dt} - x_{0_t}(\omega) + p\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin(W_t(\omega))\right] = 0 \tag{21}$$

Now, substituting the approximated solution (12) in Equation (21), one can get

$$H(u,p) = \sum_{i=0}^{\infty} p^i \frac{du_{i_t}(\omega)}{dt} + \sum_{i=0}^{\infty} p^i u_{i_t}(\omega) - \frac{dx_{0_t}(\omega)}{dt} - x_{0_t}(\omega) + p\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin\big(W_t(\omega)\big)\right] = 0.$$

$$= \left[\frac{du_{0_t}(\omega)}{dt} + u_{0_t}(\omega)\right] + p\left[\frac{du_{1_t}(\omega)}{dt} + u_{1_t}(\omega)\right] + \sum_{i=2}^{\infty} p^j \left[\frac{du_{j_t}(\omega)}{dt} + u_{j_t}(\omega)\right] - \frac{dx_{0_t}(\omega)}{dt} - x_{0_t}(\omega) + p\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin\big(W_t(\omega)\big)\right] = 0.$$

Thus, by equating the coefficients of like powers of p will yields to:

$$p^0:\ \frac{du_{0_t}(\omega)}{dt} + u_{0_t}(\omega) - \frac{dx_{0_t}(\omega)}{dt} - x_{0_t}(\omega) = 0$$

$$p^1:\ \frac{du_{1_t}(\omega)}{dt} + u_{1_t}(\omega) + \frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin(W_t(\omega)) = 0$$

$$p^j:\ \frac{du_{j_t}(\omega)}{dt} + u_{j_t}(\omega) = 0,\ \text{ for all } j = 2,3,\dots.$$

Consequently, by solving the above differential equations for $u_{i_t}(\omega), \textit{for all } i = 0,1,2,\dots$, one may get:

$$u_{0_t}(\omega) = 1;$$

$$u_{1_t}(\omega) = e^{-t} + e^{-t}\int_0^t e^s \sin\big(W_s(\omega)\big)\, ds - 1;$$

$$u_{j_t}(\omega) = 0,\ \text{ for all } j = 2,3,\dots.$$

Now, the approximated solution of Equation (20) is given when $p \longrightarrow 1$ as:

$$x_t(\omega) = u_{0_t}(\omega) + u_{1_t}(\omega) + \cdots$$

$$= 1 + e^{-t} + e^{-t}\int_0^t e^s \sin(W_s(\omega))\, ds - 1$$

$$= e^{-t} + e^{-t}\int_0^t e^s \sin(W_s(\omega))\, ds.$$

which is the exact solution for Equation (20).

Now, by using the LHPM, it possible to find the handy approximate solution as follows:

By assuming $L(x) = \frac{dx_t(\omega)}{dt} + x_t(\omega)$, $N(x) = 0$, $g(t, x_t(\omega, p), W_t(\omega)) = \sin(W_t(\omega))$ and in order to obtain an approximate analytical solution for Equation (20), we must construct a homotopy function in accordance to Equation (7) as:

$$\frac{dx_t(\omega)}{dt} + x_t(\omega) - \frac{dx_{0_t}(\omega)}{dt} - x_{0_t}(\omega) + p\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin\big(W_t(\omega)\big)\right] = 0. \quad (22)$$

Applying the Laplace transformation on Equation (22), give:

$$sL[x_t(\omega)] - x(0) + L[x_t(\omega)] = L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega)\right] - p\,L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin\big(W_t(\omega)\big)\right],$$

and then by solving the last above equation for $L[x_t(\omega)]$, getting:

$$L[x_t(\omega)] = \frac{1}{s+1}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega)\right]\right] - \frac{p}{s+1}L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin\big(W_t(\omega)\big)\right],$$

Applying Laplace invers transformation implies to:

$$x_t(\omega) = L^{-1}\left[\frac{1}{s+1}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega)\right]\right]\right] - p\,L^{-1}\left[\frac{1}{s+1}\,L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin\big(W_t(\omega)\big)\right]\right]. \quad (23)$$

Also, we assume a series solution for $x_t(\omega)$ in the form of Equation (12), and we choose $u_{0_t}(\omega) = x_{0_t}(\omega) = 1$ as a first approximation for the solution for Equation (20) and substituting Equation (12) in Equation (23) will give:

$$\sum_{i=0}^{\infty} p^i\, u_{i_t}(\omega) = L^{-1}\left[\frac{1}{s+1}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega)\right]\right]\right] - p\,L^{-1}\left[\frac{1}{s+1}\,L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \right.\right.$$

Now, comparing the coefficients of like power of p, we have:

$$p^0: u_{0_t}(\omega) = L^{-1}\left[\frac{1}{s+1}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega)\right]\right]\right];$$

$$p^1:\ u_{1_t}(\omega) = -L^{-1}\left[\frac{1}{s+1}L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin\big(W_t(\omega)\big)\right]\right];$$

$$p^j:\ u_{j_t}(\omega) = 0,\ \text{for all}\ j = 2,3,\dots.$$

Consequently, by solving the above differential equations getting:

$$u_{0_t}(\omega) = L^{-1}\left[\frac{1}{s+1}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega)\right]\right]\right]$$
$$= L^{-1}\left[\frac{1}{s+1}\left[1 + L[1]\right]\right]$$
$$= L^{-1}\left[\frac{1}{s+1} + \frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s}\right] = 1;$$

$$u_{1_t}(\omega) = -L^{-1}\left[\frac{1}{s+1}\,L\left[\frac{dx_{0_t}(\omega)}{dt} + x_{0_t}(\omega) - \sin(W_t(\omega))\right]\right]$$
$$= -L^{-1}\left[\frac{1}{s+1}\,L\left[1 - \sin(W_t(\omega))\right]\right]$$
$$= -L^{-1}\left[\frac{1}{s(s+1)} - \frac{\sin(W_t(\omega))}{s(s+1)}\right]$$

$$= -1 + e^{-t} + \sin(W_t(\omega)) - \sin(W_t(\omega))e^{-t};$$

$$u_{j_t}(\omega) = 0\ ,\ \text{for all}\ \ j = 2,3, \dots .$$

By substituting the results of $u_{0_t}(\omega),\ u_{1_t}(\omega), \dots$ back into Equation (12) and calculating the limit when $p \longrightarrow 1$, we get the approximate solution of Equation (20) to be as:

$$x_t(\omega) = \sin(W_t(\omega)) - \sin(W_t(\omega))e^{-t} + e^{-t},$$

which is the same as the exact solution of Equation (20).

Figure 2 illustrates the approximations results for $N = 500$ and 1000 generations of Brownian motion that are given in Figure 1.

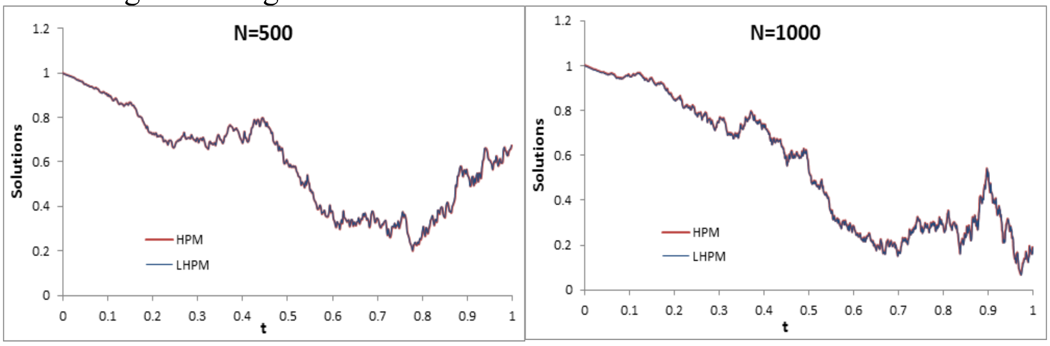

Figure 2: Approximate results of Example 1 with 100 and 1000 generations of Brownian motion.

Example 6.2: Consider the problem of solving the second order nonlinear RODE:

$$\frac{dx_t(\omega)}{dt} + W_t(\omega)\ \ x_t^2(\omega) = 0,\ t \in [0,1], \tag{24}$$

with initial condition $x_{t_0}(\omega)\ = x_{0_t}(\omega) = 1$.

Similarly in the first example, using the HPM with the initial guess approximate solution $x_{0_t}(\omega) = 1$ and by assuming that $L(x) = \frac{dx_t(\omega)}{dt}$, $N(x) = W_t(\omega)\, x_t^2(\omega)$, $g(t, x_t(\omega, p), W_t(\omega)) = 0$. Hence, the homotopy function *H(u,p)* take the form:

$$H(u,p) = \frac{du_t(\omega,p)}{dt} - \frac{dx_{0_t}(\omega)}{dt} + p\left[\frac{dx_{0_t}(\omega)}{dt} + W_t(\omega)\ u_t^2(\omega,p)\right] = 0, \tag{25}$$

Now, substituting the approximated solution (12) in Equation (25), one can get:

$$H(u,p) = \sum_{i=0}^{\infty} p^i \frac{du_{i_t}(\omega)}{dt} - \frac{dx_{0_t}(\omega)}{dt} + p\left[\frac{dx_{0_t}(\omega)}{dt} + W_t(\omega)\left[\sum_{i=0}^{\infty} p^i\, u_{i_t}(\omega)\right]^2\right]$$

$$= \frac{du_{0_t}(\omega)}{dt} + p\frac{du_{1_t}(\omega)}{dt} + \sum_{j=0}^{\infty} p^j \frac{du_{j_t}(\omega)}{dt} - \frac{dx_{0_t}(\omega)}{dt} +\ p\left[\frac{dx_{0_t}(\omega)}{dt}\right] +$$

$$pW_t(\omega)\left[\sum_{i=0}^{\infty} p^i\, u_{i_t}(\omega)\right]^2 = 0.$$

$$H(u,p) = \frac{du_{0_t}(\omega)}{dt} + p\frac{du_{1_t}(\omega)}{dt} + \sum_{j=2}^{\infty} p^j \frac{du_{j_t}(\omega)}{dt} - \frac{dx_{0_t}(\omega)}{dt} +\ p\left[\frac{dx_{0_t}(\omega)}{dt}\right] +$$

$$pW_t(\omega)\left[u_{0_t}(\omega) + p\, u_{1_t}(\omega) + p^2 u_{2_t}(\omega) + \cdots\right]^2 = 0.$$

$$\begin{aligned} H(u,p) = {} & \frac{du_{0_t}(\omega)}{dt} + p\frac{du_{1_t}(\omega)}{dt} + \sum_{j=2}^{\infty} p^j \frac{du_{j_t}(\omega)}{dt} - \frac{dx_{0_t}(\omega)}{dt} \\ & +p\ \left[\frac{dx_{0_t}(\omega)}{dt} + W_t(\omega) u_{0_t}^2(\omega)\right] + p^2 W_t(\omega)\left[2u_{0_t}(\omega)\ u_{1_t}(\omega)\right] \\ & +p^3\ W_t(\omega)\ \left[2u_{0_t}(\omega)\ u_{2_t}(\omega) + u_{1_t}^2(\omega)\right] \\ & +p^5\ W_t(\omega)\ \left[2u_{1_t}(\omega) u_{3_t}(\omega) + u_{2_t}^2(\omega) + \dots\right] \\ & +p^6\ W_t(\omega)\ \left[2u_{2_t}(\omega) u_{3_t}(\omega) + \cdots\right] + p^7\ W_t(\omega)\ \left[u_{3_t}^2(\omega) + \cdots\right] + \dots = 0. \end{aligned}$$

Thus, by equating the coefficients of like powers of p will yields to:

$$p^0\colon \frac{du_{0_t}(\omega)}{dt} = \frac{dx_{0_t}(\omega)}{dt};$$

$$p^1\colon \frac{du_{1_t}(\omega)}{dt} = -\left[\frac{dx_{0_t}(\omega)}{dt} + W_t(\omega) u_{0_t}^2(\omega)\right];$$

p^2: $\frac{du_{2_t}(\omega)}{dt} = -W_t(\omega)[2u_{0_t}(\omega)u_{1_t}(\omega)]$;
p^3: $\frac{du_{3_t}(\omega)}{dt} = -W_t(\omega)[2u_{0_t}(\omega)u_{2_t}(\omega) + u_{1_t}^2(\omega)]$;
p^4: $\frac{du_{4_t}(\omega)}{dt} = -W_t(\omega)[2u_{0_t}(\omega)u_{3_t}(\omega) + u_{1_t}(\omega)u_{2_t}(\omega)]$;
$\vdots$

Consequently, by applying the first integral operator to the above differential equations, to calculate $u_{0_t}, u_{1_t}, u_{2_t}, \ldots$, we get:

p^0: $u_{0_t}(\omega) = 1$;
p^1: $u_{1_t}(\omega) = -W_t(\omega)t$;
p^2: $u_{2_t}(\omega) = W_t^2(\omega)t^2$;
p^3: $u_{3_t}(\omega) = -W_t^3(\omega)t^3$;
p^4: $u_{4_t}(\omega) = W_t^4(\omega)t^4$;
$\vdots$

Now, the approximated solution of Equation (24) is given when $p \longrightarrow 1$ as:

$$\begin{aligned} x_t(\omega) &= u_{0_t}(\omega) + u_{1_t}(\omega) + u_{3_t}(\omega) + u_{4_t}(\omega) + \cdots \\ &= 1 - W_t(\omega)\, t + W_t^2(\omega)\, t^2 - W_t^3(\omega)\, t^3 + W_t^4(\omega)\, t^4 + \cdots \\ &= \sum_{i=0}^{\infty}(-W_t(\omega)\, t)^i \\ &= \frac{1}{1+W_t(\omega)\, t}. \end{aligned}$$

Which is the exact solution for Equation (24).

Also, by using the LHPM, it is possible to find them by hand approximate solution, as follows:

By assuming that $L(x) = \frac{dx_t(\omega)}{dt}$, $N(x) = W_t(\omega)\, x_t^2(\omega)$, $g(t, x_t(\omega, p), W_t(\omega)) = 0$, and to obtain an approximate analytical solution for (24), we must construct a homotopy function in accordance with Equation (7) as:

$$\frac{dx_t(\omega)}{dt} - \frac{dx_{0_t}(\omega)}{dt} + p\left[\frac{dx_{0_t}(\omega)}{dt} + W_t(\omega)\, x_t^2(\omega)\right] = 0. \tag{26}$$

Applying the Laplace transformation on Equation (26), give:

$$s\mathrm{L}[x_t(\omega)] - x(0) = \mathrm{L}\left[\frac{dx_{0_t}(\omega)}{dt}\right] - p\mathrm{L}\left[\frac{dx_{0_t}(\omega)}{dt}\right] - W_t(\omega)p\, \mathrm{L}[x_t^2(\omega)],$$

and then by solving the last equation for $\mathrm{L}[x_t(\omega)]$, we get:

$$\mathrm{L}[x_t(\omega)] = \frac{1}{s}\left[x(0) + \mathrm{L}\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right] - \frac{p}{s}\mathrm{L}\left[\frac{dx_{0_t}(\omega)}{dt}\right] - \frac{1}{s}\, W_t(\omega)\, p\, \mathrm{L}[x_t^2(\omega)].$$

Applying Laplace invers transformation getting:

$$x_t(\omega) = \mathrm{L}^{-1}\left[\frac{1}{s}\left[x(0) + \mathrm{L}\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right]\right] - p\mathrm{L}^{-1}\left[\frac{1}{s}\mathrm{L}\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right] W_t(\omega)p\, \mathrm{L}^{-1}\left[\frac{1}{s}\, \mathrm{L}[x_t^2(\omega)]\right]. \tag{27}$$

Also, we assume a series solution for $x_t(\omega)$ in the form of (12), and we choose $u_{0_t}(\omega) = x_{0_t}(\omega) = 1$ as a first approximation for the solution for Equation (24) and substituting Equation (12) in Equation (27), will give:

$$\sum_{i=0}^{\infty} p^i u_{i_t}(\omega) = \mathrm{L}^{-1}\left[\frac{1}{s}\left[x(0) + \mathrm{L}\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right]\right] - p\mathrm{L}^{-1}\left[\frac{1}{s}\mathrm{L}\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right] - W_t(\omega)p\, \mathrm{L}^{-1}\left[\frac{1}{s}\, \mathrm{L}\left[\left[\sum_{i=0}^{\infty} p u_{i_t}(\omega)\right]^2\right]\right]^i,$$

and hence:

$$u_{0_t}(\omega) + p\, u_{1_t}(\omega)+\ldots = L^{-1}\left[\frac{1}{s}\left[x(0) + L\left[\frac{dx_0(0;\omega)}{dt}\right]\right]\right] - pL^{-1}\left[\frac{1}{s}L\left[\frac{dx_0(0;\omega)}{dt}\right]\right] -$$

$$W_t(\omega)\, p\, L^{-1}\left[\frac{1}{s}\, L\left[\left[u_{0_t}(\omega) + p\, u_{1_t}(\omega) + p^2\, u_{2_t}(\omega)+\ldots\right]^2\right]\right]$$

$$= L^{-1}\left[\frac{1}{s}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right]\right] - pL^{-1}\left[\frac{1}{s}L\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right] -$$
$$W_t(\omega)\, p\, L^{-1}\left[\frac{1}{s}\, L[u_{0_t}^2(\omega) + 2pu_{0_t}(\omega)\, u_{1_t}(\omega) + p^2\, u_{1_t}^2(\omega) +\right.$$
$$2p^2u_{0_t}(\omega)\, u_{2_t}(\omega) + 2p^3u_{0_t}(\omega)u_{3_t}(\omega) + 2p^3u_{1_t}(\omega)\, u_{2_t}(\omega) +$$
$$p^4u_{2_t}^2(\omega) + 2p^4u_{1_t}(\omega)u_{3_t}(\omega) + 2p^5u_{2_t}(\omega)u_{3_t}(\omega) + p^6u_{3_t}^2(\omega)+\ldots]$$
$$= L^{-1}\left[\frac{1}{s}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right]\right] - pL^{-1}\left[\frac{1}{s}L\left[\frac{dx_{0_t}(\omega)}{dt} +\right.\right.$$
$$\left.\left.W_t(\omega)\, u_{0_t}^2(\omega)\right]\right] - p^2\, W_t(\omega)\, L^{-1}\left[\frac{1}{s}L[2\, u_{0_t}(\omega)\, u_{1_t}(\omega)]\right] -$$
$$p^3\, W_t(\omega)\, L^{-1}\left[\frac{1}{s}L[u_{1_t}^2(\omega) +\right.$$
$$\left.2u_{0_t}(\omega)\, u_{2_t}(\omega)]\right] - p^4\, W_t(\omega)\, L^{-1}\left[\frac{1}{s}L[2u_{0_t}(\omega)\, u_{3_t}(\omega) +\right.$$
$$\left.2u_{1_t}(\omega)\, u_{2_t}(\omega)]\right] - p^5W_t(\omega)\, L^{-1}\left[\frac{1}{s}L[u_{2_t}^2(\omega) + 2u_{1_t}(\omega)u_{3_t}(\omega)+\ldots]\right] -$$
$$p^6W_t(\omega)\, L^{-1}\left[\frac{1}{s}L[2u_{2_t}(\omega)\, u_{3_t}(\omega)+\ldots]\right] -$$
$$p^7W_t(\omega)\, L^{-1}\left[\frac{1}{s}L[u_{3_t}^2(\omega)+\ldots]\right] + \ldots .$$

Now, by comparing the coefficients of like power of p, we get:

$$p^0: u_{0_t}(\omega) = L^{-1}\left[\frac{1}{s}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right]\right];$$

$$p^1:\ u_{1_t}(\omega) = -L^{-1}\left[\frac{1}{s}L\left[\frac{dx_{0_t}(\omega)}{dt} + W_t(\omega)u_{0_t}^2(\omega)\right]\right];$$

$$p^2: u_{2_t}(\omega) = -W_t(\omega)L^{-1}\left[\frac{1}{s}L[2\, u_{0_t}(\omega)u_{1_t}(\omega)]\right];$$

$$p^3:\ u_{3_t}(\omega) = -\, W_t(\omega)L^{-1}\left[\frac{1}{s}L[u_{1_t}^2(\omega) + 2u_{0_t}(\omega)u_{2_t}(\omega)]\right];$$

$$p^4:\ u_{4_t}(\omega) = -\, W_t(\omega)L^{-1}\left[\frac{1}{s}L[2u_{0_t}(\omega)u_{3_t}(\omega) + 2u_{1_t}(\omega)u_{2_t}(\omega)]\right];$$

$$p^5: u_{5_t}(\omega) = -W_t(\omega)L^{-1}\left[\frac{1}{s}L[u_{2_t}^2(\omega) + 2u_{1_t}(\omega)u_{3_t}(\omega) + \cdots]\right];$$

and so on.

Consequently, by solving the above differential equations for $u_{0_t}(\omega), u_{1_t}(\omega), \ldots$, getting:

$$u_{0_t}(\omega) = L^{-1}\left[\frac{1}{s}\left[x(0) + L\left[\frac{dx_{0_t}(\omega)}{dt}\right]\right]\right]$$
$$= L^{-1}\left[\frac{1}{s}[1 + L[0]]\right]$$
$$= L^{-1}\left[\frac{1}{s}\right] = 1;$$

$$u_{1_t}(\omega) = -L^{-1}\left[\frac{1}{s}L\left[\frac{dx_{0_t}(\omega)}{dt} + W_t(\omega)\, u_{0_t}^2(\omega)\right]\right] = -L^{-1}\left[\frac{1}{s}L[0 + W_t(\omega)\, [1]^2]\right]$$
$$= -W_t(\omega)\, L^{-1}\left[\frac{1}{s}L[1]\right]$$

$$= -W_t(\omega)\mathrm{L}^{-1}\left[\frac{1}{s^2}\right] = -W_t(\omega)\, t;$$

$$u_{2_t}(\omega) = -W_t(\omega)\,\mathrm{L}^{-1}\left[\frac{1}{s}\mathrm{L}[2\, u_{0_t}(\omega)\, u_{1_t}(\omega)]\right]$$

$$= -W_t(\omega)\,\mathrm{L}^{-1}\left[\frac{1}{s}\mathrm{L}[2\,(1)\,(-W_t(\omega)\ t)]\right]$$

$$= W_t^2(\omega)\,\mathrm{L}^{-1}\left[\frac{1}{s}\mathrm{L}[2t\,]\right]$$

$$= W_t^2(\omega)\,\mathrm{L}^{-1}\left[\frac{1}{s}\left[\frac{2}{s^2}\right]\right]$$

$$= W_t^2(\omega)\mathrm{L}^{-1}\left[\left[\frac{2!}{s^{2+1}}\right]\right] = W_t^2(\omega)\, t^2.$$

By the same process, the results of $u_{3_t}(\omega), u_{4_t}(\omega)$ are founded to be:

$$u_{3_t}(\omega) = -W_t^3(\omega)\, t^3;$$

$$u_{4_t}(\omega) = W_t^4(\omega)\, t^4;$$

and so on.

By substituting the results of $u_0, u_1, \ldots$ in (12) and calculating the limit when $p \longrightarrow 1$, we get the approximate solution of Equation (24) as:

$$x_t(\omega) = 1 - W_t(\omega)t + W_t^2(\omega)t^2 - W_t^3(\omega)t^3 + W_t^4(\omega)t^4 + \cdots.$$

$$= \sum_{i=0}^{\infty}(-\,W_t(\omega)\, t)^i$$

$$= \frac{1}{1+\,W_t(\omega)\, t}$$

which is the same as the exact solution for Equation (24).

Figure 3 illustrates the approximation results for N = 500 and 1000 generations of Brownian motion that are given in Figure 1.

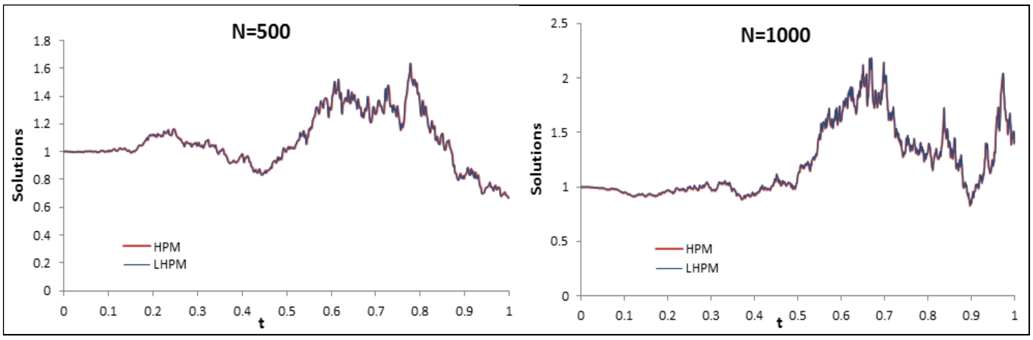

Figure 3: Approximate results of Example 2 with 100 and 1000 generations of Brownian motion

7. Conclusions

The HPM, gives us too accurate results as expected, while the hybrid method which combines the HPM and Laplace transformation, give exact results. But, the random part (or stochastic processes) in the suggested differential equation makes the behavior of the solution change according to the total numbers of Brownian motion generations.

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