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New Mittag-Leffler Spectral Method Based on Fractional Chelyshkov Polynomial to Solve Multi-Type of Fractional Ordinary Differential Equations

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Abstract

In this research, a new weighting method was presented based on the truncated fractional Mittag-Leffler function. Alternatives to Jacobi polynomials represented by the Chelyshkov polynomial with the orthogonal property and the fractional degree were relied upon. This approach is based on weighted residual methods. The differential equations are converted into a system of linear or nonlinear algebraic equations. Accurate results were obtained in various applications, and the convergence of the proposed method was studied and the Chelyshkov polynomial was compared with other functions. In addition, weighting methods were compared. In comparison with other methods, the results showed the efficiency of the proposed method in solving such types of fractional equations.

Keywords: Chelyshkov polynomials, Mittag-Leffler weight Method, Weighted residual method, Fractional derivative, Spectral method.

طريقة ميتاج-ليفلر الطيفية الجديدة المعتمدة على كثيرات حدود تشيليشكوف الكسرية لحل أنواع متعددة من المعادلات التفاضلية الاعتيادية الكسرية

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الخلاصة

في هذا البحث تم تقديم طريقة جديدة للترجيح تعتمد على دالة ميتاج-ليفلر الكسرية المبتورة. تم الاعتماد على بدائل كثيرات حدود جاكوبي الممثلة في كثيرة حدود تشيليشكوف ذات الخاصية التعامدية والدرجة الكسرية. يعتمد هذا النهج على الطرائق المتبقي المرجح، اذ يتم تحويل المعادلات التفاضلية إلى أنظمة من المعادلات الجبرية الخطية أو غير الخطية. تم الحصول على نتائج دقيقة في تطبيقات مختلفة، وتمت دراسة تقارب الطريقة المقترحة ومقارنة متعددة حدود تشيليشكوف مع الدوال الأخرى. بالإضافة إلى ذلك، تمت مقارنة

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طرائق الترجيح. وبالمقارنة مع الطرق الأخرى أظهرت النتائج كفاءة الطريقة المقترحة في حل مثل هذا النوع من المعادلات الكسرية.

1. Introduction

Many real-world applications result in differential equations with various differential operators, and the issues become more complex according to the nature of natural phenomena. Despite the availability of many analytical methods, nevertheless, there are more difficulty in determining the exact solution to many mathematical models [1].

The numerical solution is considered one of the most important alternative methods to obtain a solution that is close to the analytical solution. One of the numerical methods is to reduce the residual error function to zero. These methods are known as weighted residual methods (WRM). The WRMs depend on calculus of variation in finding the minimum approximation [2]. Orthogonal polynomials are often chosen due to two properties: the first is that they are analytical in wide fields and the second is that the orthogonality property reduces and facilitates mathematical operations [3].

Chelyshkov polynomials (CPs) are introduced as a suitable alternative to orthogonal Jacobi polynomials in the interval [0,1] [4]. CPs used as a basic function to solve different types of differential equations. The Tau method was applied to solve the Bagley-Torvik equation [5]. The collocation method (COM) has also been used to solve the system of FODEs [6]. The results obtained from applying CPs were compared with Bessel, Chebyshev, and Legendre polynomials, and the results show superiority to fractional Chelyshkov polynomials [7], [8], [9].

In this work, instead of using the collocation method, different types of weighted residual methods will be applied, in addition to proposing a new weight function, and a comparison will be made between the most commonly used polynomials. Furthermore, the importance of Chelyshkov fractional polynomials will be highlighted. In addition, the Mittag- Leffler function is proposed to be the weight function in Petrov- Galerkin method. The field of applying this proposed method are linear and nonlinear FODEs with constant and variable coefficient. Furthermore, the solution delay FODEs and system of FODEs are also provided.

1.1 Fractional Derivative.

Let $f(t)$ be a function defined on $[a,b]$, and suppose $\alpha > 0$, the Caputo fractional derivative is defined as [10]

$$ {}_{a}^{C}D_t^{\ \alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} \frac{d^n}{du^n} f(u)du \tag{1}$$

where $n-1 < \alpha < n$.

If $f(t) = t^p$ then,

$$ {}_{0}^{C}D_t^{\ \alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, \quad p \geq n \tag{2}$$

In Caputo's concept, the function $f(t)$ must be continuous and differentiable n times on $[a,b]$, in addition, the initial conditions in the fractional differential equation must be of the integer order. Furthermore, the derivative of the constant function is always zero [11].

1.2 Weighted Residual Method (WRM)

Let us assume the following differential equation [2]:

$$Au(x) = q(x), \tag{3}$$

where A is a fractional differential operator and u is an unknown function defined on the Hilbert space $H(a,b)$, and this function is approximated by writing it as a linear combination of a expansion functions $N_k(x)$ as follows:

$$u(x) \approx U_n(x) = \sum_{k=0}^{n} c_k N_k(x) \tag{4}$$

Substitute $U_n(x)$ in relation (3) to get the following residual function:

$$R(x) = AU_n(x) - q(x) \neq 0 \tag{5}$$

The following theorem is a fundamental theorem to reduce $R(x)$ to the minimum.

Theorem (1) [12]. Suppose $R(x)$ be a function defined on $H(a,b)$, let the value of the following integral be verified for any given function $w(x)$ defined on $H(a,b)$, i.e.

$$\int_a^b R(x)w(x)dx = 0, \tag{6}$$

then,

$$R(x) = 0, \forall\, x \in (a,b)$$

These methods are divided into several types, the most important of which are:

1.2.1 Galerkin Method (GM) [2]

In this method the weighted function depends on the expansion function, that is:

$$w_k(x) = N_k(x) \tag{7}$$

1.2.2 Petrov-Galerkin Method (PGM) [2]

In this method, the expansion function is not relied upon, but other functions analytical in $H(a,b)$ are used. Several methods fall under this method, for instance, the collocation method (CM), subdomain method (SM), and momentum method (MM).

1.2.3. Least Squares Method (LSM) [2]

This method relies primarily on making the slope of the norm function of the residual for the unknown coefficients equal to zero, i.e. $\frac{\partial}{\partial c_k}\int_\Omega R^2(x)dx = 0$, which leads to $2\int_\Omega R\,(x)\frac{\partial R(x)}{\partial c_k}dx = 0$, so the weighted function is defined by: $w_k(x) = \frac{\partial R(x)}{\partial c_k}$.

1.3 Chelyshkov Polynomials.

The Chelyshkov polynomials are defined by [8]:

$$C_{Nn}(x) = \sum_{k=n}^{N} \gamma_{Nnk}\, x^k, n \in I^N \tag{8}$$

where, $x \in I = [0,1]$, $I^N = \{0,1,\dots,N\}$, and $\gamma_{Nnk} = (-1)^{k-n}\binom{N-n}{k-n}\binom{N+k+1}{N-n}$.

The relation (8) represents CPs of Degree N, it can be generalized to a fractional order vN , where $0 < v \leq 1$ as follows:

$$C_{Nnv}(x) = \sum_{k=n}^{N} \gamma_{Nnk}\, x^{kv}, \qquad n \in I^N \tag{9}$$

The interval I can be generalized $to\ I^b = [0,b]$ according to the following relationship [9]:

$$C_{Nnv}(x) = \sum_{k=n}^{N} \gamma_{Nnk}\left(\frac{x^v}{b}\right)^k, \qquad n \in I^N \tag{10}$$

By applying the fractional Caputo derivative to relation (9), we obtain the following:

$$D^{\alpha}C_{Nnv}(x) = \begin{Bmatrix} \sum_{k=n}^{N} \gamma_{Nnk} \frac{\Gamma(kv+1)}{\Gamma(kv-\alpha+1)} x^{kv-\alpha} & , & kv \geq m+1 \\ 0 & , & kv < m+1 \end{Bmatrix}, \tag{11}$$

where, $n \in I^N$, $m < \alpha < m+1$

One of the most important properties that characterize these functions is the orthogonality property with weighting $w_v(x) = x^{v-1}$, which is given according to the following relationship:

$$\int_0^b C_{Nn}(x)C_{Nk}(x)w_v(x)dx = \begin{Bmatrix} \frac{b}{v(k+n+1)} & , & n = k \\ 0 & , & n \neq k \end{Bmatrix}, n,k \in I^N \tag{12}$$

1.3 .1 The Expansion Approximation and Errors.

Consider the weighted space $L^2_{w_v}(I)$, which is defined by[13]:

$$L^2_{w_v}(I) = \left\{ f: I \to \mathbb{R}; f \text{ is measurable on } I, \int_0^1 |f(x)|^2 w_v(x)dx < \infty \right\} \tag{13}$$

The inner product and the norm are provided by:

$$\langle f, g \rangle_{w_v} = \int_0^1 f(x)g(x)w_v(x)dx \ , \|f\|_{w_v} = \langle f, f \rangle^{\frac{1}{2}}_{w_v} \tag{14}$$

Suppose $S_N = span\{C_{N0v}(x), C_{N1v}(x), \dots, C_{NNv}(x)\}$ a finite-dimensional base and a subspace of $L^2_{w_v}(I)$. For any function $y(x) \in L^2_{w_v}(I)$ there exists a unique approximation $y_N(x) \in S_N$ and satisfied the following conditions:

$$\|y - y_N\|_{w_v} \leq \|y - Y\|_{w_v}, \forall\, Y \in S_N \tag{15}$$

Furthermore, it can be expanded $y_N(x)$ by a fractional Chelyshkov polynomials as:

$$y_N(x) = \sum_{k=0}^{N} a_k C_{Nkv}(x) \tag{16}$$

Multiply both sides of relation (16) by $C_{Nnv}(x)w_v(x)$ and integrated the results from 0 to b, so we get:

$$\int_0^b y_N(x)C_{Nnv}(x)w_v(x)dx = \sum_{k=0}^{N} a_k \int_0^b C_{Nkv}(x)C_{Nnv}(x)w_v(x)\, dx$$

The following result is obtained by applying the orthogonality property (12):

$$a_n = \frac{v(2n+1)}{b} \int_0^b y_N(x)C_{Nnv}(x)w_v(x)dx\, , n \in I^N, \tag{17}$$

that is:

$$a_n = \frac{\langle y, C_{Nnv} \rangle_{w_v}}{\langle C_{Nnv}, C_{Nnv} \rangle_{w_v}}, n \in I^N \tag{18}$$

Theorem (2) [6]. Suppose $D^{nv}y(x) \in C[0,1], n \in I^{N+1}$ and let $y_N(x)$ be the best approximate to the function $y(x)$ then the bound of error is given by:

$$\|y - y_N\|_{w_v} \leq \frac{Q_v}{\Gamma(1+(N+1)v)} \frac{1}{\sqrt{(2N+3)v}}, \text{ where, } Q_v = \sup_{0<x\leq 1} \left\{ \left| D^{(N+1)v}y(x) \right| \right\}. \tag{19}$$

2. New Mittag-Leffler Weight Method (MLWM)

The Mittag- Leffler function of two parameters α *and* β is defined by:

$$E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha j + \beta)} \tag{20}$$

In this research, the truncated Mittag- Leffler function is suggested to be the weight function. Furthermore, the parameter α is computed by the minimum fractional order that appears in the governing differential equation. Since the degree of the Chelyshkov polynomials may be a fraction, it is multiplied by the fraction part x^{v-1}. So, the Mittag-leffler weights are defined by:

$$M_k(x) = \sum_{j=0}^{k} \frac{x^{j+v-1}}{\Gamma(\alpha j + 1)} \tag{21}$$

Proposition (1). If the weight function is defined as $\mathrm{w_k(x)} = M_k(x)$ in the integral represent by equation (6), then the residual $R(x)$ is vanished.

Proof: for all $k = 1,2,\dots,N-m$, the functions $M_k(x)$ are analytic on $H(0,b)$. Suppose $R(x) \neq 0$, then either $R(x) > 0$, or $R(x) < 0$, if it is positive for some subinterval $[a_1, b_1]$ in $[0, b]$. Since $M_k(x) > 0, \forall\, k$, in addition, the value of $M_k(x) = 0$ only if $x = 0$. But the value of the following integral is positive and not equal to zero,

$$\int_0^b R(x) M_k(x) dx = \int_{a_1}^{b_1} R(x) \sum_{j=0}^{k} \frac{x^{j+v-1}}{\Gamma(\alpha j + 1)} dx > 0, \tag{22}$$

which is a contradiction with the hypotheses. Therefore $R(x) = 0$.

3. The Proposed Method (MLWM-CPs)

Consider the following fractional differential equation of order m:

$$Au(x) = L^{\alpha} u + Lu + F(u) = q(x), x \in [0, b] \tag{23}$$

where L, L^{α} be a linear differential operator of integer and fractional order respectively, and F be a nonlinear operator. This differential equation satisfies the following initial or boundary conditions

$$L_0 u(0) = a_i,\ i = 1,2,\dots,m \tag{24}$$

$$L_0 u(0) = a_i, L_1 u(b) = b_j, i + j = m, \text{where } L_0, L_1\ linear\ operator \tag{25}$$

Suppose the approximate solution of equation (23) can be written as a Chelyshkov polynomials as follows:

$$u_N(x) = \sum_{k=0}^{N} c_k C_{Nkv}(x) \tag{26}$$

Substituting (26) in (23) we get:

$$\sum_{k=0}^{N} c_k \{L^{\alpha} + L\}\, C_{Nkv}(x) + F\left(\sum_{k=0}^{N} c_k C_{Nkv}(x)\right) = q(x), \tag{27}$$

The following is computed by multiplying relation (27) by the Mittag-Leffler weights $M_j(x), j = 1,2 \dots, N - m$ and integrate the result, so

$$\int_0^b \left[\sum_{k=0}^{N} c_k \{L^{\alpha} + L\}\, C_{Nkv}(x) + F\left(\sum_{k=0}^{N} c_k C_{Nkv}(x)\right)\right] M_j(x)\ dx = \int_0^b q(x) M_j(x)\, dx, \tag{28}$$

which leads to $N - \mathrm{m}$ of a nonlinear system of equations, by applying the initial or boundary conditions, it can obtain

$$L_0 \textstyle\sum_{k=0}^{N} c_k C_{Nkv}(0) = a_i,\ i = 1,2,\dots,m \tag{29}$$

$$L_0 \sum_{k=0}^{N} c_k C_{Nkv}(0) = a_i \,, L_1 \sum_{k=0}^{N} c_k C_{Nkv}(b) = b_j, i + j = m. \tag{30}$$

Thus, the required coefficients are obtained by solving the given system of $N + 1$ nonlinear equations. If $F(u) = 0$, then the differential equation is linear and it is reduced to a system of linear algebraic equations.

4. Numerical Examples.

The examples illustrate the efficiency of the proposed method in solving fractional differential equations, in addition to studying the efficiency of the CPs as an expansion function, and comparing the results with other famous polynomials (see **Ex1** and **Ex3**). The effect of fractional degrees is also studied (see **Ex2** and **Ex3**). Side by side, the effect of weighted residual methods on the solution results are discussed, and the Mittag-Leffler method is compared with other famous methods in Petrov- Galerkin, Galerkin, and the least squares method (see **Ex1** and **Ex3**). Finally, the effect of the length of the interval I on the accuracy of the approximate solution is detected (see **Ex4**). The remaining examples are presented to illustrate the efficiency of the method in solving different types of fractional differential equations. The following experiments are implemented using **MATLAB 2020a**. The root mean squares error (RMS) is used to compare the errors. The RMS is determined by [14]:

$$RMS = \sqrt{\sum_{i=1}^{M} \frac{\left(y(x_i) - y_N(x_i)\right)^2}{M}} \tag{31}$$

where $x_i \in [0, b], \forall i, y$ the exact solution, and y_N is the approximate solution.

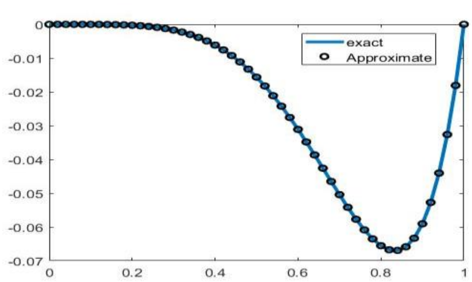
Example 1. Consider the following Fractional Dynamic Model of Bagley-Torvik [5]:

$$u'' + \frac{1}{2} u^{(\alpha)} + u = f(x), \qquad 0 < \alpha < 2,$$

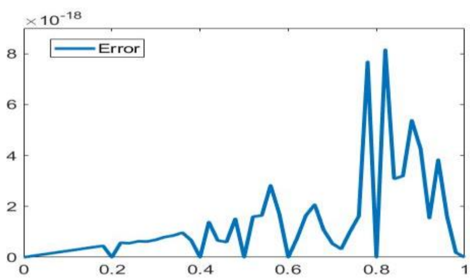
where, $f(x) = x^6 - x^5 + \frac{360x^{6-\alpha}}{\Gamma(7-\alpha)} - \frac{60x^{5-\alpha}}{\Gamma(6-\alpha)} + 30x^4 - 20x^3$, with boundary conditions: $u(0) = u(1) = 0$, and the exact solution is $u(x) = x^6 - x^5$.

Using the proposed method (MLWM-CPs), when $N = 6$, and $\alpha = 0.5$, the approximate solution is given by: $U(x) = -2.1503e - 18x + 3.7569e - 19x^2 - 9.0427e - 19x^3 + 4.4813e - 18x^4 - x^5 + x^6$.

The difference between exact and approximate solutions are showed in Figure 1, as can be seen, the error is of the order 10^{-18} in most values.



(a)



(b)

Figure 1: Describes the performance of the solution for Ex. 1, where (a) presents both the exact and approximate solutions with parameters set to $\mathrm{N} = 6$, and $\alpha = 0.5$. Furthermore, (b) displays the graph representing the absolute error related to Ex. 1.

Error! Reference source not found.) shows the effect of weighted residual methods on the a pproximate solution using CPs when $N = 6, \alpha = 0.5$. It should be noted that these methods provide approximate solutions with fairly close accuracy, but the Galerkin method has the orthogonality property. Furthermore, the Mittag-Leffler method is considered the fastest method to reach the approximate solution, which reduces the significant computational effort compared to other methods. If there is complexity in using integrals in linear and nonlinear systems, then it is preferable to use the collocation method because it is free from calculating integrals, but it depends on the appropriate choice of collocation points.

Table 1: Explain the effect of weighted residual methods on the approximate solution.

Weighted residual method	RMS	Time consumed (second)
GM -CPs	$\mathbf{2.745e-19}$	7.47
CM -CPs	$3.1081e-19$	12.18
SM -CPs	$3.1079e-19$	15.50
MM -CPs	$3.1081e-19$	$\mathbf{3.88}$
MLWM -CPs	$3.1079e-19$	$\mathbf{2.23}$
LSM -CPs	$3.1080e-19$	16.86

Error! Reference source not found.) explain the effect of the value of α and N on the a pproximate solution using the proposed method. The root means square error (RMS) is calculated in each case. We note that in this example, the value of α has a slight effect, while the accuracy of the approximate solution increases with the value of N, and the solution stabilizes when $\boldsymbol{N \geq 6}$. It should be noted that when using fractional CPs with effect v, they must also be $\boldsymbol{Nv \geq 6}$ to obtain a suitable approximate solution.

Table 2: shows the effect of the value of α on the approximate solution using MLWM-CPs

	$\boldsymbol{\alpha = 0.3}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.7}$	$\boldsymbol{\alpha = 0.9}$	$\boldsymbol{\alpha = 1.5}$
$N = 3$	0.0352	0.0358	0.0364	0.0365	0.0324
$N = 4$	0.0239	0.0242	0.0246	0.0249	0.0223
$N = 5$	0.0076	0.0077	0.0079	0.0080	0.0071
$N = 6$	$2.732e-19$	$2.745e-19$	$3.261e-19$	$5.014e-19$	$3.965e-19$
$N = 7$	$2.674e-19$	$2.541e-19$	$2.543e-19$	$2.544e-19$	$9.801e-19$
$N = 8$	$4.607e-19$	$2.723e-19$	$1.036e-18$	$2.287e-18$	$1.565e-18$

Error! Reference source not found.) provides a numerical comparison using the Galerkin m ethod based on the polynomials of Chelyshkov, Legendre, Chebyshev of the first and second kind, Jacobi, Bernstein about $f(t) = e^t$, Gegenbauer, Hermite, Laguerre, Taylor, and Mittag-Leffler function $\alpha = \beta = 1$, when $N = 6$ and $\alpha = 0.5$. The comparison shows that the best results appear when using Taylor, Jacobi, and Chelyshkov polynomials. It is known that the relationship between Chelyshkov and Jacoby polynomials has been clarified (Ref. [4]), and all polynomials are close to Taylor polynomial.

Table 3: shows the effect of the expansion function on the approximate solution using Galerkin method, with N=6 and α=0.5.

Polynomials	RMS	Time consumed (s)
Chelyshkov	$\mathbf{2.7449e-19}$	$\mathbf{7.47}$
Chebyshev first kind[15]	4.8608e − 19	7.82
Chebyshev 2nd kind[16]	5.3628e − 19	8.20
Bernstein[17]	$8.1550e-14$	14.58
Legendre, [18]	4.3166e − 19	8.23
Jacobi[19]	$\mathbf{2.5794e-19}$	8.74
Gegenbauer[20]	$5.3628e-19$	7.98
Hermite[21]	$6.6202e-19$	7.87
Laguerre [22]	$1.4551e-14$	10.82
Taylor [23]	$\mathbf{2.5457e-19}$	$\mathbf{6.41}$
Mittag-Leffler [24]	2.5642e − 19	10.21

Example 2. Suppose the following initial value problem

$$u^{(0.5)}+u=\sqrt{x}+\sqrt{\frac{\pi}{2}},u(0)=0,$$

with exact solution $u(x)=\sqrt{x}$. The approximate solution using MLWM-CPs, when $\mathrm{N}=4$ $,v=\frac{1}{2}$ is $U(x)=\sqrt{x}$,. **Error! Reference source not found.**) shows a numerical comparison b y taking different values of v and observing the change in error.

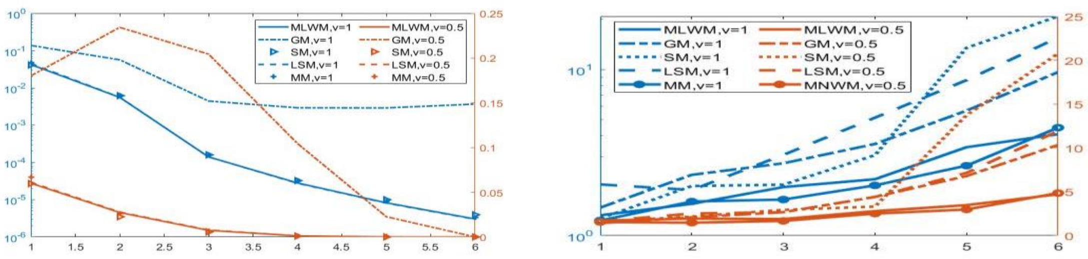
Table 4: shows the effect of the value of v on the approximate solution.

	N = 1	N = 2	N = 3	N = 4	N = 5
$v=\frac{1}{4}$	0.3064	1.0425e − 73	1.0425e − 73	1.0425e − 73	1.0425e − 73
$v=\frac{1}{2}$	1.0425e − 73	1.0425e − 73	1.0425e − 73	1.0425e − 73	1.0425e − 73
$v=\frac{3}{4}$	0.0311	0.0329	0.0502	0.0953	0.2061
$v=1$	0.0447	0.0551	0.0907	0.1836	0.4162

Example 3. Let the following FODEs

$$u^{(\alpha)}+u=\frac{15}{8}\frac{\sqrt{\pi}}{\Gamma(3.5-\alpha)}x^{2.5-\alpha}+6\frac{x^{3-\alpha}}{\Gamma(4-\alpha)}+x^2(\sqrt{x}+x),u(0)=0,$$

with exact solution $u(x)=x^2(\sqrt{x}+x)$.
The effect of the expansion function on the approximate solution using MLWM, with, $v=1$ and $\alpha=0.5$. when $N=3$, $RMS=1.4058\mathrm{e}-04$, when $N=6$, $RMS=3.0316\mathrm{e}-06$.
Figure (2) shows the accuracy and speed of the approximate solution resulting from applying the MLWM-CPs proposed method compared with others method of WRM when using $N=6$ and $v=0.5$. In this case, the approximate solution is given by $U(x)=-1.8795e-22x^{0.5}+1.201e-21x-7.7567e-21x^{1.5}+6.30795e-20x^2+0.08\overline{3}x^{2.5}+2.08\overline{3}x^3$.
This implementation with RMS=$1.0442e-17$.



(a) (b)

Figure (2) comparison among the weighted residual method based on CPs in Ex 3(a) Comparison between WRM in Ex3 using $v = 0.5, 1$, and $N = 1, \dots, 6$ (b) time consumed in Ex3. using $v = 0.5, 1$, and $N = 1, \dots, 6$

Example 4. Suppose the following linear FODEs [25]

$$u''' + u^{(2.5)} + u'' + 4u' + u^{(2.5)} + 4u = 6\cos(x), x \in I^b = [0, b],$$

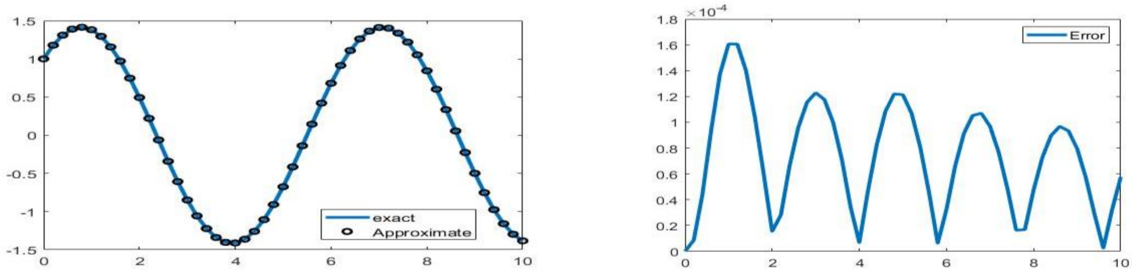
with initial condition $u(0) = 1, u'(0) = 1, and\ u''(0) = -1.$

The exact solution of this equation is $u(x) = \sqrt{2}\sin(x + \frac{\pi}{4})$.

In this problem, increasing the length of the period I^b requires increasing the number of unknown coefficients N. Table (1) shows the effect of the length of I^b and the number of elements of Chelyshkov's polynomials on the accuracy of the approximate solution. Figure 3 shows the approximate solution using MLWM-CPs when $N = 14, b = 10$ and $v = 1$.

Table (1): shows the effect of the value of b on the approximate solution using MLWM, when $v = 1$.

	N = 3	N = 5	N = 7	N = 9
$b = 1$	0.0023	1.828e − 05	3.803e − 08	4.283e − 11
$b = 2$	0.0222	5.616e − 04	4.743e − 06	2.044e − 08
$b = 3$	0.0621	0.0018	5.383e − 05	5.384e − 07
$b = 4$	0.1075	0.0022	1.177e − 04	2.2584e − 06
$b = 5$	0.1404	0.0032	4.5015e − 04	1.8576e − 05
$b = 6$	0.1501	0.0128	0.0019	2.6140e − 04



(a) (b)

Figure 3: The diagram demonstrates the efficacy of the solution for Ex.4 In part (a), it presents both the precise and estimated solutions with parameters set at when $v = 1$, $b = 10$ and $N = 14$. Furthermore, the accompanying graph in part (b) illustrates the absolute error pertaining to Ex.4.

Example 5. Consider the variable coefficient fractional diffusion equation [26]

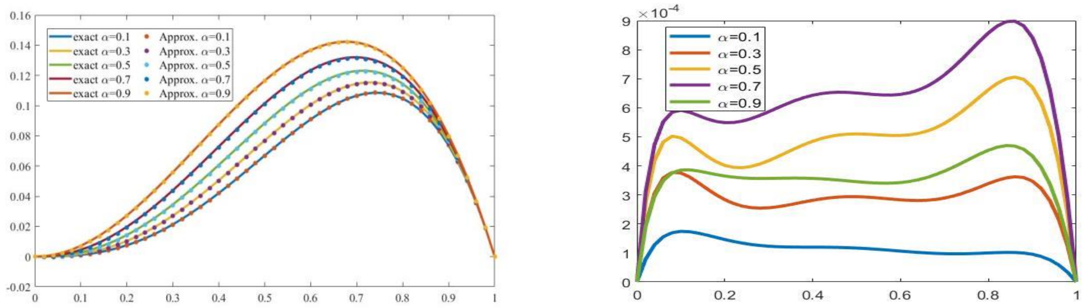
$$-\frac{d}{dx}\left(K(x)u^{(1-\alpha)}\right)=f(x),\alpha\in(0,1),$$

Where the diffusivity coefficient $K(x)=e^{0.5\sin(\pi x)}$ and

$f(x)=\Gamma(4-\alpha)e^{0.5\sin(\pi x)}\left(\frac{\pi}{12}\cos(\pi x)\,(4-\alpha)x^3-3x^2\right)+\frac{1}{2}((4-\alpha)x^2-2x)$

with boundary conditions $u(0)=u(1)=0$.The exact solution is $u(x)=x^{3-\alpha}(1-x)$

Figure 4 illustrate the approximate solutions in many value of α, wih $N=5,v=1$.



(a) (b)

Figure 4: comparison between approximate and exact solution using different value of α. (a) Approximate solution in Ex.5 with $\mathrm{N}=5,\mathrm{v}=1$, $\alpha=0.1,\ldots,0.9$ (b) absolute errors.

Example 6. Consider the following system of Bagley-Torvik FODEs [19]

$$u_1^{(0.8)}-u_1+3u_2=4x^4-x^5-3x^3+\frac{15625}{924\Gamma(0.2)}x^{\frac{21}{5}}-\frac{625}{44\Gamma(0.2)}x^{\frac{16}{5}},$$

$$u_2^{(0.8)}+4u_1-2u_2=4x^5-6x^4+2x^3+\frac{625}{44\Gamma(0.2)}x^{\frac{16}{5}}-\frac{125}{11\Gamma(0.2)}x^{\frac{11}{5}},$$

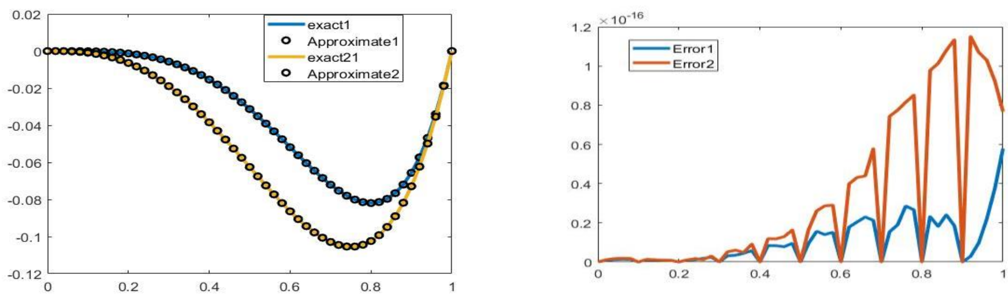
with initial conditions $u_1(0)=u_2(0)=0$. The exact solutions are $u_1(x)=x^5-x^4,u_2(x)=x^4-x^3$.

When $N=6,v=1$, the following approximate solutions are illustrated in Figure 5,

$$\boldsymbol{U_1(x)=2.4e-16x^6+x^5-x^4-7.4e-17x^3+1.2e-17x^2-8.7e-19x}$$

$$\boldsymbol{U_2(x)=-7.3e-16x^6+7.2e-16x^5+x^4-x^3-1.2e-17x^2+9.4e-19x}$$

with $RMS1 = 2.1566\mathrm{e}-18$ and $RMS2 = 7.1135\mathrm{e}-18$.



(a) (b)

Figure 5 : (a) Illustrates the approximate solution in Ex.4.6 using $N=6$, $v=1$, (b) errors.

Example 7. Consider the fractional nonlinear delay pantograph differential equations [27]

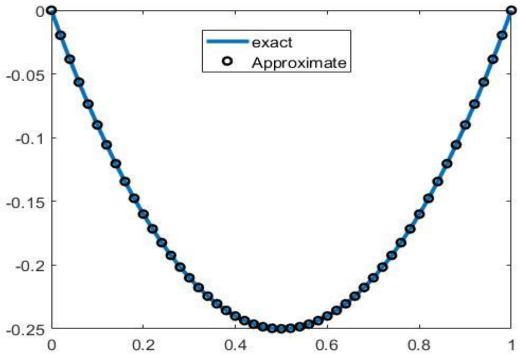
$$u^{\left(\frac{1}{2}\right)} = u^2 + u\left(\frac{1}{3}x\right) - \frac{8}{3\sqrt{(\pi)}}x^{\frac{3}{2}} - \frac{2}{\sqrt{(\pi)}}x^{\frac{1}{2}} - x^4 + 2x^3 - \frac{10}{9}x^2 + \frac{1}{3}x$$

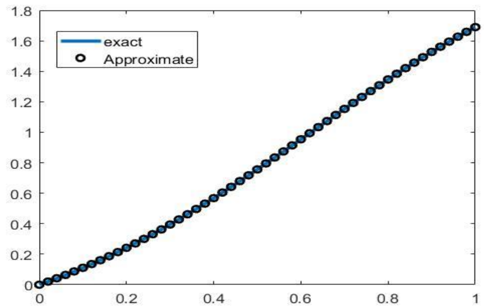
with initial condition $u(0) = 0$. The exact solution is $u(x) = x^2 - x$.
When $N = 3, v = \frac{1}{3}$, the approximate solution generated by MLWM-CPs method is equivalent to the exact solution.

Example 8. Consider the following nonlinear Riccati FODEs

$$u^{(\alpha)} - 2u + u^2 = 1, x \in I,$$

with initial condition $u(0) = 0$. The exact solution when $\alpha = 1$ is $u(x) = 1 - \sqrt{2}\frac{\sqrt{2}\tanh(\sqrt{2}\,x)-1}{\tanh(\sqrt{2}\,x)-\sqrt{2}}$. When $N = 5, v = 1$, the approximate solution is $U(x) = 1.0218x - 0.7477x^2 + 1.3574x^3 - 2.1189x^4 + 0.6816x^5$ with $RMS = 3.4965e - 05$





(a) Ex7, $\boldsymbol{N = 3, v = 1\ and\ \alpha = 0.5}$ (b) Ex8, $\boldsymbol{N = 5, v = 1\ and\ \alpha = 1}$

Figure 6 : Draw the approximate and exact solution in Ex7 to Ex8

Example 9. Suppose the following non-linear fractional-order Van der Pol ODEs [1]

$$u^{(1+\alpha)} + \mu(u^2 - 1)u^{(\alpha)} + u = 0, \mu > 0, \alpha \in (0,1],$$

with initial conditions $u(0) = 0, u'(0) = 1$.
When $N = 4, v = 1, \mu = 0.05$ and $\alpha = 1$ the approximate solution is $U(x) = x + 0.0351x^2 - 0.1951x^3 + 0.0202x^4$ comparing this solution with the approximate solution generated by RK4 method in MATLAB using (ode45) solver,$RMS = 4.5079e - 04$.
Figure 7: the approximate solution of Van der Pol FODE using the proposed method, when μ=0.05 comparing with the numerical solution. when $\mu = 0.05$ and α equal to 0.2,0.4,0.6,0.8.

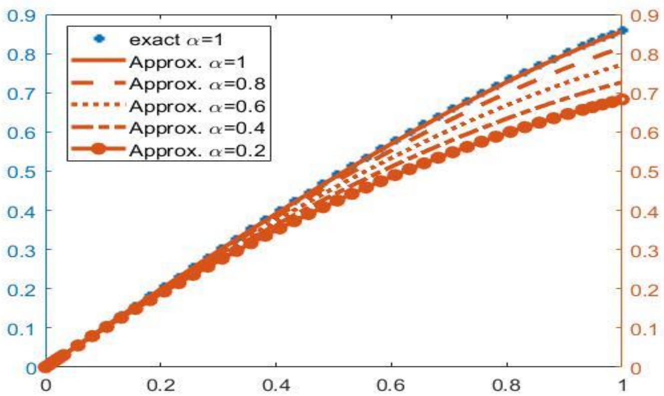


Figure 7: the approximate solution of Van der Pol FODE using the proposed method, when $\mu = 0.05$ comparing with the numerical solution.

5. Conclusions

In this article, the proposed method was studied, compared, and applied to a wide range of linear and nonlinear fractional differential equations with constant and variable coefficients. Orthogonal functions were also compared with the Chelyshkov polynomial. In addition, the convergence of the proposed method was studied (see proposition (1)), and the famous weighted residual methods were compared. The results show the effectiveness of the method and its efficiency in solving different types of differential equations. The effect of the fractional degree on Chelyshkov polynomials was studied, in contrast to the results obtained in previous literature. We point out that the fractional degree in polynomials is useful if the problem requires it, while some results show divergence when using the fractional degree (see Ex 2). The results show that the proposed Mittag-Leffler method gives accurate and fast convergence results compared to other methods, which reduces significant computational efforts. In addition, the effect of fractional orders was studied and the results showed that the effect is very small (see table (2)).

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