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Leveraging Loss Functions to Bayesian Estimations of (X-G) distribution

Wasnaa J. Hmood(1)
Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq
wasnaagh2024@uomustansiriyah.edu.iq

Shurooq A.K.S. AL-Sultany(2)
Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq
shuroq80@uomustansiriyah.edu.iq

Abstract:
This paper, focused on evaluate the performance of Bayesian estimation for estimating the scale parameter, reliability and hazard functions of X-Gamma (X-G) distribution. Bayesian procedure involving information Gamma and non-information Gamma prior. The Lindley's approximation has been used to calculate Bayes estimates utilizing two different loss functions, namely the squared error and the entropy loss function. The comparisons between the scale parameter estimates were based on values from bias and Mean Square Error and for reliability and hazard functions were integrated mean squared error.
Keywords:X-Gamma (X-G) distribution; squared error loss function (SLF); Entropy loss function (ELF); bias (BAS); Mean Square Error (MSE).

Introduction:
Probability theory is a theory that was born in the middle of the nineteenth century in France and was developed by leading scientists to become today one of the most important and widespread mathematical theories. The importance of probability theory is not only highlighted by its important application in various fields, but there are other sciences and theories that have come to focus primarily on this theory, including statistics, decision theory, queueing theory, and reliability theory. The theory of probability has entered into various applications in various scientific, technical, engineering, humanitarian, economic and military fields, and its study in university studies has become an indisputable matter.[3].
The failure behavior of any system can be considered a random variable due to variations from one system to another, resulting from the different types and natures of systems or processes. Therefore, it is reasonable to seek a statistical model of system failure under various loss functions, represented by common life distributions. However, these distributions sometimes fail to compete with newly developed life distributions, as the data is more accurately captured and represented by these newer models. Recently, a new life distribution, the

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$X - Gamma$ distribution, which has a single parameter, has been introduced and studied. The $X - Gamma$ distribution, due to the simplicity of its density function and its advantageous properties, serves as a promising candidate for modeling lifetimes.

In this paper, various Bayesian estimates of the parameter (α) have been obtained using information Gamma prior and non-information Gamma prior under different loss function, represented by (the square error as symmetric and Entropy as asymmetric).

$X - Gamma$ Distribution

$X - Gamma\ (X - G)$ distribution **with one parameter introduced by** Sen et al. (2016) [9]. The probability density function (pdf) of the $(X - G)$ distribution is given, respectively, by

$$f(x;\alpha) = \frac{\alpha^2}{(\alpha+1)}\left(\frac{\alpha}{2}x^2 + 1\right)e^{-\alpha x} \quad , x > 0, \alpha > 0 \qquad (1)$$

At time (t), the reliability of the $(X - G)$ distribution is given, respectively.

$$R_e(t) = \left(\frac{\alpha+\alpha t+\frac{\alpha^2 t^2}{2}+1}{\alpha+1}e^{-\alpha t}\right) \quad ; t > 0 \qquad (2)$$

Figure (1): offer $R_e(t)$ of the $(X - G)$ distribution for different values of the scale parameter α.

$$h_a(t) = \left(\frac{\alpha^2\left(\frac{\alpha}{2}t^2+1\right)}{\alpha+\alpha t+\frac{\alpha^2 t^2}{2}+1}\right) \quad ; t > 0 \qquad (3)$$

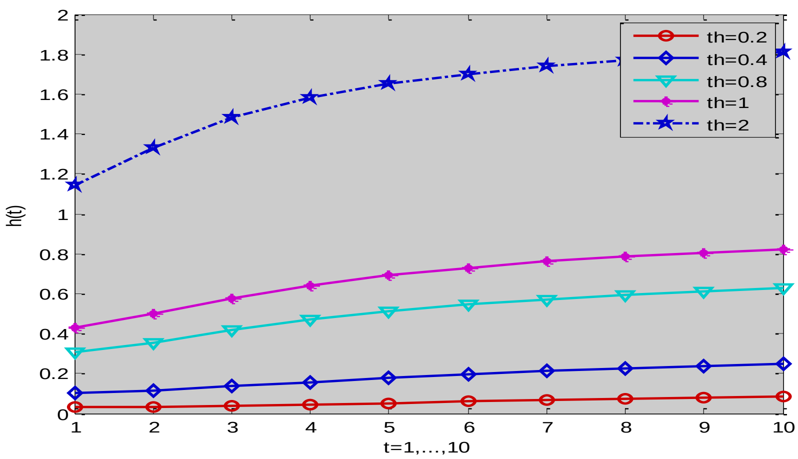

Figure (2): offer $h_a\,(t)$of the $(X-G)$distribution for different values of the scale parameter α.

Maximum Likelihood Estimator (MLE)

Let $(X_1,X_2,\ldots,X_k)$ be an independent and identically distributed (i.i.d.) sample of size (k) from an $(X-G)$ distribution, as defined in (1).

the complete-data likelihood function:[1]

$$L(\alpha|\underline{x}) = \prod_{i=1}^{k} f_x(x_i\,;\alpha) = = \frac{\alpha^{2k}}{(\alpha+1)^k}\,.\,\prod_{i=1}^{k}\left(\frac{\alpha}{2}x_i{}^2+1\right).\,e^{-\alpha\sum_{i=1}^{k}x_i} \quad (4)$$

the natural log likelihood function will be

$$\ln L(\alpha|\underline{x}) \;= 2k\ln(\alpha) - k\ln(1+\alpha) + \sum_{i=1}^{k}\ln\left(\frac{\alpha}{2}x_i{}^2+1\right) - \alpha\sum_{i=1}^{k}x_i \quad (5)$$

the first derivative of $\ln L(\alpha|\underline{x})$ with respect to the scale parameter (α) will be

$$\frac{\partial \ln L(\alpha|\underline{x})}{\partial\alpha} \Rightarrow \frac{2k}{\alpha} - \frac{k}{\alpha+1} + \frac{1}{2}\sum_{i=1}^{k}\frac{x_i^2}{\left(\frac{\alpha}{2}x_i{}^2+1\right)} - \sum_{i=1}^{k}x_i \quad (6)$$

And the second-order derivative of $\ln L(\alpha|\underline{x})$ with respect to the scale parameter (α) will be

$$\frac{\partial^2 \ln L(\alpha|\underline{x})}{\partial\,\alpha^2} = -\frac{2k}{\alpha^2} + \frac{k}{(\alpha+1)^2} - \frac{1}{4}\sum_{i=1}^{k}\frac{x_i^4}{\left(\frac{\alpha}{2}x_i^2+1\right)^2} \quad (7)$$

to solve the equation (7), we use the Newton-Raphson method(N-RA) because it gives a solution that is very close to the optimal solution, better than other numerical iterative methods. We will symbolize Maximum Likelihood Estimator for the parameter α using N-RA as $\hat{\alpha}_{MA}$

Bayes Estimations of the scale Parameter (α), Reliability and Hazard Functions

The Bayesian method of estimation is based on the assumption that the initial information available about the parameter to be estimated can be formulated in the form of a probability density function called (Prior P.d.f.), The standard Bayes estimator is based on the (posterior p.d.f.), This function in the parameter $\alpha$is known as the conditional function of the domain of this parameter in the presence of the current sample.

To obtain a Bayesian estimate of the unknown scale parameter, we need a prior distribution for this parameter. Bayesian procedure involving information Gamma prior and non-information Gamma prior.

consider informative Gamma prior with pdf,

$$h(\alpha) = \frac{\mathcal{B}^{\mathcal{A}}\,\alpha^{\mathcal{A}-1}\,e^{-\mathcal{B}\alpha}}{\Gamma(\mathcal{A})}\;;\alpha>0,\mathcal{A},\mathcal{B}>0 \quad (8)$$

It can be taken as non-informative Gamma prior by setting hyper-parameters $\mathcal{A}, \mathcal{B}$ very small non-negative values. [8]
Now, combining the Gamma prior (8) with Likelihood function (4) we have the posterior distribution of α ,as

$$\mathcal{P}(\alpha|\underline{x}) = \frac{L(\alpha|\underline{x})\hbar(\alpha)}{\int_0^{\infty} L(\alpha|\underline{x})\hbar(\alpha)d\alpha} \tag{9}$$

$$\Rightarrow \mathcal{P}(\alpha|\underline{x}) = \frac{\frac{\alpha^{2k+\mathcal{A}-1}}{(\alpha+1)^k} \cdot \prod_{i=1}^{k} \left(\frac{\alpha}{2}x_i{}^2+1\right). e^{-\alpha\left(\sum_{i=1}^{k} x_i+\mathcal{B}\right)}}{\int_0^{\infty} \frac{\alpha^{2k+\mathcal{A}-1}}{(\alpha+1)^k} \cdot \prod_{i=1}^{k} \left(\frac{\alpha}{2}x_i{}^2+1\right). e^{-\alpha\left(\sum_{i=1}^{k} x_i+\mathcal{B}\right)} d\alpha} \tag{10}$$

There are two types of loss functions: the first type is the symmetric loss function and the second type is the asymmetric loss function. we used the square error as symmetric and Entropy as asymmetric loss function.

Legendre (1805) and Gauss (1810) proposed the squared error loss function (SLF) for α of the following form: [2]

$$L(\hat{\alpha}_{SLF}, \alpha) = (\hat{\alpha}_{SLE} - \alpha)^2 \tag{11}$$

Where $\hat{\alpha}_{SLF}$ is an estimation of α based on SLF.
Properties SLF:
1- Symmetrical: Underestimation and overestimation are punished equally.
2- Major fouls lead to severe penalties.
3- It is easy to derive and widely used in practice.
So, the Bayes estimator of α is,

$$\hat{\alpha}_{BSLF} = \int_0^{\infty} \alpha \ \mathcal{P}(\alpha|\underline{x})d\alpha \tag{12}$$

Entropy loss function (ELF) use to estimate the parameter simultaneously, and the formula of ELF for α is,[5]

$$L(\hat{\alpha}_{ELF}, \alpha) = \frac{\hat{\alpha}_{ELF}}{\alpha} - \ln\left(\frac{\hat{\alpha}_{ELF}}{\alpha}\right) - 1 \tag{13}$$

Where $\hat{\alpha}_{ELF}$ is an estimation of α based on ELF.
Properties ELF:
1- A Symmetrical
2-Determine the expected value of the information in the random variable.
3- used in various applications as a useful indicator of information content.
4-It is also called weighted squared error loss function.[6]
So, the Bayes estimator of α is,

$$\hat{\alpha}_{BELF} = \left[\int_0^{\infty} \frac{1}{\alpha} \mathcal{P}(\alpha|\underline{x})d\alpha\right]^{-1} \tag{14}$$

Where, $\int_0^{\infty} \frac{1}{\alpha} \mathcal{P}(\alpha|\underline{x})d\alpha$ exist and finite.

So, Bayes estimation of any function of α ,say $\tau(\alpha)$,based on squared error loss function, can be written as:

$$\hat{\tau}_{BSLF}\,(\alpha) = \frac{\int_0^{\infty} \tau(\alpha) L(\alpha|\underline{x}) \hbar(\alpha)\, d\alpha}{\int_0^{\infty} L(\alpha|\underline{x}) \hbar(\alpha) d\alpha} \tag{15}$$

$$\Rightarrow \hat{\tau}_{BSLF}\,(\alpha) = \frac{\int_0^{\infty} \tau(\alpha)\, \frac{\alpha^{2k+\mathcal{A}-1}}{(\alpha+1)^k} \cdot \prod_{i=1}^{k} \left(\frac{\alpha}{2} x_i{}^2 + 1\right) . e^{-\alpha\left(\sum_{i=1}^{k} x_i + \mathcal{B}\right)}\, d\alpha}{\int_0^{\infty} \frac{\alpha^{2k+\mathcal{A}-1}}{(\alpha+1)^k} \cdot \prod_{i=1}^{k} \left(\frac{\alpha}{2} x_i{}^2 + 1\right) . e^{-\alpha\left(\sum_{i=1}^{k} x_i + \mathcal{B}\right)}\, d\alpha} \tag{16}$$

Bayes estimation of $\tau(\alpha)$,based on Entropy loss function can be written as:

$$\hat{\tau}_{BELF}\,(\alpha) = \left[\int_0^{\infty} \frac{1}{\tau(\alpha)}\, \mathcal{P}(\alpha|\underline{x}) d\alpha\right]^{-1}$$

Where,

$$\int_0^{\infty} \frac{1}{\tau(\alpha)}\, \mathcal{P}(\alpha|\underline{x}) d\alpha = \frac{\int_0^{\infty} \frac{1}{\tau(\alpha)} L(\alpha|\underline{x}) \hbar(\alpha)\, d\alpha}{\int_0^{\infty} L(\alpha|\underline{x}) \hbar(\alpha) d\alpha}$$

(17)

$$\Rightarrow \int_0^{\infty} \frac{1}{\tau(\alpha)}\, \mathcal{P}(\alpha|\underline{x}) d\alpha = \frac{\int_0^{\infty} \frac{1}{\tau(\alpha)} \cdot \frac{\alpha^{2k+\mathcal{A}-1}}{(\alpha+1)^k} \cdot \prod_{i=1}^{k} \left(\frac{\alpha}{2} x_i{}^2 + 1\right) . e^{-\alpha\left(\sum_{i=1}^{k} x_i + \mathcal{B}\right)}\, d\alpha}{\int_0^{\infty} \frac{\alpha^{2k+\mathcal{A}-1}}{(\alpha+1)^k} \cdot \prod_{i=1}^{k} \left(\frac{\alpha}{2} x_i{}^2 + 1\right) . e^{-\alpha\left(\sum_{i=1}^{k} x_i + \mathcal{B}\right)}\, d\alpha} \tag{18}$$

We note that the integrals (16) and (18) cannot be solved directly; therefore, we employ Lindley's approximation.

5. Lindley's approximation (L-A)

It is an approximate method suitable for solving problems that are in the form of an integral that appears in both the numerator and the denominator. It approaches the results of calculating the ratio of integrals and gives single numerical results. [4]

Equation (15) can be converted to the following form: (see [7][10])

$$\mathcal{T}(\underline{x}) = \int_0^{\infty} \tau(\alpha)\, \mathcal{P}(\alpha|\underline{x}) d\alpha = \frac{\int_0^{\infty} \tau(\alpha) e^{\ln L(\alpha|\underline{x}) + \mathrm{H}(\alpha)}\, d\alpha}{\int_0^{\infty} e^{\ln L(\alpha|\underline{x}) + \mathrm{H}(\alpha)} d\alpha} \tag{19}$$

Where,

$\tau(\alpha)$ is a function of α only,

$\ln L(\alpha|\underline{x})$ is the natural log-likelihood function as in the equation(5),

$\mathrm{H}(\alpha)$ is the natural log- prior density function.

If n is sufficiently large and $let\ \mathcal{L} = \ln L(\alpha|\underline{x})$,

, then $\mathcal{T}(\underline{x})$ by using L-A for one parameter can be computed as,

$$\mathcal{T}(\underline{x}) = \tau(\hat{\alpha}) + \frac{1}{2}\left(\hat{\tau}_{\alpha\alpha} + 2\hat{\tau}_{\alpha}\,.\hat{\mathrm{H}}_{\alpha}\right).\hat{\delta}_{\alpha\alpha} + \frac{1}{2}\left(\hat{\tau}_{\alpha}\,.\left(\hat{\delta}_{\alpha\alpha}\right)^{2}.\hat{\mathcal{L}}_{\alpha\alpha\alpha}\right) \quad (20)$$

Where,

$\hat{\alpha} = \hat{\alpha}_{MA}$

$$\hat{\delta}_{\alpha\alpha} = \left[\left(\frac{-\partial^2 \mathcal{L}}{\partial \alpha^2}\right)^{-1}\right]_{\hat{\alpha}} = \left(\frac{2k}{\hat{\alpha}^2} - \frac{k}{(\hat{\alpha}+1)^2} + \frac{1}{4}\sum_{i=1}^{k} \frac{x_i^4}{\left(\frac{\hat{\alpha}}{2}x_i^2+1\right)^2}\right)^{-1} \quad (21)$$

$$\hat{\tau}_{\alpha} = \frac{d\tau}{d\alpha}\Big|_{\alpha=\hat{\alpha}};\ \hat{\tau}_{\alpha\alpha} = \frac{d^2\tau}{d\alpha^2}\Big|_{\alpha=\hat{\alpha}},\ \hat{\mathrm{H}}_{\alpha} = \frac{d\ln h(\alpha)}{d\alpha}\Big|_{\alpha=\hat{\alpha}} = \frac{\mathcal{A}-1}{\hat{\alpha}} - \mathcal{B}.$$

Now,

$$\hat{\mathcal{L}}_{\alpha\alpha\alpha} = \frac{\partial^3 \mathcal{L}}{\partial \alpha^3}\Big|_{\alpha=\hat{\alpha}} = \frac{4k}{\hat{\alpha}^3} - \frac{2k}{(\hat{\alpha}+1)^3} + \frac{1}{4}\sum_{i=1}^{k} \frac{x_i^6}{\left(\frac{\hat{\alpha}}{2}x_i^2+1\right)^3} \quad (22)$$

Hence, according to the expressions defined above, we can derive the values of the approximate Bayesian estimators of the parameter α and the reliability and hazard function of the $(X-G)$ distribution under SLF using the following:

- For the approximate Bayes estimator for parameter α:

Let τ(α)=α and we will find, $\tau_{\alpha} = 1, \tau_{\alpha\alpha} = 0$.

$$\hat{\alpha}_{BSLF} = \int_0^{\infty} \alpha\, \mathcal{P}(\alpha|\underline{x})d\alpha = \hat{\alpha} + \frac{1}{2}\left(2\,.\hat{\mathrm{H}}_{\alpha}\right).\hat{\delta}_{\alpha\alpha} + \frac{1}{2}\left(\left(\hat{\delta}_{\alpha\alpha}\right)^{2}.\hat{\mathcal{L}}_{\alpha\alpha\alpha}\right) \quad (23)$$

- For the approximate Bayes estimator for reliability function:

Let τ(α)= $R_e(t) = \left(\frac{\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1}{\alpha+1} e^{-\alpha t}\right)$ and we will find,

$$\tau_{\alpha} = \frac{e^{-\alpha t}}{(\alpha+1)^2}\left[(\alpha+1)(1+t+\alpha t^2) - (\alpha t+t+1)\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)\right],$$

$$\tau_{\alpha\alpha} = \frac{-e^{-\alpha t}}{(\alpha+1)^2}\left[\left[\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)(t) + (\alpha t^2+t+1)(\alpha t+t) - (\alpha+1)t^2\right] + \frac{(\alpha t+t+2)}{(\alpha+1)}\left[(\alpha+1)+(\alpha t^2+t+1) - \left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)(\alpha t+t+1)\right]\right].$$

$$\widehat{R_e}_{BSLF}(t) = \int_0^{\infty} R_e(t).\, \mathcal{P}(\alpha|\underline{x})d\alpha$$
$$= \left(\frac{\hat{\alpha}+\hat{\alpha}t+\frac{\hat{\alpha}^2t^2}{2}+1}{\hat{\alpha}+1}e^{-\hat{\alpha}t}\right)+\frac{1}{2}\left(\hat{\tau}_{\alpha\alpha}+2\hat{\tau}_{\alpha}.\widehat{\mathrm{H}}_{\alpha}\right).\hat{\delta}_{\alpha\alpha}$$
$$+\frac{1}{2}\left(\hat{\tau}_{\alpha}.(\hat{\delta}_{\alpha\alpha})^2.\hat{\mathcal{L}}_{\alpha\alpha\alpha}\right)(24)$$

- For the approximate Bayes estimator for hazard function:

Let $\tau(\alpha)= h_a(t) = \left(\frac{\alpha^2\left(\frac{\alpha}{2}t^2+1\right)}{\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1}\right)$ and we will find,

$$\tau_{\alpha} = \frac{2\alpha\left(\frac{3}{4}\alpha t^2+1\right)}{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)} - \frac{\alpha^2\left(\frac{\alpha}{2}t^2+1\right)(\alpha t^2+t+1)}{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)^2},\ \tau_{\alpha\alpha} = \tau_1 - \tau_2$$

where,

$$\tau_1 = \frac{\left[\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)(3\alpha t^2+2)-\left(\frac{\alpha^3}{2}t^2+\alpha^2\right)t^2\right]}{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)^2},$$

$$\tau_2 = \frac{2\ (\alpha t^2+t+1)\left[\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)\left(\frac{2}{3}\alpha^2t^2+2\alpha\right)-\left(\frac{\alpha^3}{2}t^2+\alpha^2\right)(1+t+\alpha t^2)\right]}{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)^3}$$

$$\widehat{h_a}_{BSLF}(t) = \int_0^{\infty} h_a(t)\ \mathcal{P}(\alpha|\underline{x})d\alpha$$
$$= \left(\frac{\hat{\alpha}^2\left(\frac{\hat{\alpha}}{2}t^2+1\right)}{\hat{\alpha}+\hat{\alpha}t+\frac{\hat{\alpha}^2t^2}{2}+1}\right)+\frac{1}{2}\left(\hat{\tau}_{\alpha\alpha}+2\hat{\tau}_{\alpha}.\widehat{\mathrm{H}}_{\alpha}\right).\hat{\delta}_{\alpha\alpha}$$
$$+\frac{1}{2}\left(\hat{\tau}_{\alpha}.(\hat{\delta}_{\alpha\alpha})^2.\hat{\mathcal{L}}_{\alpha\alpha\alpha}\right) \quad (25)$$

Now, the approximate Bayesian estimators of the parameter α and the reliability and hazard function of the $(X-G)$ distribution under ELF using the following:

- For the approximate Bayes estimator for parameter α:

Let $\tau(\alpha)=\frac{1}{\alpha}$ and we will find, $\tau_{\alpha} = \frac{-1}{\alpha^2}$, $\tau_{\alpha\alpha} = \frac{2}{\alpha^3}$.

$$\mathcal{T}(\underline{x}) = \int_0^{\infty} \frac{1}{\alpha}\, \mathcal{P}(\alpha|\underline{x})d\alpha$$

$$= \frac{1}{\hat{\alpha}} + \frac{1}{2}\left(\hat{\tau}_{\alpha\alpha} + 2\hat{\tau}_{\alpha}\,.\widehat{\mathrm{H}}_{\alpha}\right).\hat{\delta}_{\alpha\alpha} + \frac{1}{2}\left(\hat{\tau}_{\alpha}\,.\left(\hat{\delta}_{\alpha\alpha}\right)^{2}.\hat{\mathcal{L}}_{\alpha\alpha\alpha}\right) \quad (26)$$

$$\hat{\alpha}_{BELF} = \left[\int_0^{\infty} \frac{1}{\alpha}\,\mathcal{P}(\alpha|\underline{x})d\alpha\right]^{-1} \quad (27)$$

- For the approximate Bayes estimator for reliability function:

Let $\tau(\alpha)=\frac{1}{R_e(t)} = \left(\frac{\alpha+1}{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right).e^{-\alpha t}}\right)$ and we will find,

$$\tau_{\alpha} = \frac{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)-(\alpha+1)\left[(\alpha t^2+t+1)-\left(\alpha t+\alpha t^2+\frac{\alpha^2t^3}{2}+t\right)\right]}{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)^2 e^{-\alpha t}},$$

$$\tau_{\alpha\alpha} = \tau_1 - \omega\tau_2$$

where,

$$\tau_1 = \frac{(\alpha t^2+t+1)-(\alpha+1).(\alpha t^3-t)+(\alpha t+\frac{\alpha^2t^3}{2}-1)}{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)^2 e^{-\alpha t}},$$

$$\omega = \left[\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)(-t)+2(\alpha t^2+t+1)\right],$$

$$\tau_2 = \frac{\left\{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)-(\alpha+1)\left[(\alpha t^2+t+1)+(-t)\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)\right]\right\}}{\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)^3 e^{-\alpha t}},$$

$$\mathcal{T}(\underline{x}) = \int_0^{\infty}\frac{1}{R_e(t)}\,\mathcal{P}(\alpha|\underline{x})d\alpha$$

$$= \left(\frac{\hat{\alpha}+1}{\left(\hat{\alpha}+\hat{\alpha}t+\frac{\hat{\alpha}^2t^2}{2}+1\right).e^{-\hat{\alpha}t}}\right) + \frac{1}{2}\left(\hat{\tau}_{\alpha\alpha}+2\hat{\tau}_{\alpha}\,.\widehat{\mathrm{H}}_{\alpha}\right).\hat{\delta}_{\alpha\alpha} + \frac{1}{2}\left(\hat{\tau}_{\alpha}\,.\left(\hat{\delta}_{\alpha\alpha}\right)^2.\hat{\mathcal{L}}_{\alpha\alpha\alpha}\right) \quad (28)$$

$$\widehat{R_e}_{BELF}(t) = \left[\int_0^{\infty}\frac{1}{R_e(t)}\,\mathcal{P}(\alpha|\underline{x})d\alpha\right]^{-1}$$

- For the approximate Bayes estimator for hazard function:

Let $\tau(\alpha)=\frac{1}{h_a(t)}=\left(\frac{\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1}{\alpha^2\left(\frac{\alpha}{2}t^2+1\right)}\right)$ and we will find,

$$\tau_\alpha=\frac{\left(\frac{\alpha^3}{2}t^2+\alpha^2\right).(\alpha t^2+t+1)-\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right).\left(\frac{3\alpha^2t^2}{2}+2\alpha\right)}{\alpha^4\left(\frac{\alpha}{2}t^2+1\right)^2},\ \tau_{\alpha\alpha}=\tau_1-\tau_2$$

where,

$$\tau_1=\frac{\left(\frac{\alpha^3}{2}t^4+\alpha^2t^2\right)-\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right)(3\alpha t^2+2)}{\alpha^4\left(\frac{\alpha}{2}t^2+1\right)^2},$$

$$\tau_2=\frac{\alpha^2(3\alpha t^2+4).\left(\alpha\left(\frac{\alpha}{2}t^2+1\right).(\alpha t^2+t+1)-\left(\alpha+\alpha t+\frac{\alpha^2t^2}{2}+1\right).\left(\frac{3\alpha}{2}t^2+2\right)\right)}{\alpha^6\left(\frac{\alpha}{2}t^2+1\right)^3}$$

$$\mathcal{T}(\underline{x})=\int_0^\infty\frac{1}{h_a(t)}\mathcal{P}(\alpha|\underline{x})d\alpha$$

$$=\left(\frac{\hat{\alpha}+\hat{\alpha}t+\frac{\hat{\alpha}^2t^2}{2}+1}{\hat{\alpha}^2\left(\frac{\hat{\alpha}}{2}t^2+1\right)}\right)+\frac{1}{2}\left(\hat{\tau}_{\alpha\alpha}+2\hat{\tau}_\alpha.\hat{\mathrm{H}}_\alpha\right).\hat{\delta}_{\alpha\alpha}+\frac{1}{2}\left(\hat{\tau}_\alpha.\left(\hat{\delta}_{\alpha\alpha}\right)^2.\hat{\mathcal{L}}_{\alpha\alpha\alpha}\right)\quad(29)$$

$$\widehat{h_a}_{BELF}(t)=\left[\int_0^\infty\frac{1}{h_a(t)}\mathcal{P}(\alpha|\underline{x})d\alpha\right]^{-1}$$

Numerical Experiments

Simulation is a quantitative methodology used to conduct computational experiments that clarify, model, and analyze the dynamics of complex real-world systems through specific frameworks of mathematical and logical relationships.

In this section, we focus on evaluate the behavior of Bayesian estimators for one scale parameter, Reliability and Hazard Functions of the X-G distribution using the information Gamma when the values of hyperparameter are chosen to be $\mathcal{A}=\mathcal{B}=1.5$ and non-information Gamma prior when the values of hyperparameter are chosen to be $\mathcal{A}=\mathcal{B}=0.0001$. A random sample was generated according to the X-Gamma (X-G) distribution using the inverse transformation method, and six sample sizes (k = 10, 20, 30,50,75 and 100) were chosen for small, medium, and large data sets.

The default values of the scale parameter for five cases $\alpha=0.02,0.4,0.7,1.3,2$.

The initial value of the scale parameter (α) followed with the algorithms under study was chosen to be the moment estimator.

$\hat{\alpha}_{me} = \frac{-(m-1)+\sqrt{(m-1)^2+12m}}{2m}$, where $m = \frac{(3+\alpha)}{\alpha(1+\alpha)}$

The comparisons between the scale parameter estimates were based on values from bias (BAS), Mean Square Error (MSE) and for reliability and hazard functions were integrated mean squared error.

$$bias(\alpha) = \sum_{I=1}^{F} \frac{(\hat{\alpha}_I - \alpha)}{F} \quad (30)$$

$$\text{MSE}(\alpha) = \sum_{I=1}^{F} \frac{(\hat{\alpha}_I - \alpha)^2}{F} \quad (31)$$

$$\text{IMSE}(\hat{R}(t)) = \frac{1}{F}\sum_{J=1}^{F} \frac{1}{k_t}\sum_{I=1}^{k_t}\left(\hat{R}_J(t_I) - R(t_I)\right)^2 \quad (32)$$

$$\text{IMSE}(\hat{h}(t)) = \frac{1}{F}\sum_{J=1}^{F} \frac{1}{k_t}\sum_{I=1}^{k_t}\left(\hat{h}_J(t_I) - h(t_I)\right)^2 \quad (33)$$

where $\hat{\alpha}_I$ is the estimate of α , α is the true value,
The number of samples replicated chosen to be ($F = 5000$).
k_t is the number of times where t=1,2,3,4.
$\hat{R}_J(t_I)$ is the estimates of reliability function at the replicate J and the time I.
$\hat{h}_J(t_I)$ is the estimates of hazard function at the replicate J and the time I.
The tables below display the outcomes of the estimates.

Table (1)

Bias results of the Estimates the Parameter α associated with Bayes Estimates of (X-G) distribution for Different Sample Sizes.

k	α	Non-Informative Priors $\mathcal{A} = \mathcal{B} = 0.0001$		Informative Priors $\mathcal{A} = \mathcal{B} = 1.5$		THE BEST
		$\hat{\alpha}_{BSLF}$	$\hat{\alpha}_{BELF}$	$\hat{\alpha}_{BSLF}$	$\hat{\alpha}_{BELF}$	
10	0.02	0.0001068	0.0001063	0.0001076	0.0001071	NBELF
	0.4	0.1435676	0.1439119	0.1431217	0.1434664	IBSLF
	0.7	0.4699834	0.4709857	0.4688186	0.4698228	IBSLF
	1.3	1.8043033	1.8074236	1.8013549	1.8044841	IBSLF
	2	4.6838885	4.6905794	4.6788055	4.6855171	IBSLF
20	0.02	0.0001041	0.0001040	0.0001041	0.0001040	BELF
	0.4	0.1423327	0.1424205	0.1423327	0.1424205	IBSLF
	0.7	0.4657291	0.4659857	0.4657291	0.4659857	IBSLF
	1.3	1.7900520	1.7908520	1.7900520	1.7908520	IBSLF
	2	4.6371592	4.6388764	4.6371592	4.6388764	IBSLF
30	0.02	0.0001037	0.0001036	0.0001036	0.0001035	IBELF
	0.4	0.1417967	0.1418360	0.1417457	0.1417850	IBSLF
	0.7	0.4647811	0.4648959	0.4646472	0.4647620	IBSLF

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	1.3	1.7846939	1.7850523	1.7843514	1.7847099	IBSLF
	2	4.6050197	4.6057921	4.6044307	4.6052034	IBSLF
50	0.02	0.0001031	0.0001030	0.0001030	0.0001029	IBELF
	0.4	0.1416488	0.1416630	0.1416303	0.1416445	IBSLF
	0.7	0.4641137	0.4641552	0.4640651	0.4641066	IBSLF
	1.3	1.7807560	1.7808858	1.7806315	1.7807614	IBSLF
	2	4.5918962	4.5921764	4.5916820	4.5919622	IBSLF
75	0.02	0.0001022	0.0001022	0.0001022	0.0001022	BELF
	0.4	0.1415755	0.1415818	0.1415672	0.1415736	IBSLF
	0.7	0.4628216	0.4628401	0.4627999	0.4628184	IBSLF
	1.3	1.7760845	1.7761424	1.7760289	1.7760869	IBSLF
	2	4.5829401	4.5830651	4.5828445	4.5829695	IBSLF
100	0.02	0.0001021	0.0001021	0.0001021	0.0001021	BELF
	0.4	0.1415139	0.1415175	0.1415093	0.1415129	IBSLF
	0.7	0.4627046	0.4627150	0.4626924	0.4627028	IBSLF
	1.3	1.7718465	1.7718792	1.7718152	1.7718479	IBSLF
	2	4.5795175	4.5795879	4.5794636	4.5795340	IBSLF

Table (2)

MSE results of the Estimates the Parameter α associated with Bayes Estimates of (X-G) distribution for Different Sample Sizes.

k	α	Non-Informative Priors $\mathcal{A} = \mathcal{B} = 0.0001$		Informative Priors $\mathcal{A} = \mathcal{B} = 1.5$		THE BEST
		$\hat{\alpha}_{BSLF}$	$\hat{\alpha}_{BELF}$	$\hat{\alpha}_{BSLF}$	$\hat{\alpha}_{BELF}$	
10	0.02	0.0001068	0.0001063	0.0001076	0.0001071	IBELF
	0.4	0.1435676	0.1439119	0.1431217	0.1434664	IBSLF
	0.7	0.4699834	0.4709857	0.4688186	0.4698228	IBSLF
	1.3	1.8043033	1.8074236	1.8013549	1.8044841	IBSLF
	2	4.6838885	4.6905794	4.6788055	4.6855171	IBSLF
20	0.02	0.0001041	0.0001040	0.0001043	0.0001042	IBELF
	0.4	0.1423327	0.1424205	0.1422188	0.1423067	IBSLF
	0.7	0.4657291	0.4659857	0.4654301	0.4656868	IBSLF
	1.3	1.7900520	1.7908520	1.7892896	1.7900902	IBSLF
	2	4.6371592	4.6388764	4.6358477	4.6375662	IBSLF
30	0.02	0.0001037	0.0001036	0.0001036	0.0001035	IBELF
	0.4	0.1417967	0.1418360	0.1417457	0.1417850	IBSLF
	0.7	0.4647811	0.4648959	0.4646472	0.4647620	IBSLF
	1.3	1.7846939	1.7850523	1.7843514	1.7847099	IBSLF

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	2	4.6050197	4.6057921	4.6044307	4.6052034	IBSLF
50	0.02	0.0001031	0.0001031	0.0001032	0.0001032	IBELF
	0.4	0.1416488	0.1416630	0.1416303	0.1416445	IBSLF
	0.7	0.4641137	0.4641552	0.4640651	0.4641066	IBSLF
	1.3	1.7807560	1.7808858	1.7806315	1.7807614	IBSLF
	2	4.5918962	4.5921764	4.5916820	4.5919622	IBSLF
75	0.02	0.0001023	0.0001023	0.0001023	0.0001023	IBELF
	0.4	0.1415755	0.1415818	0.1415672	0.1415736	IBSLF
	0.7	0.4628216	0.4628401	0.4627999	0.4628184	IBSLF
	1.3	1.7760845	1.7761424	1.7760289	1.7760869	IBSLF
	2	4.5829401	4.5830651	4.5828445	4.5829695	IBSLF
100	0.02	0.0001022	0.0001022	0.0001023	0.0001022	IBELF
	0.4	0.1415139	0.1415175	0.1415093	0.1415129	IBSLF
	0.7	0.4627046	0.4627150	0.4626924	0.4627028	IBSLF
	1.3	1.7718465	1.7718792	1.7718152	1.7718479	IBSLF
	2	4.5795175	4.5795879	4.5794636	4.5795340	IBSLF

Table (3)

The IMSE results of the Estimates the reliability function associated with Bayes Estimates of (X-G) distribution for Different Sample Sizes.

k	α	Non-Informative Priors $\mathcal{A} = \mathcal{B} = 0.0001$		Informative Priors $\mathcal{A} = \mathcal{B} = 1.5$		THE BEST
		$\widehat{R}_{e_{BSLF}}(t)$	$\widehat{R}_{e_{BELF}}(t)$	$\widehat{R}_{e_{BSLF}}(t)$	$\widehat{R}_{e_{BELF}}(t)$	
10	0.02	0.0044743	0.0044886	0.0044721	0.0044882	IBSLF
	0.4	0.0682574	0.0689950	0.0676039	0.0680792	IBSLF
	0.7	0.1995898	0.2045793	0.1975558	0.2018656	IBSLF
	1.3	0.3109459	0.3337172	0.3076091	0.3292042	IBSLF
	2	0.2987701	0.3370325	0.2957578	0.3325507	IBSLF
20	0.02	0.0038912	0.0038238	0.0038910	0.0038236	IBELF
	0.4	0.0681320	0.0680102	0.0676000	0.0680048	IBSLF
	0.7	0.1919129	0.2031192	0.1913807	0.2014095	IBSLF
	1.3	0.3103884	0.3253061	0.3018491	0.3241195	IBSLF
	2	0.2978421	0.3211722	0.2948123	0.3200275	IBSLF
30	0.02	0.0036327	0.0036334	0.0036325	0.0036330	IBSLF
	0.4	0.0680451	0.0679023	0.0674702	0.0679013	IBSLF
	0.7	0.1909122	0.2029399	0.1908726	0.2026202	IBSLF
	1.3	0.3009858	0.3226035	0.3005792	0.3220672	IBSLF
	2	0.2872588	0.3130577	0.2818835	0.3125486	IBSLF

50	0.02	0.0024598	0.0024447	0.0024594	0.0024445	IBELF
	0.4	0.0651098	0.0671376	0.0650828	0.0670993	IBSLF
	0.7	0.1827827	0.2029290	0.1802695	0.2025524	IBSLF
	1.3	0.2994533	0.3203844	0.2993044	0.3201884	IBSLF
	2	0.2764599	0.3085712	0.2706323	0.3083870	IBSLF
75	0.02	0.0018889	0.0019789	0.0018887	0.0019787	IBELF
	0.4	0.0631815	0.0651937	0.0631695	0.0651767	IBSLF
	0.7	0.1624903	0.2025728	0.1624512	0.2025206	IBSLF
	1.3	0.2181302	0.3185441	0.2180636	0.3184565	IBSLF
	2	0.2051249	0.3060676	0.2050642	0.3059857	IBSLF
100	0.02	0.0010632	0.0010645	0.0010630	0.0010643	IBELF
	0.4	0.0622096	0.0632164	0.0620028	0.0632068	IBSLF
	0.7	0.1025555	0.2023737	0.1025335	0.2023723	IBSLF
	1.3	0.2167996	0.3170329	0.2167620	0.3169835	IBSLF
	2	0.2043866	0.3049170	0.2043524	0.3048709	IBSLF

Table (4)

The IMSE results of the Estimates the hazard function associated with Bayes Estimates of (X-G) distribution for Different Sample Sizes.

k	α	Non-Informative Priors $\mathcal{A} = \mathcal{B} = 0.0001$		Informative Priors $\mathcal{A} = \mathcal{B} = 1.5$		THE BEST
		$\hat{h}_{a_{BSLF}}(t)$	$\hat{h}_{a_{BELF}}(t)$	$\hat{h}_{a_{BSLF}}(t)$	$\hat{h}_{a_{BELF}}(t)$	
10	0.02	0.0030045	0.0032364	0.0030042	0.0032345	IBELF
	0.4	0.0143508	0.0157133	0.0142362	0.0153100	IBSLF
	0.7	0.0836583	0.0985438	0.0830690	0.0966829	IBSLF
	1.3	0.4739917	0.5970023	0.4714472	0.5893078	IBSLF
	2	1.4274350	1.8613362	1.4215805	1.8445743	IBSLF
20	0.02	0.0027789	0.0025766	0.0027680	0.0025765	IBELF
	0.4	0.0143385	0.0135911	0.0141509	0.0132644	IBELF
	0.7	0.0835922	0.0877715	0.0824423	0.0862193	IBSLF
	1.3	0.4691254	0.5417214	0.4684834	0.5349858	IBSLF
	2	1.4263540	1.7125421	1.3435037	1.6954895	IBSLF
30	0.02	0.0025076	0.0023421	0.0025055	0.0022421	IBELF
	0.4	0.0142794	0.0121094	0.0140665	0.0118221	IBELF
	0.7	0.0820076	0.0797027	0.0819406	0.0782755	IBELF
	1.3	0.4034511	0.5046214	0.4031642	0.4990979	IBSLF
	2	1.3622992	1.5949389	1.5616663	1.5800763	NBSLF
50	0.02	0.0019981	0.0004576	0.0019980	0.0004562	IBELF

	0.4	0.0142162	0.0101934	0.0140169	0.0099702	*IBELF*
	0.7	0.0813208	0.0674481	0.0802966	0.0661805	*IBELF*
	1.3	0.3998192	0.4456748	0.3057150	0.4403961	*IBSLF*
	2	1.3227788	1.4510614	1.1725499	1.4419084	*IBSLF*
75	0.02	0.0007764	0.0003892	0.0007762	0.0003888	*IBELF*
	0.4	0.0140397	0.0088691	0.0136376	0.0087114	*IBELF*
	0.7	0.0803297	0.0558202	0.0800189	0.0546453	*IBELF*
	1.3	0.3059023	0.3879563	0.2958557	0.3833111	*IBSLF*
	2	1.2749235	1.3219781	1.1574821	1.3138414	*IBSLF*
100	0.02	0.0006547	0.0002430	0.0006545	0.0002328	*IBELF*
	0.4	0.0136474	0.0061658	0.0126462	0.0060562	*IBELF*
	0.7	0.0763861	0.0170184	0.0763800	0.0159857	*IBELF*
	1.3	0.2055179	0.3447508	0.1054915	0.3410340	*IBSLF*
	2	1.1757885	1.2041836	1.0757309	1.1956556	*IBSLF*

Conclusions and Recommendations

Tables (1-4) offer a numerical comparison through using a Monte Carlo style to obtained estimates the one unknown parameter α, reliability and hazard rate functions of X-G distribution by MSE, IMSE respectively.

The most essential concluding observations of the numerical results are:

Concluding from Tables (1-3): with different values of α, the Bayes estimate under squared error loss function using informative priors showed the best performance in comparison with other estimates for all sample sizes except for α=0.02, where approximate Bayes estimate under Entropy loss priors was the best .

Concluding from Table (4): with different values of α, the Bayes estimate under Entropy error loss function using informative priors showed the best performance in comparison with other estimates for sample sizes $k = 30,50,27,100$ with cases α=0.02,0.4,0.7.

Based on our findings, we recommend primarily employing the Bayes estimate under squared error loss function using informative priors for estimating the scale parameter and reliability function of the X-G distribution. For the hazard rate function of the X-G distribution, we recommend using the Bayes estimate under Entropy error loss function using informative priors.

Further Work

Bayesian estimation of the X-G distribution with other loss functions not used in research or estimation by other methods.

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الاستفادة من دوال الخسارة في التقديرات البايزية لتوزيع (X-G)

وسناء جليل حمود [1]
قسم علوم الرياضيات، كلية العلوم، الجامعة المستنصرية
wasnaagh2024@uomustansiriyah.edu.iq

شروق احمد كريم صادق السلطاني [2]
قسم علوم الرياضيات، كلية العلوم، الجامعة المستنصرية
shuroq80@uomustansiriyah.edu.iq

مستخلص البحث:
ركز هذا البحث على تقييم اداء طريقة التقدير البيزي لتقدير معلمة القياس ودالة المعولية ودالة الخطورة لتوزيع اكس كاما. الاجراء البيزي يتضمن معلومات كاما السابقة ومعلومات كاما غير السابقة. تم اعتماد تقريب لاندلي لحساب تقديرات بيز بالاستفادة من دالتي خسارة وهما الخطا التربيعي والانتروبيا. تمت المقارنة بين تقديرات معلمة القياس وفقا لقيم متوسطات الخطا ومقارنة اداء تقديرات دالة المعولية والخطورة وفقا لقيم متوسط مربعات الخطا التكاملي

الكلمات المفتاحية :توزيع اكس كاما ، دالة الخسارة التربيعية ،دالة خسارة انتروبي، بيز ،متوسط مربعات الخطا.

ملاحظة : هل البحث مستل من رسالة ماجستير او اطروحة دكتوراه ؟ نعم