

New Applications of a Generalization of Tremblay Fractional Differential Operator for Defining Subclasses of Analytic

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Abstract— In this paper, we study basic geometric properties of analytic and univalent functions in the open unit disk. A new family $\mathcal{W}_{m,n,\sigma}^{\zeta,\lambda,\kappa}(\omega, \rho, \Upsilon; \aleph)$ of analytic and univalent functions is defined by using a generalization of Tremblay fractional differential operator associated with the convolution product of a multiplier transformation and a Ruscheweyh derivative, especially. Coefficient-related studies and extreme points, distortion properties for the functions in class $\mathcal{W}_{m,n,\sigma}^{\zeta,\lambda,\kappa}(\omega, \rho, \Upsilon; \aleph)$ are given and properties of starlikeness and the convexity of this class are also presented.

Keywords— Analytic and univalent function, generalization of Tremblay fractional differential operator, open unit disk.

1. INTRODUCTION

Fractional calculus of complex order has become a novel field of study, in order to define and create new subfamilies, a number of researchers have extended fractional calculus operators on popular families of analytic and univalent functions. They have also investigated various interesting properties of these new families such as (see, [1, 2, 3, 4]). In our research, the family of functions in the open $U = \{z \in \mathbb{C} : |z| < 1\}$ that are analytic, regular, or holomorphic will be represented by the notation $H(U)$. Assuming that $a \in \mathbb{C}$ and a positive integer s , $H[a, s]$ is the subclass of H consisting of functions with the following formula:

$$f(z) = a + a_s z^s + a_{s+1} z^{s+1} + \dots, z \in U,$$

and $\mathbb{A}_s = \{f \in H(U), f(z) = z + a_{s+1} z^{s+1} + a_{s+2} z^{s+2} + \dots, z \in U\}$ with $\mathbb{A}_1 = \mathbb{A}$ the class of the following function

$$f(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad (1)$$

we also denote by S the subfamily of $\mathbb{A}$ consisting of functions satisfying (1.1) which are also univalent in U . The Hadamard product (or convolution) of two analytic functions in the open unit disk U

$$f(z) = z + \sum_{s=2}^{\infty} a_s z^s \text{ and } g(z) = z + \sum_{s=2}^{\infty} b_s z^s$$

denoted by $f * g$ and is defined as:

$$f(z) * g(z) = z + \sum_{s=2}^{\infty} a_s b_s z^s = (f * g)(z).$$

Also, let T the subfamily of S whose functions represented with negative coefficients by

$$f(z) = z - \sum_{s=2}^{\infty} a_s z^s, \quad a_s \geq 0, z \in U. \quad (2)$$

Remark 1.1:

- The condition $\sum_{s=2}^{\infty} s|a_s| \leq 1$ is sufficient for all f of the form (1) to be in S .
- The condition $\sum_{s=2}^{\infty} s a_s \leq 1$ is sufficient for all f of the form (2) to be in T (see[5]).

We consider the following multiplier transformations.

For $f \in \mathbb{A}$, $m \in \mathbb{N} \cup \{0\}$, $\zeta, \sigma \geq 0$, the multiplier transformation $I(m, \sigma, \zeta)$ was defined by Cătaş [6] in the following infinite series:

$$I(m, \sigma, \zeta) = z + \sum_{s=2}^{\infty} \left(\frac{1 + \sigma(s-1) + 1}{1 + \zeta} \right)^m a_s z^s, z \in U.$$

In addition, Ruscheweyh [7] defined derivative operator $R^n: \mathbb{A} \rightarrow \mathbb{A}$, by

$$R^0 f(z) = f(z),$$

$$R^1 f(z) = z f'(z),$$

$$\vdots$$

$$(n+1)R^{n+1}f(z) = z(R^n f(z))' + n R^n f(z), \quad z \in U.$$

Remark 1.2: If $f \in \mathbb{A}$, then

$$R^n f(z) = z + \frac{1}{\Gamma(n+1)} \sum_{s=2}^{\infty} \frac{\Gamma(n+s)}{\Gamma(s)} a_s z^s, \quad z \in U.$$

Moreover, by using the Hadamard product methods of multiplier transformation $I(m, \sigma, \zeta)$ and the Ruscheweyh derivative R^n , the operator $IR_{\sigma, \zeta}^{m, n}: \mathbb{A} \rightarrow \mathbb{A}$ defined by Alp Lupas[8] as following form:

$$IR_{\sigma, \zeta}^{m, n} f(z) = (I(m, \sigma, \zeta) * R^n) f(z),$$

$$= z + \frac{1}{\Gamma(n+1)} \sum_{s=2}^{\infty} \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m \frac{\Gamma(n+s)}{\Gamma(s)} a_s^2 z^s, \quad z \in U$$

for each $m, n \in \mathbb{N}$, $\sigma, \zeta \geq 0$.

Remark 1.3: For $\zeta = 0$, $\sigma \geq 0$, then

1. Operator $D_{\sigma}^m = I(m, \sigma, 0)$ was introduced and utilized by Al-Oboudi [9].

2. Operator $S^m = I(m, 1, 0)$ for $\sigma = 1$ was reduced to the Sălăgean differential operator [10].

3. $R^n = I(0, \sigma, 0)$ for $m = 0$ reduced to the Ruscheweyh derivative operator [7].

Researchers in geometric function theory have shown considerable interest in topic of fractional differential and integral operators (see, for example, [10,11,12 and13]). The generalization of the Tremblay fractional calculus operators [14, 15] were introduced and investigated as follows:

Definition 1.1: [14, 15] For $0 < \lambda \leq 1$, $0 < \rho \leq 1$ such that $1 \geq \rho - \lambda > 0$ and $\kappa > -1$. The generalized fractional integral operator $T_z^{\rho, \lambda, \kappa}$ of three parameters is defined by:

$$\begin{aligned} T_z^{\rho, \lambda, \kappa} f(z) &= \frac{(\kappa+1)^{\rho-\lambda} \Gamma(\rho)}{\Gamma(\rho) \Gamma(\rho-\lambda)} z^{1-\rho-\kappa} \int_0^z \frac{t^{\kappa+\lambda-1} f(t)}{(z^{1+\kappa} - t^{1+\kappa})^{1-\rho+\lambda}} dt, \quad z \in U. \end{aligned} \quad (3)$$

For $0 < \lambda \leq 1$, $0 < \rho \leq 1$ such that $1 > \rho - \lambda \geq 0$ and $\kappa > -1$, the generalized fractional differential operator $\mathbb{T}_z^{\rho, \lambda, \kappa}$ of three parameters, is defined by:

$$\begin{aligned} \mathbb{T}_z^{\rho, \lambda, \kappa} f(z) &= \frac{(\kappa+1)^{\rho-\lambda} \Gamma(\lambda)}{\Gamma(\rho) \Gamma(1-\rho+\lambda)} \left(z^{1-\lambda-\kappa} \frac{d}{dz} \right) \int_0^z \frac{t^{\kappa+\rho-1} f(t)}{(z^{1+\kappa} - t^{1+\kappa})^{\rho-\lambda}} dt, \end{aligned} \quad (4)$$

where $f(z) (z \in U)$ is an analytic function in a simply-connected region of the z -plane that is simply connected and contains the origin C . The multiplicity of $(z^{1+\kappa} - t^{1+\kappa})^{\rho-\lambda-1}$ in (3) and $(z^{1+\kappa} - t^{1+\kappa})^{-\rho+\lambda}$ in (4) are removed by requiring $\log(z^{1+\kappa} - t^{1+\kappa})$ to be real when $(z^{1+\kappa} - t^{1+\kappa}) > 0$, respectively.

Example 1.1: [15] For $f(z) = z^s$, we obtain

$$\begin{aligned} \mathbb{T}_z^{\rho, \lambda, \kappa} f(z) &= \frac{(\kappa+1)^{\rho-\lambda} \Gamma(\lambda) \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(\rho) \Gamma\left(\frac{\kappa(\rho-\lambda+1)+\lambda+s}{\kappa+1} - 1\right)} z^{\kappa(1-\rho+\lambda)+s}, \quad z \in U. \end{aligned}$$

By utilizing the fractional differential operator defined in (4) and implementing Example 1, a new fractional differential operator associated with the convolution product of a multiplier transformation and a Ruscheweyh derivative can be defined as follows:

Definition 1.2: For $\kappa > -1$, $\lambda, \zeta \geq 0$, $m, n \in \mathbb{N}$ and $0 < \lambda \leq 1$, $0 < \rho \leq 1$ such that $1 \geq \rho - \lambda > 0$. Let $f(z)$ given in (1), the fractional differential associated with the convolution product of a multiplier transformation and a Ruscheweyh derivative is defined by:

$$\begin{aligned} \mathbb{T}_z^{\rho, \lambda, \kappa} IR_{\sigma, \zeta}^{m, n} f(z) &= \frac{(\kappa+1)^{\rho-\lambda} \Gamma(\lambda)}{\Gamma(\rho) \Gamma(1-\rho+\lambda)} \left(z^{1-\lambda} \frac{d}{dz} \right) \int_0^z \frac{t^{\kappa+\rho+s-1} IR_{\sigma, \zeta}^{m, n} f(t)}{(z^{\kappa+1} - t^{\kappa+1})^{\rho-\lambda}} dt \\ &= \frac{(\kappa+1)^{\rho-\lambda} \Gamma(\lambda)}{\Gamma(\rho) \Gamma(1-\rho+\lambda)} \left(z^{1-\lambda} \frac{d}{dz} \right) \int_0^z \frac{t^{(\kappa+1)+\rho-1}}{(z^{\kappa+1} - t^{\kappa+1})^{\rho-\lambda}} dt + \\ &\quad \frac{(\kappa+1)^{\rho-\lambda} \Gamma(\lambda)}{\Gamma(\rho) \Gamma(n+1) \Gamma(1-\rho+\lambda)} \\ &\quad \sum_{s=2}^{\infty} \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m \frac{\Gamma(n+s)}{\Gamma(s)} a_s^2 \\ &\quad \times \left(z^{1-\lambda} \frac{d}{dz} \right) \int_0^z \frac{t^{\kappa+\rho+s-1}}{(z^{\kappa+1} - t^{\kappa+1})^{\rho-\lambda}} dt \end{aligned}$$

after a simple calculation, we obtain the following formula:

$$\begin{aligned} \mathbb{T}_z^{\rho, \lambda, \kappa} IR_{\sigma, \zeta}^{m, n} f(z) &= \frac{(\kappa+1)^{\rho-\lambda} \Gamma(\lambda) \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma(\rho) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} z^{(1-\rho+\lambda)\kappa+1} + \\ &\quad \frac{(\kappa+1)^{\rho-\lambda} \Gamma(\lambda)}{\Gamma(\rho) \Gamma(n+1)} \times \\ &\quad \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \Gamma\left(\frac{\rho+s}{\kappa+1}\right) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m}{\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 z^{(1-\rho+\lambda)\kappa+s}, \quad z \in U. \end{aligned} \quad (5)$$

The operator $\mathbb{T}_z^{\rho, \lambda, \kappa} IR_{\sigma, \zeta}^{m, n} f(z)$ given in (1.4) is now used to define a new subclass of analytic functions as follows:

Definition 1.3: For $\kappa > -1$, $\lambda, \sigma, \zeta \geq 0$, $\omega \geq 1$, $m, n \in \mathbb{N}$, $Y \in \mathbb{C} - \{0\}$, $0 < \lambda \leq 1$, $0 < \rho \leq 1$ such that $1 \geq \rho - \lambda > 0$ and

$\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1})}{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1}) + |Y| \Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})} < \aleph \leq 1$, let $\mathcal{W}_{m,n,\sigma}^{\rho,\lambda,\kappa}(\omega, \rho, Y; \aleph)$ be the subclass of $\mathbb{A}$ consisting of functions that satisfy the inequality:

$$\left| \frac{(1-\rho+\lambda)\kappa\Psi(\rho, \lambda, \kappa) \left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right) + \omega\Psi(\rho, \lambda, \kappa) (\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z))'}{(1-\rho+\lambda)\kappa\Psi(\rho, \lambda, \kappa) \left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right) + \omega\Psi(\rho, \lambda, \kappa) (\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z))' - Y} \right| < \aleph. \quad (6)$$

where

$$\Psi(\rho, \lambda, \kappa) = \frac{(\kappa+1)^{-\rho+\lambda} \Gamma(\rho)}{\Gamma(\lambda)}.$$

By using the above definition the new family $\mathcal{W}_{m,n,\sigma}^{\rho,\lambda,\kappa}(\omega, \rho, Y; \aleph)$ of analytic functions involving the modification of fractional differential operator associated with operator $\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n}$ is defined in the open unit disk. We presented some basic geometric properties such as finding the coefficient bounds, distortion bounds, and radius of starlike and convexity.

2. COEFFICIENT INEQUALITIES

In the following result the coefficient bounds and extreme points for functions in $\mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ are obtained.

Theorem 2.1: The function $f \in \mathbb{A}$ belongs to the class $\mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ if and only if

$$\sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma(\frac{\rho+s}{\kappa+1}) a_s^2}{\Gamma(s) \Gamma(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1})} a_s^2 < \frac{\aleph|Y| \Gamma(n+1)}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1}) \Gamma(n+1)}{\Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})}, \quad (7)$$

the result is sharp for the function

$$\mathcal{F}(z) = z + \sqrt{\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1})}{\Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})} \right) \Gamma(n+1) \times \Gamma(s) \Gamma(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1})}{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m \times [(1-\rho+\lambda)\kappa+\omega s] \Gamma(\frac{\rho+s}{\kappa+1})}} z^s,$$

$s \geq 2$.

Proof. Let the function $f \in \mathbb{A}$ and the inequality in (6) satisfies, then we obtain:

$$\left| \frac{(1-\rho+\lambda)\kappa\Psi(\rho, \lambda, \kappa) \left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right) + \omega\Psi(\rho, \lambda, \kappa) (\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z))'}{(1-\rho+\lambda)\kappa\Psi(\rho, \lambda, \kappa) \left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right) + \omega\Psi(\rho, \lambda, \kappa) (\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z))' - Y} \right|$$

=

$$\left| \frac{\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1})}{\Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})} z^{(1-\rho+\lambda)\kappa} + \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma(\frac{\rho+s}{\kappa+1}) a_s^2 z^{(1-\rho+\lambda)\kappa+s-1}}{\Gamma(n+1) \Gamma(s) \Gamma(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1})}}{\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1})}{\Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})} z^{(1-\rho+\lambda)\kappa} + \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma(\frac{\rho+s}{\kappa+1}) a_s^2 z^{(1-\rho+\lambda)\kappa+s-1}}{\Gamma(n+1) \Gamma(s) \Gamma(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1})} - Y} \right|$$

=

$$\left| \frac{\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1})}{\Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})} z^{(1-\rho+\lambda)\kappa} + \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma(\frac{\rho+s}{\kappa+1}) a_s^2 z^{(1-\rho+\lambda)\kappa+s-1}}{\Gamma(n+1) \Gamma(s) \Gamma(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1})}}{\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1})}{\Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})} z^{(1-\rho+\lambda)\kappa} + \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma(\frac{\rho+s}{\kappa+1}) a_s^2 z^{(1-\rho+\lambda)\kappa+s-1}}{\Gamma(n+1) \Gamma(s) \Gamma(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1})} - Y} \right|$$

<

$$\left| \frac{\left| \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1})}{\Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})} z^{(1-\rho+\lambda)\kappa} \right| + \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma(\frac{\rho+s}{\kappa+1}) a_s^2 z^{(1-\rho+\lambda)\kappa+s-1}}{\Gamma(n+1) \Gamma(s) \Gamma(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1})}}{|Y| \left| \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma(\frac{\rho+1}{\kappa+1})}{\Gamma(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1})} z^{(1-\rho+\lambda)\kappa} \right| - \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma(\frac{\rho+s}{\kappa+1}) a_s^2 z^{(1-\rho+\lambda)\kappa+s-1}}{\Gamma(n+1) \Gamma(s) \Gamma(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1})}} \right|$$

=

$$\frac{\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \Big|_{z^{(1-\rho+\lambda)\kappa}} + \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(n+1)\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 \Big|_{z^{(1-\rho+\lambda)\kappa+s-1}}}{|Y| - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \Big|_{z^{(1-\rho+\lambda)\kappa}} - \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(n+1)\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 \Big|_{z^{(1-\rho+\lambda)\kappa+s-1}}}$$

$$< \aleph, z \in U.$$

Choosing values of z on the real axis and considering $z \rightarrow -1$, we have:

$$\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} + \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(n+1)\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 <$$

$$\aleph |Y| - \aleph \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} - \aleph \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(n+1)\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2,$$

which is equivalent to

$$\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \sum_{s=2}^{\infty} \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 <$$

$$\frac{\aleph |Y| \Gamma(n+1)}{\aleph + 1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right) \Gamma(n+1)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}. \quad (8)$$

Getting that $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$.

Conversely, let assuming that $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$, then we obtain the following inequality:

$$\Re \left(\frac{(1-\rho+\lambda)\kappa\psi(\rho, \lambda, \kappa)}{\left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right)} + \frac{\omega\psi(\rho, \lambda, \kappa) (\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z))'}{(1-\rho+\lambda)\kappa\psi(\rho, \lambda, \kappa)} \right. \\ \left. \frac{\left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right)}{\left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right)} + \frac{\omega\psi(\rho, \lambda, \kappa) (\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z))' - Y}{\left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right)} \right) > -\aleph$$

we have

$$\Re \left(\frac{(1-\rho+\lambda)\kappa\psi(\rho, \lambda, \kappa)}{\left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right)} + \frac{\omega\psi(\rho, \lambda, \kappa) (\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z))'}{(1-\rho+\lambda)\kappa\psi(\rho, \lambda, \kappa)} + \aleph \right. \\ \left. \frac{\left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right)}{\left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right)} + \frac{\omega\psi(\rho, \lambda, \kappa) (\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z))' - Y}{\left((1-\omega) \frac{\mathbb{T}_z^{\rho,\lambda,\kappa} IR_{\sigma,\zeta}^{m,n} f(z)}{z} \right)} \right) > 0,$$

and

$$\Re \left(\frac{(1+\aleph) \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \Big|_{z^{(1-\rho+\lambda)\kappa}} + \frac{1+\aleph}{\Gamma(n+1)} \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 \Big|_{z^{(1-\rho+\lambda)\kappa+s-1}} - \aleph Y}{\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \Big|_{z^{(1-\rho+\lambda)\kappa}} + \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 \Big|_{z^{(1-\rho+\lambda)\kappa+s-1}} - Y}$$

$$> 0$$

Taking account that $\Re(-e^{i\lambda}) \geq -|e^{i\lambda}| = -1$, the above inequality is due to:

$$\frac{(1+\aleph) \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \Big|_{r^{(1-\rho+\lambda)\kappa}} + \frac{1+\aleph}{\Gamma(n+1)} \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 \Big|_{r^{(1-\rho+\lambda)\kappa+s-1}} - \aleph Y}{\frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \Big|_{r^{(1-\rho+\lambda)\kappa}} + \frac{1}{\Gamma(n+1)} \times \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m \frac{[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 \Big|_{r^{(1-\rho+\lambda)\kappa+s-1}} - Y}$$

$$> 0.$$

Letting $r \rightarrow -1$ and by using the mean value theorem, we have the desired inequality (7). This completes the proof of Theorem 2.1.

Corollary 2.1: Function $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ indicates that:

$$a_s \leq \sqrt{\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}\right) \Gamma(n+1)\Gamma(s)\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}{\Gamma(n+s)\left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}, \quad s \geq 2 \quad (9)$$

with equality only for functions defined by (8).

Theorem 2.2: Let $f_1(z) = z$ and

$$f_s(z) = z - \sqrt{\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}\right) \Gamma(n+1)\Gamma(s)\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}{\Gamma(n+s)\left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)} z^s, \quad s \geq 2. \quad (10)$$

For $\kappa, \lambda, \sigma, \zeta, \omega \geq 1$, $m, n \in \mathbb{N}$, $0 < \lambda \leq 1, 0 < \rho \leq 1, Y \in \mathbb{C} - \{0\}$ and $0 < \aleph \leq 1$, then $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ if and only if it can be presented in the following form:

$$f(z) = \sum_{s=1}^{\infty} p_s f_s(z), \quad (11)$$

where

$$p_s \geq 0, \text{ and } \sum_{s=1}^{\infty} p_s = 1.$$

Proof. Assume f can be presented as in (11). Then,

$$f(z) = z - \sum_{s=2}^{\infty} p_s \sqrt{\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}\right) \Gamma(n+1)\Gamma(s)\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}{\Gamma(n+s)\left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)} z^s.$$

Now, let

$$\sum_{s=2}^{\infty} \sqrt{\frac{\Gamma(n+s)\left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}\right) \Gamma(n+1)\Gamma(s)\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} p_s} \times$$

$$\sqrt{\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}\right) \Gamma(n+1)\Gamma(s)\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}{\Gamma(n+s)\left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}} = \sum_{s=2}^{\infty} p_s = 1 - p_1 \leq 1.$$

Therefore, $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$.

Conversely, let $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$. Then by utilizing (9), setting

$$p_s = \sqrt{\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}\right) \Gamma(n+1)\Gamma(s)\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}{\Gamma(n+s)\left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}} a_s, \quad s \geq 2$$

and $p_1 = 1 - \sum_{s=2}^{\infty} p_s$, we obtain $p_1 = 1 - \sum_{s=2}^{\infty} p_s$, we obtain $f(z) = \sum_{s=1}^{\infty} p_s f_s(z)$, the proof of Theorem 2.2 is completed.

3. DISTORTION BOUNDS

In the following theorem, we establish distortion inequalities for functions belonging to the class $\mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$. These inequalities are illustrated.

Theorem 3.1: For $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$, inequality

$$\begin{aligned} & r - \sqrt{\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}\right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}} r^2 \\ & \leq |f(z)| \leq \\ & r + \sqrt{\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)}\right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}} r^2 \end{aligned} \quad (12)$$

holds if the sequence $\{\alpha_s(\rho, \lambda, \omega, \kappa, m, n, \sigma, \zeta)\}_{j=2}^{\infty}$ is non-decreasing, and

$$1 - \frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}} r^2$$

$$\leq |f'(z)| \leq 1 + \frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}} r^2 \quad (13)$$

holds if the sequence $\left\{ \frac{\alpha_s(\rho, \lambda, \omega, \kappa, m, n, \sigma, \zeta)}{s} \right\}_{s=2}^{\infty}$ is non- decreasing, where

$$\alpha_s(\rho, \lambda, \omega, \kappa, m, n, \sigma, \zeta) = \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}.$$

The bounds in (1.11) and (1.12) are sharp, for f given by

$$f(z) = z + \frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}} z^2, \quad z = \pm r. \quad (14)$$

Proof: By applying Theorem 1, we obtain:

$$\sum_{s=2}^{\infty} a_s \leq \frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}}, \quad (15)$$

we get

$$|z| - |z|^2 \sum_{s=2}^{\infty} a_2 \leq |f(z)| \leq |z| + |z|^2 \sum_{s=2}^{\infty} a_2.$$

Thus,

$$r - \frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}} r^2$$

$$\leq |f(z)| \leq r +$$

$$\frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}} r^2. \quad (16)$$

Thus, (12) follows from (16).

Furthermore,

$$\sum_{s=2}^{\infty} sa_s \leq \frac{\left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+1}{\kappa+1}\right)}{\sqrt{(n+1)[(1-\rho+\lambda)\kappa+2\omega] \Gamma\left(\frac{\rho+2}{\kappa+1}\right) \left(\frac{1+\sigma+1}{1+\zeta}\right)^m}}.$$

Hence (13) follows from

$$1 - r \sum_{s=2}^{\infty} sa_s \leq |f'(z)| \leq 1 + r \sum_{s=2}^{\infty} sa_s.$$

4. RADIUS OF STARLIKENESS AND CONVEXITY

In this section we give the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$.

Theorem 4.1: The function $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ is close-to-convex of the order ∂ ($0 \leq \partial < 1$) in $|z| < r$, where:

$$r := \inf_{s \geq 2} \frac{\sqrt{(1-\partial)^2(n+1)[(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right) \left(\frac{1+\sigma(s-1)+1}{1+\zeta}\right)^m}}{s \Gamma(s+1) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}. \quad (17)$$

The result is sharp, with the extremal function f given by (14).

Proof: For the function $f \in \mathbb{A}$, we have to show that:

$$|f'(z) - 1| < 1 - \partial. \quad (18)$$

By a simple calculation we obtain

$$|f'(z) - 1| \leq \sum_{s=2}^{\infty} sa_s |z|,$$

which is less than $1 - \partial$ if

$$\sum_{s=2}^{\infty} \frac{s}{1-\partial} a_s |z| < 1.$$

Function $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ if and only if

$$\frac{1}{\Gamma(n+1)} \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m \times [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right)} a_s^2 < 1.$$

$$\times \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)$$

The relation (16) is true if

$$\frac{s}{1-\delta} |z| \leq \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m \times [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right)} \sqrt{\times \Gamma(n+1) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)},$$

or, equivalent to

$$|z| \leq \sum_{s=2}^{\infty} \frac{(1-\delta)^2 \Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m \times [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{s \Gamma(s+1) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right)} \sqrt{\times \Gamma(n+1) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)},$$

which completes the proof.

Theorem 4.2: Let $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$, then:

H(1)- f is starlike of order ∂ ($0 \leq \partial < 1$), in $|z| < r_1$ where:

$$r_1 = \inf_{s \geq 2} \frac{(1-\delta)^2 \Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) (s+\partial-2)^2 \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right)} \sqrt{\Gamma(n+1) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}.$$

H(2)- f is convex of order ∂ ($0 \leq \partial < 1$), in $|z| < r_2$ where:

$$r_2 = \inf_{s \geq 2} \frac{(1-\delta)^2 \Gamma(n+s) [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m}{s \Gamma(s+1) (s-1)^2 \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma(n+1) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}.$$

Each of these results is sharp for the extremal function f given by (17).

Proof. 1. For $0 \leq \partial < 1$ we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \partial. \quad (19)$$

We obtain

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{s=2}^{\infty} (s-1) a_s |z|}{1 + \sum_{s=2}^{\infty} a_s |z|},$$

which is less than $1 - \partial$ if

$$\sum_{s=2}^{\infty} \frac{(s+\partial-2)}{1-\partial} a_s |z| < 1.$$

Function $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ if and only if

$$\frac{1}{\Gamma(n+1)} \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 < 1, \quad (20)$$

relation (19) holds if:

$$\frac{(s+\partial-2)}{1-\delta} |z| < \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m \times [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \sqrt{\times \Gamma(n+1) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}} |z|.$$

Equivalently,

$$|z| < \frac{(1-\delta)^2 \Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m \times [(1-\rho+\lambda)\kappa+\omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) (s+\partial-2)^2 \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa+\omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right)} \sqrt{\Gamma(n+1) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)},$$

which yields the starlikeness of the family.

H(2)- The function f is convex if and only the function zf' is starlike; therefore it is enough to prove H(2) with a similar method as that of the proof of H(1). Thus, the function f is convex if and only if:

$$|zf''(z)| < 1 - \partial. \quad (20)$$

We obtain

$$|zf''(z)| \leq \left| \sum_{s=2}^{\infty} s(s-1) a_s |z| \right| < 1 - \partial.$$

Equivalent to,

$$\sum_{s=2}^{\infty} \frac{s(s-1)}{1-\partial} a_s |z| < 1.$$

Function $f \in \mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ if and only if:

$$\frac{1}{\Gamma(n+1)} \sum_{s=2}^{\infty} \frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa + \omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa + \omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)} a_s^2 < 1.$$

The relation (20) holds if

$$\frac{s(s-1)}{1-\partial} |z| < \sqrt{\frac{\Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa + \omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{\Gamma(s) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa + \omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma(n+1) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}} |z|.$$

Equivalent to

$$|z| < \sqrt{\frac{(1-\partial)^2 \Gamma(n+s) \left(\frac{1+\sigma(s-1)+1}{1+\zeta} \right)^m [(1-\rho+\lambda)\kappa + \omega s] \Gamma\left(\frac{\rho+s}{\kappa+1}\right)}{s(s-1)^2 \Gamma(s+1) \left(\frac{\aleph|Y|}{\aleph+1} - \frac{[(1-\rho+\lambda)\kappa + \omega] \Gamma\left(\frac{\rho+1}{\kappa+1}\right)}{\Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda}{\kappa+1}\right)} \right) \Gamma(n+1) \Gamma\left(\frac{\kappa(\lambda-\rho)+\lambda+s-1}{\kappa+1}\right)}},$$

which leads the convexity of the family.

CONCLUSION

We examine fundamental geometric characteristics of both analytic and univalent functions in the open unit disk in this research. In particular, a generalization of the Tremblay fractional differential operator associated with the convolution product of a multiplier transformation and a Ruscheweyh derivative is used to establish a novel family $\mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ of analytic and univalent functions. Studies on coefficients, extreme points, distortion qualities, and properties of starlikeness and convexity for the functions in class $\mathcal{W}_{m,n,\sigma,\zeta}^{\rho,\lambda,\kappa}(\omega, \sigma, Y; \aleph)$ are provided.

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REFERENCES

- [1] M. A. Jasim and Z. E. Abdunaby, Some certain properties on analytic p-valent functions connected to fractional differential operator. *Journal of interdisciplinary Mathematics*, vol. 27, no. 4, pp. 939-945, 2024.
- [2] Z. E. Abdunaby and Rabha W. Ibrahim, Applications of differential inequalities employing a new convoluted operator constructed by the supertrigonometric function, *Iraqi Journal of Science*, vol.56, no. 6, pp. 5302-5312, 2024.
- [3] A. Fernandez and B. Dumitru, Classes of operators in fractional calculus: a case study, *Math. Methods Appl. Sci.* vol. 44, no.11, pp.9143-9162, 2021.
- [4] H. M. Srivastava, Operators of Fractional Calculus and Their Multidisciplinary Applications, *Fractal Fract.* vol. 7, no.5, pp.415, 2023.
- [5] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* vol. 51, pp. 109-116, 1975.
- [6] A. Catas, On certain classes of p-valent functions defined by multiplier transformations. In Proceedings of the international symposium on geometric function theory and applications: GFTA (2007), 241-250.
- [7] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* Vol. 49, 109-115, 1975.
- [8] A. Alb Lupas, About some differential sandwich theorems using a multiplier transformation and Ruscheweyh derivative. *Journal of Computational Analysis and Applications*, vol. 21, pp.1218-1224, 2016.
- [9] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator. *Int J Math Math Sci.*, vol. 27, pp. 1429-1436, 2004.
- [10] G. S. Sălăgean, Subclasses of univalent functions, complex analysis-fifth Romanian-Finnish seminar, part 1 (Bucharest, 1981). *Lect Notes Math* 1013:362-372.
- [11] S. S. Al-Azawee and S. S. Alhily, Some Geometric Properties of a Hyperbolic Univalent Function, *Iraqi Journal of Science*, vol. 2021, pp 2022-2028, 2021.
- [12] S. S. Alhily and S. S. Al-Azawee, Some invariant differential operator properties for hyperbolic univalent convex functions, *Journal of Interdisciplinary Mathematics*, vol. 25, pp. 2641-2651, 2022.
- [13] AL-khafaji A. K., Abed S. S. and Atshan W. G., On differential subordination of a certain subclass of univalent functions, *Journal of kufa for mathematics and computer*, vol. 5, no. 3 (2018).
- [14] Z. Esa, b, H. M. Srivastava, A. Kılıçman, and R. W. Ibrahim, A Novel Subclass of Analytic Functions Specified by a Family of Fractional Derivatives in the Complex Domain, *Filomat*, vol.31,no.9,pp. 2837-2849, 2017.
- [15] A. Kilicman and Z. E. Abdunaby, Some applications for generalized fractional operators in analytic functions spaces, *Korean Journal of Mathematics*, vol.27, no.3, pp. 581-594, 2019.