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# $\mathcal{R}_{as}$ -Projective Modules

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#### Abstract

A submodule A of an  $\mathcal{R}$ -module M is  $\mathcal{R}$ -Annihilator-small submodule  $(\mathcal{R}_{as}-submodule)$  if whenever  $A+A_1=M$ ,  $A_1$  is a faithful submodule of M, implies that  $Ann(A_1)=0$ . In this paper, we introduce the notation of  $\mathcal{R}_{as}$ -projective as a generalization of projective modules. Also, we define and present some properties of  $\mathcal{R}_{as}$ -epimorphism.

**Keywords:** Projective modules,  $\mathcal{R}_{as}$  – submodules,  $\mathcal{R}_{as}$  –epimorphisim,  $\mathcal{R}_{as}$  –projective modules.

# $\mathcal{R}$ المقاسات التالفة الصغيرة الاسقاطية من النمط

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#### الخلاصة

المقاس الجزئي A من المقاس M على الحلقة  $\mathcal{R}$  هو مقاس تالف صغير من النمط  $\mathcal{R}$  اذا متى ماكان A البحث،  $A+A_1=M$  مقاس جزئي تالف من A يقود الى A على البحث،  $A+A_1=M$  قدمنا مفهوم المقاسات التالفة الصغيرة الاسقاطية من النمط A كتعميم للمقاسات الاسقاطية. وكذلك، عرفنا وعرضنا بعض خواص التماثل الشامل الصغير التالف من النمط A.

#### 1. Introduction

In the present paper, the ring  $\mathcal{R}$  is associative with identity and every  $\mathcal{R}$  —module is left and unitary. A submodule A of an  $\mathcal{R}$  —module M is small submodule (denoted by  $A \ll M$ ) if whenever  $A + A_1 = M$ , implies that  $A_1 = M$ , where  $A_1 \subseteq M$  [1-5]. Let  $\mathcal{R}$  be an integral domain, a module M is called a torsion-free  $\mathcal{R}$  —module if Ann(x) = 0, for every non-zero element x in M [6-8]. An  $\mathcal{R}$  —module M is called faithful if Ann(M) = 0 [9-10]. A submodule A of an  $\mathcal{R}$  — module M is  $\mathcal{R}$  —Annihilator—small submodule  $(\mathcal{R}_{as}$  —submodule and denoted by  $A \ll^a M$ ) if whenever  $A + A_1 = M$ ,  $A_1$  is a faithful submodule of M, subsequently  $Ann(A_1) = 0$ , where  $Ann(A_1) = \{r \in \mathcal{R} : r, A_1 = 0\}$  [11],

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[12]. An  $\mathcal{R}$  -module M is an  $\mathcal{R}$  -Annihilator -hollow module ( $\mathcal{R}_{as}$  -hollow) if every proper submodule of M is an  $\mathcal{R}_{as}$  -submodule in M [13]. An  $\mathcal{R}$  -module P is projective, if for any  $f \in Hom(P,B)$  and any epimorphism  $g:A \to B$ , there exists  $f_1 \in Hom(P,A)$  such that  $gof_1 = f$  [14-16]. An  $\mathcal{R}$  -module P is called small projective module, if for any epimorphism  $g:A \to B$  with  $ker(g) \ll A$  then goHom(P,A) = Hom(P,B) [17]. Let P be a projective module and  $f:P \to M$  be an epimorphism with  $ker(f) \ll P$ , then a pair (P,f) is a projective cover of M [18], [19]. A ring  $\mathcal{R}$  is right-perfect if every right  $\mathcal{R}$  -module have a projective cover [18]. An epimorphism  $f:A \to B$  is called split if there exists a homomorphism  $f_1:B \to A$  with  $fof_1 = I_B$  [20]. In this paper the  $\mathcal{R}_{as}$  -Projective modules and  $\mathcal{R}_{as}$  -epimorphism are studied and their properties are obtained.

# 2. $\mathcal{R}_{as}$ -epimorphism

## **Definition 2.1:**

For any  $\mathcal{R}$  —modules A and B. An epimorphism  $g: A \to B$  is an  $\mathcal{R}_{as}$  — epimorphism as long as ker(g) is an  $\mathcal{R}_{as}$  —submodule of A.

## Example 2.2:

Consider the module Z as Z—module. Let  $\pi: Z \to \frac{Z}{nZ}$  be the natural epimorphism. To show that  $ker(\pi) = nZ \ll^a Z$ , let Z = nZ + mZ, where  $Ann(mZ) = \{r \in Z; r.mZ = 0\} = 0$ , hence mZ is a faithful submodule of Z. Thus,  $ker \pi$  is  $Z_{as}$ —submodule of Z. Thus  $\pi$  is an  $Z_{as}$ —epimorphism.

# **Proposition 2.3:**

Let M, M' and M'' be  $\mathcal{R}$  —modules. If  $f: M \to M'$  and  $g: M' \to M''$  are two epimorphisms, then:

- 1. If  $g \circ f$  is an  $\mathcal{R}_{as}$  epimorphism, then f is an  $\mathcal{R}_{as}$  –epimorphism.
- 2. If g is an  $\mathcal{R}_{as}$  epimorphism, then  $g \circ f$  is an  $\mathcal{R}_{as}$  –epimorphism.

# **Proof:**

- 1. Let  $g \circ f$  be an  $\mathcal{R}_{as}$  epimorphism and suppose that M = K + ker(f), where  $K \subseteq M$ . We claim that  $M = K + ker(g \circ f)$ . Clearly,  $ker(g \circ f) + K \subseteq M$ , so it is enough to show that  $M \subseteq ker(g \circ f) + K$ . Let  $x \in M$  then  $f(x) \in M'$  and  $g(f(x)) \in M''$ , but M = ker(f) + K, then x = a + b; where a belongs to ker(f) and b belongs to K, then f(x) = f(b), hence g(f(x)) = g(f(b)) and  $M = K + ker(g \circ f)$ . But  $ker(g \circ f) \ll^a M$ , therefore K is faithful submodule of M. Thus, ker(f) is an  $\mathcal{R}_{as}$  –epimorphisim.
- 2. Let g be  $\mathcal{R}_{as}$  epimorphism and suppose that  $M = K + ker(g \circ f)$ , where  $K \subseteq M$ . Claim that M' = ker(g) + f(K). It is clear that  $ker(g) + f(K) \subseteq M'$ . Now, let  $y \in M'$ , then  $f^{-1}(y) \in M$ , so  $f^{-1}(y) = a + b$ , where a belongs to  $ker(g \circ f)$  and b belongs to K. As  $y = f \circ f^{-1}(y) = f(a) + f(b)$ , where  $f(a) \in ker(g)$  and  $f(b) \in f(K)$ . Hence, M' = f(K) + ker(g). But  $ker(g) \ll^a M'$ , then  $ker(g \circ f) = (f^{-1}(ker(g)) \ll^a M)$  by [11, Proposition 2.1.5], and hence K is faithful submodule of M. Thus,  $ker(g \circ f)$  is an  $\mathcal{R}_{as}$  –epimorphisim.

## Corollary 2.4:

Every split  $\mathcal{R}_{as}$  –epimorphism is an isomorphism.

## **Proof:**

Let  $f: M \to N$  be an  $\mathcal{R}_{as}$  -epimorphisim which is splits. To prove f is an isomorphism, it is enough to show that ker(f) = 0. Let  $x \in ker(f)$ , so f(x) = 0. Now, since f is splits so there exists  $f_1: N \to M$  such that  $f_1 \circ f = I_M$ . Thus,  $f_1 \circ f(x) = f_1(f(x)) = f_1(0) = 0$ , and so x = 0. Hence, ker(f) = 0.

# **Proposition 2.5:**

Consider the following commutative diagram of modules  $A_1$ ,  $B_1$ ,  $C_1$  and  $A_2$ ,  $B_2$ ,  $C_2$ :

$$0 \to A_1 \stackrel{f_1}{\to} B_1 \stackrel{g_1}{\to} C_1 \to 0$$

$$\alpha \qquad \beta \qquad \gamma$$

$$0 \to A_2 \stackrel{f_2}{\to} B_2 \stackrel{g_2}{\to} C_2 \to 0$$

With both rows are exact then:

- 1. If  $\alpha, \beta$  are epimorphisms and  $g_2$  is an  $\mathcal{R}_{as}$  -epimorphism, then  $g_1$  is an  $\mathcal{R}_{as}$  -epimorphism.
- 2. If  $\beta$  is an epimorphism,  $\gamma$  is monomorphism and  $g_2$  is an  $\mathcal{R}_{as}$  -epimorphism, then  $g_1$  an  $\mathcal{R}_{as}$  -epimorphism.

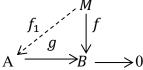
#### **Proof:**

- 1. Let  $B_1 = ker(g_1) + D$ , where D is a submodule of  $B_1$ . Since the diagram is commute, then  $f_2 \circ \alpha(A_1) = \beta \circ f_1(A_1)$ . But  $\alpha$  is onto, then  $f_2(A_2) = \beta(ker(g_1))$ . Also, the sequences are exact, hence  $ker(g_2) = \beta(ker(g_1))$  and  $\beta^{-1}(ker(g_2)) = ker(g_1)$ . Since  $g_2$  is an  $\mathcal{R}_{as}$  -epimorphisim and by [11, Proposition 2.1.5], we get  $ker(g_1) = \beta^{-1}(ker(g_2)) \ll^a B_1$  and hence  $g_1$  is an  $\mathcal{R}_{as}$  -epimorphisim.
- 2. Since the diagram is commute, then  $\gamma \circ g_1 = g_2 \circ \beta$ . But  $\gamma$  is monomorphism, so  $\beta^{-1}(ker(g_2)) = ker(g_1)$ . Since  $ker(g_2) \ll^a B_1$  ( $g_2$  is an  $\mathcal{R}_{as}$  -epimorphism), then  $ker(g_1) \ll^a B_1$  by [11, Proposition 2.1.5]. Therefore,  $g_1$  is an  $\mathcal{R}_{as}$  -epimorphism.

# 3. $\mathcal{R}_{as}$ -Projective Modules

# **Definition 3.1:**

If the following diagram is commute:

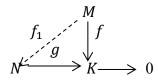


Where g is  $\mathcal{R}_{as}$  -epimorphism (i.e.,  $ker(g) \ll^a A$ ) and f is any homomorphism. Then an  $\mathcal{R}$  - module M is called an  $\mathcal{R}_{as}$  -projective.

**NOTE:** Obviously that every projective module is an  $\mathcal{R}_{as}$  -projective.

#### Remarks 3.2:

Let M, N and K be  $\mathcal{R}$  —modules and consider the following diagram:



Then:

- 1. If N is projective (free), then M is projective if and only if it is an  $\mathcal{R}_{as}$  -projective by [11, Proposition 2.1.12].
- 2. If N is faithful, then every small projective module M is an  $\mathcal{R}_{as}$  -projective by [11, Proposition 2.1.10].
- 3. Let N be a torsion-free with  $\mathcal{R}$  is an integral domain. Therefore, M is projective if and only if it is an  $\mathcal{R}_{as}$  -projective by [11, Proposition 2.1.11].
- 4. Let N be a faithful with  $ker(g) \ll^e \mathcal{R}$ . So, M is projective if and only if it is  $\mathcal{R}_{as}$  -projective, by [11, Proposition 2.1.13].

- 5. If M is an  $\mathcal{R}_{as}$  -projective, then N is a faithful by [11, Proposition 2.1.14].
- 6. Let N be a faithful and R is an integral domain with  $Ann(ker(g)) \neq 0$ . Then M is projective if and only if it is  $\mathcal{R}_{as}$  -projective by [11, Proposition 2.1.15].
- 7. Let N be a faithful and torsion-free and let  $\mathcal{R}$  be an integral domain such that ker(g) is finitely-generated submodule of N. So M is projective if and only if it is an  $\mathcal{R}_{as}$  -projective by [11, Proposition 2.1.16].
- 8. Let N be an  $\mathcal{R}_{as}$  -hollow module. Then M is projective if and only if M is an  $\mathcal{R}_{as}$  -projective.

# **Proposition 3.3:**

Let M be an  $\mathcal{R}$  —module, then the statements below are equivalent:

- 1. M is  $\mathcal{R}_{as}$  -projective;
- 2. For each  $\mathcal{R}_{as}$  -epimorphism  $g: N \to K$  the homomorphism  $Hom(I,g): Hom(M,N) \to Hom(M,K)$  is an epimorphism;
- 3. For any  $\mathcal{R}_{as}$  -epimorphism  $g: N \to K$ ,  $Hom(M, K) = g \circ Hom(M, N)$ .

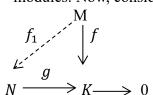
# **Proof:**

 $(1 \Longrightarrow 2)$  Let  $g: N \to K$  be an  $\mathcal{R}_{as}$  -epimorphism and  $f \in Hom(M,K)$ . Since M is an  $\mathcal{R}_{as}$  -projective, consequently there exists  $f_1 \in Hom(M,N)$  such that  $g \circ f_1 = f$ . So,  $Hom(I,g) \circ f_1 = g \circ f_1 = f$ .

$$\begin{array}{ccc}
f_1 & & M \\
f & & \downarrow f \\
N & \xrightarrow{g} K \longrightarrow 0
\end{array}$$

 $(2 \Rightarrow 3)$  Let  $g: N \to K$  be an  $\mathcal{R}_{as}$  -epimorphism. Form (2) we get  $Hom(I,g): Hom(M,N) \to Hom(M,K)$  is an epimorphism. To show that  $Hom(M,K) = g \circ Hom(M,N)$ . It is enough to show that  $Hom(M,K) \subseteq g \circ Hom(M,N)$ , since it is clear that  $g \circ Hom(M,N) \subseteq Hom(M,K)$ . Now, let  $f \in Hom(M,K)$ , then there exists  $f_1 \in Hom(M,N)$  such that  $Hom(I,g) \circ f_1 = f$ . Hence,  $f \in g \circ Hom(M,N)$ . Thus  $Hom(M,K) \subseteq g \circ Hom(M,N)$ .

 $(3 \Rightarrow 1)$  Let N and K are any  $\mathcal{R}$  – modules. Now, consider the following diagram:



Where g is  $\mathcal{R}_{as}$  -epimorphisim (i.e.,  $ker(g) \ll^a N$ ) and f is any homomorphism. From (3), we get  $Hom(M,K) = g \circ Hom(M,N)$  and since  $f \in Hom(M,K)$ , so there exists  $f_1 \in Hom(M,N)$  such that  $g \circ f_1 = f$ . Therefore, M is an  $\mathcal{R}_{as}$  -projective.

# **Proposition 3.4:**

Let M be a projective  $\mathcal{R}$  — module and let  $\mathcal{R}$  be an integral domain. Consequently, every epimorphism  $f: M \to N$  is an  $\mathcal{R}$  as —epimorphism.

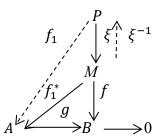
**Proof:** Let  $f: M \to N$  be an epimorphism. Since every proper submodule of projective  $\mathcal{R}$  — module M is an  $\mathcal{R}_{as}$  —submodule by [11, Proposition 2.1.12]. So,  $ker(f) \ll^a M$ . Thus f is an  $\mathcal{R}_{as}$  —epimorphism.

#### Remark 3.5:

Let M be an  $\mathcal{R}$  – module and let P be an  $\mathcal{R}_{as}$  –projective such that  $P \simeq M$ . Then M is an  $\mathcal{R}_{as}$  –projective.

#### **Proof:**

Consider the following diagram:



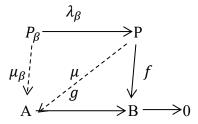
Where  $\xi\colon P\to M$  is an isomorphism, f is any homomorphism and g is an  $\mathcal{R}_{as}$  -epimorphism. Since P is an  $\mathcal{R}_{as}$  -projective, then there exists a homomorphism  $f_1\colon P\to A$  such that  $g\circ f_1=f\circ \xi$ . Define  $\xi^{-1}\colon M\to P$  be a homomorphism such that  $\xi\circ \xi^{-1}=I_M$  and  $f_1^*\colon M\to A$  by  $f_1^*=f_1\circ \xi^{-1}$ . Now, to show that  $g\circ f_1^*=f$ . Since  $g\circ f_1=f\circ \xi$ , then  $g\circ f_1^*=g\circ (f_1\circ \xi^{-1})=(f\circ \xi)\circ \xi^{-1}=f\circ I_M=f$ .

# **Proposition 3.6:**

Let  $P = \bigoplus_{\beta \in J} P_{\beta}$ . Then  $P_{\beta}$  is an  $\mathcal{R}_{as}$  -projective for all  $\beta$  belongs to J if and only if P is an  $\mathcal{R}_{as}$  -projective.

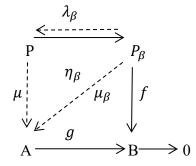
#### **Proof:**

Suppose that  $P_{\beta}$  is an  $\mathcal{R}_{as}$  -projective for all  $\beta \in J$ . Consider the following diagram:



Where  $g: A \to B$  is an  $\mathcal{R}_{as}$  -epimorphism (i.e.,  $ker(g) \ll^a A$ ),  $f: P \to B$  is any homomorphism and  $\lambda_\beta \colon P_\beta \to P$  be the injection homomorphism. Since  $P_\beta$  is an  $\mathcal{R}_{as}$  -projective for all  $\beta \in J$ , so there exists a homomorphism  $\mu_\beta \colon P_\beta \to A$  such that  $g \circ \mu_\beta = f \circ \lambda_\beta$  for all  $\beta \in J$ . Define  $\mu \colon P \to A$  such that  $\mu_\beta = \mu \circ \lambda_\beta$ . It enough to show that  $g \circ \mu = f$ . Since  $f \circ \lambda_\beta = g \circ \mu_\beta$  and so  $\mu_\beta = \mu \circ \lambda_\beta$ . Consequently,  $f \circ \lambda_\beta = g \circ \mu \circ \lambda_\beta$ . So, we get by [2, Remark 4.1.4]  $g \circ \mu = f$ . As a result,  $P = \bigoplus_{\beta \in J} P_\beta$  is an  $\mathcal{R}_{as}$  -projective.

Conversely, assume that  $P = \bigoplus_{\beta \in J} P_{\beta}$  is an  $\mathcal{R}_{as}$  -projective module and let  $\beta \in J$ . Consider the following diagram:



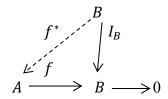
Where  $g: A \to B$  is an  $\mathcal{R}_{as}$  -epimorphism (i.e  $ker(g) \ll^a A$ ),  $f: P_\beta \to B$  is any homomorphism,  $\eta_\beta: P \to P_\beta$  is the projection homomorphism and the injection homomorphism  $\lambda_\beta: P_\beta \to P$ . Since P is  $\mathcal{R}_{as}$  -projective, so there exists  $\mu \in Hom(P,A)$  such that  $g \circ \mu = f \circ \eta_\beta$ . Define  $\mu_\beta: P_\beta \to A$  by  $\mu_\beta = \mu \circ \lambda_\beta$ . Now,  $g \circ \mu_\beta = g \circ (\mu \circ \lambda_\beta) = (f \circ \eta_\beta) \circ \lambda_\beta = f \circ I_{P_\beta} = f$ . Thus  $P_\beta$  is an  $\mathcal{R}_{as}$  -projective for all  $\beta$  belongs to J.

# **Proposition 3.7:**

Let A be an  $\mathcal{R}_{as}$  -projective module and  $f: A \to B$  be an  $R_{as}$  -epimorphism. Then f is splits if and only if  $A \oplus B$  is an  $\mathcal{R}_{as}$  -projective.

#### Proof:

- $\Rightarrow$ ) Let f be a split  $\mathcal{R}_{as}$  -epimorphism, then f is an isomorphism by Corollary 2.4 and so B is an  $\mathcal{R}_{as}$  -projective by Remark 3.5. Consequently,  $A \oplus B$  is an  $\mathcal{R}_{as}$  -projective by Proposition 3.6.
- $\Leftarrow$ ) Suppose that  $A \oplus B$  is an  $\mathcal{R}_{as}$  -projective, then B is an  $\mathcal{R}_{as}$  -projective by Proposition 3.6. Now, consider the following diagram:



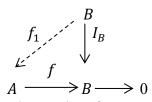
Since  $f: A \to B$  is an  $\mathcal{R}_{as}$  -epimorphism and B is an  $\mathcal{R}_{as}$  -projective, so there exists  $f^*: B \to A$  such that  $f \circ f^* = I_B$ . Thus f is split.

# Remark 3.8:

Every  $\mathcal{R}_{as}$  -epimorphism  $f: A \to B$  where B is an  $\mathcal{R}_{as}$  -projective is splits.

#### Proof:

Let  $f: A \to B$  be an  $\mathcal{R}_{as}$  -epimorphism where  $ker(f) \ll^a A$ . Consider the following diagram:



Since B is an  $\mathcal{R}_{as}$  -projective, then there exists  $f_1: B \to A$  such that  $f \circ f_1 = I_B$ . As a result, by [2, Lemma 3.9.3] the sequence

$$0 \longrightarrow ker(f) \xrightarrow{i} A \xrightarrow{f} B \longrightarrow 0$$

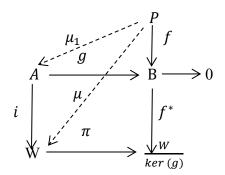
is splits. Where  $i: ker(f) \rightarrow A$  is the inclusion homomorphism.

# **Proposition 3.9:**

Let P be an  $\mathcal{R}$ —module. Then P is an  $\mathcal{R}_{as}$ —projective if and only if for every  $\mathcal{R}_{as}$ —epimorphism  $g: A \to B$  where A is an injective module and every  $f \in Hom(P,B)$ , there exists  $\mu \in Hom(P,A)$  such that  $g \circ \mu = f$ .

# **Proof:**

- $\implies$ ) Clear.
- $\iff$  Let  $g: A \to B$  be any  $\mathcal{R}_{as}$  -epimorphism (i.e.,  $ker(g) \ll^a A$ ), for any modules A, B and let  $f \in Hom(P, B)$ . Consider the below diagram:



Where  $i: A \to W$  is the inclusion homomorphism with W is injective module (every module can be embedded in W by [2, Corollary 5.5.5]), and let  $\pi: W \to \frac{W}{ker(g)}$  be the nature epimorphism. Now, define  $f^*: B \to \frac{W}{ker(g)}$  by  $f^*(b) = a + ker(g)$  for all  $b \in B$ , where g(a) = b. Let  $b_1, b \in B$  such that g(a) = b and  $g(a_1) = b_1$ . If  $b = b_1$  we obtain  $g(a) = g(a_1)$  which implies  $a - a_1 \in ker(g)$  and so  $a + ker(g) = a_1 + ker(g)$ . Hence,  $f^*$  is a well define homomorphism.

According to the assumption, there exists  $\mu \in Hom(P,W)$  such that  $\pi \circ \mu = f^* \circ f$ . Claim that  $\mu(P) \subseteq A$ . To show that, let  $s \in \mu(P)$ , then there exists  $t \in P$  such that  $s = \mu(t)$ . Now,  $\pi \circ \mu(t) = f^* \circ f(t)$  where f(t) = g(a). Means that  $\mu(t) - a \in ker(g)$  and hence  $\mu(t) \in A$ . Define  $\mu_1 : P \to A$  by  $\mu_1(p) = \mu(p)$ , for all p belongs to P. Now,  $f^* \circ f = \pi \circ \mu = \pi \circ i \circ \mu_1 = f^* \circ g \circ \mu_1$ . It enough to show that  $f^*$  is monomorphism. Let  $f^*(b) = f^*(b^*)$ , where  $b^*, b \in B$ . So,  $a + ker(g) = a^* + ker(g)$  where g(a) = b and  $g(a^*) = b^*$ . Thus  $a - a^* \in ker(g)$  this implies that  $g(a^*) = g(a)$  and so  $b^* = b$  therefore,  $f^*$  is monomorphism, so  $f = g \circ \mu_1$ . Hence, P is an  $\mathcal{R}_{as}$  -projective module.

## **Definition 3.10 [21]:**

Let  $f: P \to M$  be an epimorphism with  $ker(f) <<^a P$  where P is a projective module. Then a pair (P, f) is a projective R - a —cover of M.

#### **Proposition 3.11:**

Let  $\mathcal{R}$  be an integral domain and P is an  $\mathcal{R}$  -module. Then every projective cover (P, f) of an  $\mathcal{R}$  -module M is a projective  $\mathcal{R} - a$  -cover.

# **Proof:**

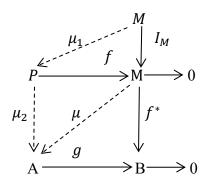
Let (P, f) be a protective cover of M where  $f: P \to M$  is an epimorphism. Since P is a projective and  $\mathcal{R}$  is an integral-domain, so by Proposition 3.4 f is an  $\mathcal{R}_{as}$  -epimorphism. Thus (P, f) is a projective  $\mathcal{R} - a$  -cover.

#### **Proposition 3.12:**

Let M be an  $\mathcal{R}_{as}$  -projective module. If (P, f) is a protective  $\mathcal{R} - a$  -cover of M, then M is projective.

## **Proof:**

Assume that M be an  $\mathcal{R}_{as}$  -projective and let (P, f) be a projective  $\mathcal{R} - a$  -cover of M. Consider the following diagram:



Such that  $g: A \to B$  is an epimorphism,  $f^* \in Hom(M, B)$  and  $I_M: M \to M$  is the identity. But f is an  $\mathcal{R}_{as}$  -epimorphism and M is an  $\mathcal{R}_{as}$  -projective. So, there exists  $\mu_1 \in Hom(M, P)$  such that  $f \circ \mu_1 = I_M$ . Since P is projective, consequently, there exists  $\mu_2 \in Hom(P, A)$  such that  $g \circ \mu_2 = f^* \circ f$ . Now, define  $\mu: M \to A$  by  $\mu = \mu_2 \circ \mu_1$ . Finally, to show that  $g \circ \mu = f^*$ . Since  $g \circ \mu = g \circ (\mu_2 \circ \mu_1) = (f^* \circ f) \circ \mu_1 = f^* \circ I_M = f^*$ . As a result, M is a projective module.

# Corollary 3.13:

Let  $\mathcal{R}$  be a right-perfect integral domain ring. Therefore, every  $\mathcal{R}$  -module is an  $\mathcal{R}_{as}$  -projective if and only if it is projective.

### **Proof:**

- $\Rightarrow$ ) clear.
- $\Leftarrow$ ) Assume that M is an  $\mathcal{R}_{as}$  -projective  $\mathcal{R}$  -module and  $\mathcal{R}$  is a right-perfect ring and I.D.Then M have a protective cover (P, f). Consequently, by Proposition 3.11 and 3.12 we get M is a projective.

## 4. Conclusions:

In this work, the concepts of projective modules and epimorphism have been generalized to new concepts called  $\mathcal{R}_{as}$  -projective modules and  $\mathcal{R}_{as}$  -epimorphism respectively. Some properties of this type of module and epimorphism have been studied. Also, we see the relation between  $\mathcal{R}_{as}$  -projective, projective and small projective modules

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