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$\mathcal{R}_{as}$ −Projective Modules

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Abstract
A submodule A of an $\mathcal{R}-module$ M is $\mathcal{R}$ −*Annihilator*−*small* submodule ($\mathcal{R}_{as} - submodule$) if whenever $A + A_1 = M$, $A_1$is a faithful submodule of M, implies that $Ann\,(A_1) = 0$. In this paper, we introduce the notation of $\mathcal{R}_{as}$ −projective as a generalization of projective modules. Also, we define and present some properties of $\mathcal{R}_{as}$ − epimorphism.

Keywords: Projective modules, $\mathcal{R}_{as}-$ submodules, $\mathcal{R}_{as}$ −epimorphisim, $\mathcal{R}_{as}$ −projective modules.

المقاسات التالفة الصغيرة الاسقاطية من النمط− $\mathcal{R}$

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الخلاصة
المقاس الجزئي A من المقاس M على الحلقة $\mathcal{R}$ هو مقاس تالف صغير من النمط $\mathcal{R}$ اذا متى ماكان $A + A_1 = M$، حيث أن A_1 مقاس جزئي تالف من M يقود الى $Ann\,(A_1) = 0$. في هذا البحث، قدمنا مفهوم المقاسات التالفة الصغيرة الاسقاطية من النمط $\mathcal{R}$ كتعميم للمقاسات الاسقاطية. وكذلك، عرفنا وعرضنا بعض خواص التماثل الشامل الصغير التالف من النمط $\mathcal{R}$.

1. Introduction

In the present paper, the ring $\mathcal{R}$ is associative with identity and every $\mathcal{R}$ −module is left and unitary. A submodule A of an $\mathcal{R}$ −module M is small submodule (denoted by $A \ll M$) if whenever $A + A_1 = M$, implies that $A_1 = M$, where $A_1 \subseteq M$ [1-5]. Let $\mathcal{R}$ be an integral domain, a module M is called a torsion-free $\mathcal{R}$ −module if $Ann(x) = 0$, for every non-zero element x in M [6-8]. An $\mathcal{R}$ −module M is called faithful if $Ann(M) = 0$ [9-10]. A submodule A of an $\mathcal{R}-$ module M is $\mathcal{R}$ −*Annihilator*−*small* submodule ($\mathcal{R}_{as}$ −submodule and denoted by $A \ll^a M$) if whenever $A + A_1 = M$, A_1 is a faithful submodule of M, subsequently $Ann(A_1) = 0$, where $Ann\,(A_1) = \{r \in \mathcal{R}\,;\, r.A_1 = 0\}$ [11],

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[12]. An $\mathcal{R}$ $-$module M is an $\mathcal{R}-Annihilator-hollow$ module ($\mathcal{R}_{as}$ $-$hollow) if every proper submodule of M is an $\mathcal{R}_{as}$ $-$submodule in M [13]. An $\mathcal{R}$ $-$module P is projective, if for any $f \in Hom(P, B)$ and any epimorphism $g: A \to B$, there exists $f_1 \in Hom(P, A)$ such that $gof_1 = f$ [14-16]. An $\mathcal{R}$ $-$module P is called small projective module, if for any epimorphism $g: A \to B$ with $ker(g) \ll A$ then $goHom(P, A) = Hom(P, B)$ [17]. Let P be a projective module and $f: P \longrightarrow M$ be an epimorphism with $\ker(f) \ll P$, then a pair (P, f) is a projective cover of M [18], [19]. A ring $\mathcal{R}$ is right-perfect if every right $\mathcal{R}$ $-$module have a projective cover [18]. An epimorphism $f: A \to B$ is called split if there exists a homomorphism $f_1: B \to A$ with $fof_1 = I_B$ [20]. In this paper the $\mathcal{R}_{as}$ $-$Projective modules and $\mathcal{R}_{as}$ $-$epimorphism are studied and their properties are obtained.

2. $\mathcal{R}_{as}$ $-$epimorphism

Definition 2.1:

For any $\mathcal{R}$ $-$modules A and B. An epimorphism $g: A \to B$ is an $\mathcal{R}_{as}-$ epimorphism as long as ker(g) is an $\mathcal{R}_{as}$ $-$submodule of A.

Example 2.2:

Consider the module Z as Z $-$module. Let $\pi: Z \to \frac{Z}{nZ}$ be the natural epimorphism. To show that $ker(\pi) = nZ \ll^{a} Z$, let $Z = nZ + mZ$, where $Ann(mZ) = \{r \in Z; r.mZ = 0\} = 0$, hence mZ is a faithful submodule of Z. Thus, $ker\,\pi$ is Z_{as} $-$submodule of Z. Thus π is an $Z_{as}-$ epimorphism.

Proposition 2.3:

Let M, M' and M'' be $\mathcal{R}$ $-$modules. If $f: M \to M'$ and $g: M' \to M''$ are two epimorphisms, then:

1. If $g \circ f$ is an $\mathcal{R}_{as}-$ epimorphism, then f is an $\mathcal{R}_{as}$ $-$epimorphism.
2. If g is an $\mathcal{R}_{as}-$ epimorphism, then $g \circ f$ is an $\mathcal{R}_{as}$ $-$epimorphism.

Proof:

1. Let $g \circ f$ be an $\mathcal{R}_{as}-$ epimorphism and suppose that $M = K + ker(f)$, where $K \subseteq M$. We claim that $M = K + ker(g \circ f)$. Clearly, $ker(g \circ f) + K \subseteq M$, so it is enough to show that $M \subseteq ker(g \circ f) + K$. Let $x \in M$ then $f(x) \in M'$ and $g(f(x)) \in M''$, but $M = ker(f) + K$, then $x = a + b$; where a belongs to $ker(f)$ and b belongs to K, then $f(x) = f(b)$, hence $g(f(x)) = g(f(b))$ and $M = K + ker(g \circ f)$. But $ker\,(g \circ f) \ll^{a} M$, therefore K is faithful submodule of M. Thus, $ker(f)$ is an $\mathcal{R}_{as}$ $-$epimorphisim.
2. Let g be $\mathcal{R}_{as}-$ epimorphism and suppose that $M = K + ker(g \circ f)$, where $K \subseteq M$. Claim that $M' = ker(g) + f(K)$. It is clear that $ker(g) + f(K) \subseteq M'$. Now, let $y \in M'$, then $f^{-1}(y) \in M$, so $f^{-1}(y) = a + b$, where a belongs to $ker(g \circ f)$ and b belongs to K. As $y = f \circ f^{-1}(y) = f(a) + f(b)$, where $f(a) \in ker(g)$ and $f(b) \in f(K)$. Hence, $M' = f(K) + ker(g)$. But $ker(g) \ll^{a} M'$, then $ker(g \circ f) = (f^{-1}(ker(g)) \ll^{a} M$ by [11, Proposition 2.1.5], and hence K is faithful submodule of M. Thus, $ker(g \circ f)$ is an $\mathcal{R}_{as}$ $-$epimorphisim.

Corollary 2.4:

Every split $\mathcal{R}_{as}$ $-$epimorphisim is an isomorphism.

Proof:

Let $f: M \to N$ be an $\mathcal{R}_{as}$ $-$epimorphisim which is splits. To prove f is an isomorphism, it is enough to show that $ker(f) = 0$. Let $x \in ker(f)$, so $f(x) = 0$. Now, since f is splits so there exists $f_1: N \to M$ such that $f_1 \circ f = I_M$. Thus, $f_1 \circ f(x) = f_1(f(x)) = f_1(0) = 0$, and so $x = 0$. Hence, $ker(f) = 0$.

Proposition 2.5:

Consider the following commutative diagram of modules A_1, B_1, C_1 and A_2, B_2, C_2:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & B_1 \arrow[r, "g_1"] \arrow[d, "\beta"] & C_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "f_2"] & B_2 \arrow[r, "g_2"] & C_2 \arrow[r] & 0
\end{tikzcd}$$

With both rows are exact then:

1. If α, β are epimorphisms and g_2 is an $\mathcal{R}_{as}$ −epimorphisim, then g_1 is an $\mathcal{R}_{as}$ −epimorphisim.
2. If β is an epimorphism, γ is monomorphism and g_2 is an $\mathcal{R}_{as}$ −epimorphisim, then g_1 an $\mathcal{R}_{as}$ −epimorphisim.

Proof:

1. Let $B_1 = ker(g_1) + D$, where D is a submodule of B_1. Since the diagram is commute, then $f_2 \circ \alpha(A_1) = \beta \circ f_1(A_1)$. But α is onto, then $f_2(A_2) = \beta(ker(g_1))$. Also, the sequences are exact, hence $ker(g_2) = \beta(ker(g_1))$ and $\beta^{-1}(ker(g_2)) = ker(g_1)$. Since g_2 is an $\mathcal{R}_{as}$ −epimorphisim and by [11, Proposition 2.1.5], we get $ker(g_1) = \beta^{-1}(ker(g_2)) \ll^a B_1$ and hence g_1 is an $\mathcal{R}_{as}$ −epimorphisim.
2. Since the diagram is commute, then $\gamma \circ g_1 = g_2 \circ \beta$. But γ is monomorphism, so $\beta^{-1}(ker(g_2)) = ker(g_1)$. Since $ker(g_2) \ll^a B_1$ (g_2 is an $\mathcal{R}_{as}$ −epimorphisim), then $ker(g_1) \ll^a B_1$ by [11, Proposition 2.1.5]. Therefore, g_1 is an $\mathcal{R}_{as}$ −epimorphisim.

3. $\mathcal{R}_{as}$ −Projective Modules

Definition 3.1:

If the following diagram is commute:

$$\begin{tikzcd}
& M \arrow[dl, dashed, "f_1"'] \arrow[d, "f"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

Where g is $\mathcal{R}_{as}$ −epimorphisim (i.e., $ker(g) \ll^a A$) and f is any homomorphism. Then an $\mathcal{R}-$ module M is called an $\mathcal{R}_{as}$ −projective.

NOTE: Obviously that every projective module is an $\mathcal{R}_{as}$ −projective.

Remarks 3.2:

Let M, N and K be $\mathcal{R}$ −modules and consider the following diagram:

$$\begin{tikzcd}
& M \arrow[dl, dashed, "f_1"'] \arrow[d, "f"] & \\
N \arrow[r, "g"] & K \arrow[r] & 0
\end{tikzcd}$$

Then:

1. If N is projective (free), then M is projective if and only if it is an $\mathcal{R}_{as}$ −projective by [11, Proposition 2.1.12].
2. If N is faithful, then every small projective module M is an $\mathcal{R}_{as}$ −projective by [11, Proposition 2.1.10].
3. Let N be a torsion-free with $\mathcal{R}$ is an integral domain. Therefore, M is projective if and only if it is an $\mathcal{R}_{as}$ −projective by [11, Proposition 2.1.11].
4. Let N be a faithful with $ker(g) \ll^e \mathcal{R}$. So, M is projective if and only if it is $\mathcal{R}_{as}$ −projective, by [11, Proposition 2.1.13].

5. If M is an $\mathcal{R}_{as}$ −projective, then N is a faithful by [11, Proposition 2.1.14].
6. Let N be a faithful and R is an integral domain with $Ann(ker(g)) \neq 0$. Then M is projective if and only if it is $\mathcal{R}_{as}$ −projective by [11, Proposition 2.1.15].
7. Let N be a faithful and torsion-free and let $\mathcal{R}$ be an integral domain such that $ker(g)$ is finitely-generated submodule of N. So M is projective if and only if it is an $\mathcal{R}_{as}$ −projective by [11, Proposition 2.1.16].
8. Let N be an $\mathcal{R}_{as}$ −hollow module. Then M is projective if and only if M is an $\mathcal{R}_{as}$ −projective.

Proposition 3.3:

Let M be an $\mathcal{R}$ −module, then the statements below are equivalent:
1. M is $\mathcal{R}_{as}$ −projective;
2. For each $\mathcal{R}_{as}$ −epimorphism $g: N \to K$ the homomorphism $Hom(I, g): Hom(M, N) \to Hom(M, K)$ is an epimorphism;
3. For any $\mathcal{R}_{as}$ −epimorphism $g: N \to K,\ Hom(M, K) = g \circ Hom(M, N)$.

Proof:

$(1 \Longrightarrow 2)$ Let $g: N \to K$ be an $\mathcal{R}_{as}$ −epimorphism and $f \in Hom(M, K)$. Since M is an $\mathcal{R}_{as}$ −projective, consequently there exists $f_1 \in Hom(M, N)$ such that $g \circ f_1 = f$. So, $Hom(I, g) \circ f_1 = g \circ f_1 = f$.

$$\begin{tikzcd} & M \arrow[d, "f"] \arrow[dl, dashed, "f_1"'] & \\ N \arrow[r, "g"] & K \arrow[r] & 0 \end{tikzcd}$$

$(2 \Longrightarrow 3)$ Let $g: N \to K$ be an $\mathcal{R}_{as}$ −epimorphism. Form (2) we get $Hom(I, g): Hom(M, N) \to Hom(M, K)$ is an epimorphism. To show that $Hom(M, K) = g \circ Hom(M, N)$. It is enough to show that $Hom(M, K) \subseteq g \circ Hom(M, N)$, since it is clear that $g \circ Hom(M, N) \subseteq Hom(M, K)$. Now, let $f \in Hom(M, K)$, then there exists $f_1 \in Hom(M, N)$ such that $Hom(I, g) \circ f_1 = f$. Hence, $f \in g \circ Hom(M, N)$. Thus $Hom(M, K) \subseteq g \circ Hom(M, N)$.

$(3 \Longrightarrow 1)$ Let N and K are any $\mathcal{R}$ − modules. Now, consider the following diagram:

$$\begin{tikzcd} & M \arrow[d, "f"] \arrow[dl, dashed, "f_1"'] & \\ N \arrow[r, "g"] & K \arrow[r] & 0 \end{tikzcd}$$

Where g is $\mathcal{R}_{as}$ −epimorphisim (i.e., $ker(g) \ll^{a} N$) and f is any homomorphism. From (3), we get $Hom(M, K) = g \circ Hom(M, N)$ and since $f \in Hom(M, K)$, so there exists $f_1 \in Hom(M, N)$ such that $g \circ f_1 = f$. Therefore, M is an $\mathcal{R}_{as}$ −projective.

Proposition 3.4:

Let M be a projective $\mathcal{R}$ − module and let $\mathcal{R}$ be an integral domain. Consequently, every epimorphism $f: M \to N$ is an $\mathcal{R}_{as}$ −epimorphism.

Proof: Let $f: M \to N$ be an epimorphism. Since every proper submodule of projective $\mathcal{R}$ − module M is an $\mathcal{R}_{as}$ −submodule by [11, Proposition 2.1.12]. So, $ker(f) \ll^{a} M$. Thus f is an $\mathcal{R}_{as}$ −epimorphism.

Remark 3.5:

Let M be an $\mathcal{R}-$ module and let P be an $\mathcal{R}_{as}$ −projective such that $P \simeq M$. Then M is an $\mathcal{R}_{as}$ −projective.

Proof:

Consider the following diagram:

$$\begin{tikzcd}
& P \arrow[d, shift right, "\xi"'] \arrow[ddl, dashed, "f_1"'] \\
& M \arrow[u, dashed, shift right, "\xi^{-1}"'] \arrow[d, "f"] \arrow[dl, "f_1^*"'] \\
A \arrow[r, "g"'] & B \arrow[r] & 0
\end{tikzcd}$$

Where $\xi: P \to M$ is an isomorphism, f is any homomorphism and g is an $\mathcal{R}_{as}$ −epimorphism. Since P is an $\mathcal{R}_{as}$ −projective, then there exists a homomorphism $f_1: P \longrightarrow A$ such that $g \circ f_1 = f \circ \xi$. Define $\xi^{-1}: M \longrightarrow P$ be a homomorphism such that $\xi \circ \xi^{-1} = I_M$ and $f_1^*: M \longrightarrow A$ by $f_1^* = f_1 \circ \xi^{-1}$. Now, to show that $g \circ f_1^* = f$. Since $g \circ f_1 = f \circ \xi$, then $g \circ f_1^* = g \circ (f_1 \circ \xi^{-1}) = (f \circ \xi) \circ \xi^{-1} = f \circ I_M = f$.

Proposition 3.6:

Let $\mathrm{P} = \oplus_{\beta \in J} P_\beta$. Then P_β is an $\mathcal{R}_{as}$ −projective for all β belongs to J if and only if P is an $\mathcal{R}_{as}$ −projective.

Proof:

Suppose that P_β is an $\mathcal{R}_{as}$ −projective for all $\beta \in J$. Consider the following diagram:

$$\begin{tikzcd}
P_\beta \arrow[r, "\lambda_\beta"] \arrow[d, dashed, "\mu_\beta"'] & \mathrm{P} \arrow[d, "f"] \arrow[dl, dashed, "\mu"'] \\
\mathrm{A} \arrow[r, "g"'] & \mathrm{B} \arrow[r] & 0
\end{tikzcd}$$

Where $g: A \to B$ is an $\mathcal{R}_{as}$ −epimorphism (i.e., $ker(g) \ll^a A$), $f: \mathrm{P} \to B$ is any homomorphism and $\lambda_\beta: P_\beta \longrightarrow \mathrm{P}$ be the injection homomorphism. Since P_β is an $\mathcal{R}_{as}$ −projective for all $\beta \in J$, so there exists a homomorphism $\mu_\beta: P_\beta \longrightarrow A$ such that $g \circ \mu_\beta = f \circ \lambda_\beta$ for all $\beta \in J$. Define $\mu: P \longrightarrow A$ such that $\mu_\beta = \mu \circ \lambda_\beta$. It enough to show that $g \circ \mu = f$. Since $f \circ \lambda_\beta = g \circ \mu_\beta$ and so $\mu_\beta = \mu \circ \lambda_\beta$. Consequently, $f \circ \lambda_\beta = g \circ \mu \circ \lambda_\beta$. So, we get by [2, Remark 4.1.4] $g \circ \mu = f$. As a result, $\mathrm{P} = \oplus_{\beta \in J} P_\beta$ is an $\mathcal{R}_{as}$ −projective.

Conversely, assume that $\mathrm{P} = \oplus_{\beta \in J} P_\beta$ is an $\mathcal{R}_{as}$ −projective module and let $\beta \in J$. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{P} \arrow[r, shift right] \arrow[d, dashed, "\mu"'] & P_\beta \arrow[l, dashed, shift right, "\lambda_\beta"'] \arrow[d, "f"] \arrow[dl, dashed, "\eta_\beta"', "\mu_\beta"] \\
\mathrm{A} \arrow[r, "g"] & \mathrm{B} \arrow[r] & 0
\end{tikzcd}$$

Where $g: A \rightarrow B$ is an $\mathcal{R}_{as}$ −epimorphism (i.e $ker(g) \ll^a A$), $f: P_\beta \rightarrow B$ is any homomorphism, $\eta_\beta: P \longrightarrow P_\beta$ is the projection homomorphism and the injection homomorphism $\lambda_\beta: P_\beta \longrightarrow$ P. Since P is $\mathcal{R}_{as}$ −projective, so there exists $\mu \in Hom(P, A)$ such that $g \circ \mu = f \circ \eta_\beta$. Define $\mu_\beta: P_\beta \longrightarrow A$ by $\mu_\beta = \mu \circ \lambda_\beta$. Now, $g \circ \mu_\beta = g \circ (\mu \circ \lambda_\beta) = (f \circ \eta_\beta) \circ \lambda_\beta = f \circ I_{P_\beta} = f$. Thus P_β is an $\mathcal{R}_{as}$ −projective for all β belongs to J.

Proposition 3.7:

Let A be an $\mathcal{R}_{as}$ −projective module and $f: A \rightarrow B$ be an R_{as} −epimorphism. Then f is splits if and only if $A \oplus B$ is an $\mathcal{R}_{as}$ −projective.

Proof:

⟹) Let f be a split $\mathcal{R}_{as}$ −epimorphism, then f is an isomorphism by Corollary 2.4 and so B is an $\mathcal{R}_{as}$ −projective by Remark 3.5. Consequently, $A \oplus B$ is an $\mathcal{R}_{as}$ −projective by Proposition 3.6.

⟸) Suppose that $A \oplus B$ is an $\mathcal{R}_{as}$ −projective, then B is an $\mathcal{R}_{as}$ −projective by Proposition 3.6. Now, consider the following diagram:

$$\begin{tikzcd} & B \arrow[dl, dashed, "f^*"'] \arrow[d, "I_B"] & \\ A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd}$$

Since $f: A \rightarrow B$ is an $\mathcal{R}_{as}$ −epimorphism and B is an $\mathcal{R}_{as}$ −projective, so there exists $f^*: B \rightarrow A$ such that $f \circ f^* = I_B$. Thus f is split.

Remark 3.8:

Every $\mathcal{R}_{as}$ −epimorphism $f: A \rightarrow B$ where B is an $\mathcal{R}_{as}$ −projective is splits.

Proof:

Let $f: A \rightarrow B$ be an $\mathcal{R}_{as}$ −epimorphism where $ker(f) \ll^a A$. Consider the following diagram:

$$\begin{tikzcd} & B \arrow[dl, dashed, "f_1"'] \arrow[d, "I_B"] & \\ A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd}$$

Since B is an $\mathcal{R}_{as}$ −projective, then there exists $f_1: B \rightarrow A$ such that $f \circ f_1 = I_B$. As a result, by [2, Lemma 3.9.3] the sequence

$$0 \longrightarrow ker(f) \xrightarrow{i} A \xrightarrow{f} B \longrightarrow 0$$

is splits. Where $i: ker(f) \rightarrow A$ is the inclusion homomorphism.

Proposition 3.9:

Let P be an $\mathcal{R}$ −module. Then P is an $\mathcal{R}_{as}$ −projective if and only if for every $\mathcal{R}_{as}$ −epimorphism $g : A \rightarrow B$ where A is an injective module and every $f \in Hom(P, B)$, there exists $\mu \in Hom(P, A)$ such that $g \circ \mu = f$.

Proof:

⟹) Clear.

⟸) Let $g: A \longrightarrow B$ be any $\mathcal{R}_{as}$ −epimorphism (i.e., $ker(g) \ll^a A$), for any modules A, B and let $f \in Hom(P, B)$. Consider the below diagram:

$$\begin{tikzcd}
& P \arrow[d, "f"] \arrow[dl, dashed, "\mu_1"'] \arrow[ddl, dashed, "\mu"'] & \\
A \arrow[r, "g"] \arrow[d, "i"'] & B \arrow[r] \arrow[d, "f^*"] & 0 \\
W \arrow[r, "\pi"] & \frac{W}{ker\,(g)} &
\end{tikzcd}$$

Where $i: A \rightarrow W$ is the inclusion homomorphism with W is injective module (every module can be embedded in W by [2, Corollary 5.5.5]), and let $\pi: W \rightarrow \frac{W}{ker(g)}$ be the nature epimorphism. Now, define $f^*: B \rightarrow \frac{W}{ker(g)}$ by $f^*(b) = a + ker(g)$ for all $b \in B$, where $g(a) = b$. Let $b_1, b \in B$ such that $g(a) = b$ and $g(a_1) = b_1$. If $b = b_1$ we obtain $g(a) = g(a_1)$ which implies $a - a_1 \in ker(g)$ and so $a + ker(g) = a_1 + ker(g)$. Hence, f^* is a well define homomorphism.
According to the assumption, there exists $\mu \in Hom(P, W)$ such that $\pi \circ \mu = f^* \circ f$. Claim that $\mu(P) \subseteq A$. To show that, let $s \in \mu(P)$, then there exists $t \in P$ such that $s = \mu(t)$. Now, $\pi \circ \mu(t) = f^* \circ f(t)$ where $f(t) = g(a)$. Means that $\mu(t) - a \in ker(g)$ and hence $\mu(t) \in A$. Define $\mu_1 : P \rightarrow A$ by $\mu_1(p) = \mu(p)$, for all p belongs to P. Now, $f^* \circ f = \pi \circ \mu = \pi \circ i \circ \mu_1 = f^* \circ g \circ \mu_1$. It enough to show that f^* is monomorphism. Let $f^*(b) = f^*(b^*)$, where $b^*, b \in B$. So, $a + ker(g) = a^* + ker(g)$ where $g(a) = b$ and $g(a^*) = b^*$. Thus $a - a^* \in ker(g)$ this implies that $g(a^*) = g(a)$ and so $b^* = b$ therefore, f^* is monomorphism, so $f = g \circ \mu_1$. Hence, P is an $\mathcal{R}_{as}$ −projective module.

Definition 3.10 [21]:
Let $f: P \rightarrow M$ be an epimorphism with $ker\,(f) <<^{a} P$ where P is a projective module. Then a pair (P, f) is a projective $R - a$ −cover of M.

Proposition 3.11:
Let $\mathcal{R}$ be an integral domain and P is an $\mathcal{R}$ −module. Then every projective cover (P, f) of an $\mathcal{R}$ −module M is a projective $\mathcal{R} - a$ −cover.
Proof:
Let (P, f) be a protective cover of M where $f: P \rightarrow M$ is an epimorphism. Since P is a projective and $\mathcal{R}$ is an integral-domain, so by Proposition 3.4 f is an $\mathcal{R}_{as}$ −epimorphism. Thus (P, f) is a projective $\mathcal{R} - a$ −cover.

Proposition 3.12:
Let M be an $\mathcal{R}_{as}$ −projective module. If (P, f) is a protective $\mathcal{R} - a$ −cover of M, then M is projective.
Proof:
Assume that M be an $\mathcal{R}_{as}$ −projective and let (P, f) be a projective $\mathcal{R} - a$ −cover of M. Consider the following diagram:

$$\begin{tikzcd}
& M \arrow[d, "I_M"] \arrow[dl, dashed, "\mu_1"'] & \\
P \arrow[r, "f"] \arrow[d, dashed, "\mu_2"'] & M \arrow[r] \arrow[d, "f^*"] \arrow[dl, dashed, "\mu"'] & 0 \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

Such that $g: A \to B$ is an epimorphism, $f^* \in Hom(M, B)$ and $I_M: M \to M$ is the identity. But f is an $\mathcal{R}_{as}$ −epimorphism and M is an $\mathcal{R}_{as}$ −projective. So, there exists $\mu_1 \in Hom(M, P)$ such that $f \circ \mu_1 = I_M$. Since P is projective, consequently, there exists $\mu_2 \in Hom(P, A)$ such that $g \circ \mu_2 = f^* \circ f$. Now, define $\mu: M \to A$ by $\mu = \mu_2 \circ \mu_1$. Finaly, to show that $g \circ \mu = f^*$. Since $g \circ \mu = g \circ (\mu_2 \circ \mu_1) = (f^* \circ f) \circ \mu_1 = f^* \circ I_M = f^*$. As a result, M is a projective module.

Corollary 3.13:
Let $\mathcal{R}$ be a right-perfect integral domain ring. Therefore, every $\mathcal{R}$ −module is an $\mathcal{R}_{as}$ −projective if and only if it is projective.
Proof:
⇒) clear.
⇐) Assume that M is an $\mathcal{R}_{as}$ −projective $\mathcal{R}$ −module and $\mathcal{R}$ is a right-perfect ring and I.D.Then M have a protective cover (P, f). Consequently, by Proposition 3.11 and 3.12 we get M is a projective.

4. Conclusions:

In this work, the concepts of projective modules and epimorphism have been generalized to new concepts called $\mathcal{R}_{as}$ −projective modules and $\mathcal{R}_{as}$ −epimorphism respectively. Some properties of this type of module and epimorphism have been studied. Also, we see the relation between $\mathcal{R}_{as}$ −projective, projective and small projective modules

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