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RESEARCH ARTICLE

Iterative Approximation of Best Proximity Points of M-T Cyclic Contraction Mappings

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ABSTRACT

In many instances, acquiring a solution for an operator equation by using conventional analytical techniques, even after it has been shown that such a solution exists. One has to know the approximate value of this solution to solve situations like these. In order to do this, the operator equation must first be restructured to take the form of a fixed-point equation (FP). On the FP equation, the most convenient iterative method is used, and the limit of the sequence that is created by this algorithm is, in fact, the value of the FP that is sought for the FP equation, as well as the solutions to the operator equation. The numerical computation of FPs for nonlinear operators is now an interesting research subject in nonlinear analysis owing to its applicability in several fields. Many researchers have developed a wide range of techniques to estimate the FP for various sorts of applications. The primary objective of this paper is to present new iteration schemes for approximating the best proximity point (BPP). The convergence of BPP for M-T cyclic contraction mappings (MTCC-mapping) has been investigated in the context of uniformly convex Banach spaces (UCB-space). The iterations approach proposed by Mann and Ishikawa was taken into account and as a result, some strong convergence results were obtained for the BPP for MTCC-mapping. Furthermore, numerical examples supporting the primary conclusion are provided, and the convergence behavior of the iterations is compared.

Keywords: Best proximity point, Iterative sequences, M-T cyclic contraction mapping, Strong convergence, Uniformly convex Banach spaces

Introduction

Functional analysis is a subject of mathematics which analyzes functions by studying the functioning of a given function and discovering connections and assumptions that can arise. It is also used to examine a variety of spaces.[1–3] One of the most difficult and quickly expanding subfields in nonlinear functional analysis is fixed-point (FP) theory. This theorem is an important resource for establishing the existence and singularity of solutions to a broad variety of mathematical models that explain phenomena across various domains. The core idea of FP theory is the approximation of FPs in various domains for non-linear mappings using various iterative procedures. Several authors put forward various iterative methods for determining the FP's approximate value.[4–6] An iterative process[7] is proposed to approximate FPs of nonexpansive mappings in Banach spaces that are uniformly convex. The iterative process is as follows:

$$\kappa_{n+1} = (1-\alpha_n)\kappa_n + \alpha_n T\kappa_n. \qquad (1)$$

The primary significance of the BPP theory is in its association with both FP theory and optimization theory. Consequently, the exploration of finding the BPP of mappings that meet various forms of contractive conditions becomes an intriguing area of study. An extensive number of authors have researched the existence of the BPP, see.[8–10] Eldred and Veeramani[11] presented iteration algorithms as a novel approach to identifying the BPP in cyclic contraction mapping.

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The convergence of BPP for noncyclic contractions is discussed by Gabeleh and Kunzi.[12] In the present article, new iteration schemes are presented for approximating the BPP and the convergence of BPP for MTCC-mapping is examined in the context of UCB-space. Moreover, the outcomes produced by iterative algorithms were evaluated and contrasted.

Preliminaries

In this part, some fundamental terminology and ideas that are relevant to the context of our findings are provided.

Assume W is a normed space and consider $\mathcal{U} \subseteq W$ and $\mathcal{V} \subseteq W$. The pair $(\mathcal{U}_\circ, \mathcal{V}_\circ)$ will represent the proximal pair of $(\mathcal{U}, \mathcal{V})$, is defined as follows

$$\mathcal{U}_\circ = \{\rho_1 \in \mathcal{U} : \|\rho_1 - \rho_2\| = \mathrm{dist}(\mathcal{U}, \mathcal{V}), \text{ for some } \rho_2 \in \mathcal{V}\},$$

$$\mathcal{V}_\circ = \{\rho_2 \in \mathcal{V} : \|\rho_1 - \rho_2\| = \mathrm{dist}(\mathcal{U}, \mathcal{V}), \text{ for some } \rho_1 \in \mathcal{U}\},$$

where $\mathrm{dist}(\mathcal{U}, \mathcal{V}) = \inf\{\|\rho_1 - \rho_2\| : \rho_1 \in \mathcal{U}, \rho_2 \in \mathcal{V}\}$.

Definition 1: [13] Assume (W, d) is metric space and $(\mathcal{U}, \mathcal{V})$ is a pair of nonempty subsets of (W, d), where $\mathcal{U}_\circ \neq \emptyset$. Then $(\mathcal{U}, \mathcal{V})$ possesses the P-property(P-p) if and only if

$$\begin{cases} d(k_1, \rho_1) = \mathrm{dist}(\mathcal{U}, \mathcal{V}) \\ d(k_2, \rho_2) = \mathrm{dist}(\mathcal{U}, \mathcal{V}) \end{cases} \Rightarrow d(k_1, k_2) = d(\rho_1, \rho_2)$$

where $k_1, k_2 \in \mathcal{U}_\circ$ and $\rho_1, \rho_2 \in \mathcal{V}_\circ$.

According to Abkar and Gabeleh[14] any closed, convex, and bounded pair in a UCB-space W possessed the P-p,

Definition 2: [15] A function $\phi : [0, \infty) \to [0, 1)$ is termed M-T if $\limsup_{s \to t+} \phi(t) < 1$ for every $t \in [0, \infty)$.

Definition 3: [13] Consider $\mathcal{U}$, $\mathcal{V}$ are nonempty subsets of (W, d). If $T : \mathcal{U} \cup \mathcal{V} \to \mathcal{U} \cup \mathcal{V}$ fulfills

(1) $T(\mathcal{U}) \subset \mathcal{V}$ and $T(\mathcal{V}) \subset \mathcal{U}$;
(2) there is $\phi : [0, \infty) \to [0, 1)$ (which is M-T function)

such that

$$d(T\rho_1, T\rho_2) \leq \phi(d(\rho_1, \rho_2))\, d(\rho_1, \rho_2) + (1 - \phi(d(\rho_1, \rho_2)))\, \mathrm{dist}(\mathcal{U}, \mathcal{V})$$

for any $\rho_1 \in \mathcal{U}$ and $\rho_2 \in \mathcal{V}$,

then T is termed as M-T cyclic contraction (MTCC-mapping) on $\mathcal{U} \cup \mathcal{V}$.

Theorem 1: [15] *Suppose $\mathcal{U}, \mathcal{V}$ are nonempty subsets of (W, d) and assume $T : \mathcal{U} \cup \mathcal{V} \to \mathcal{U} \cup \mathcal{V}$ is MTCC-map. Let $\rho_1 \in \mathcal{U}$. Define a sequence $\{\rho_n\}_{n \in N}$ by $\rho_{n+1} = T\rho_n$. Assume $\{\rho_{2n-1}\}$ possesses a convergent subsequence in $\mathcal{U}$, then there is $u \in \mathcal{U}$ with $d(u, Tu) = \mathrm{dist}(\mathcal{U}, \mathcal{V})$.*

Lemma 1: [11] *Let W be a UCB-space and $\mathcal{U} \subseteq W$, $\mathcal{V} \subseteq W$ where $\mathcal{U}$ is closed and convex and $\mathcal{V}$ is closed. Assume that the sequences $\{\rho_n\}$ and $\{u_n\}$ in $\mathcal{U}$ and $\{q_n\}$ in $\mathcal{V}$ that satisfy:*

(i) $\|u_n - q_n\| \to \mathrm{dist}(\mathcal{U}, \mathcal{V})$.
(ii) there is N_0 such that $\|\rho_m - q_n\| \leq \mathrm{dist}(\mathcal{U}, \mathcal{V}) + \varepsilon$ for each $\varepsilon > 0$ and for $m > n \geq N_0$.

Then, there is N_1 such that $\|\rho_m - u_n\| \leq \varepsilon, \forall m > n \geq N_1$.

Lemma 2: [11] *Let W be a UCB-space and let $\mathcal{U} \subseteq W$, $\mathcal{V} \subseteq W$ where $\mathcal{U}$ is closed and convex and $\mathcal{V}$ is closed. Assume that the sequences $\{\rho_n\}$ and $\{u_n\}$ in $\mathcal{U}$ and $\{q_n\}$ in $\mathcal{V}$ that satisfy:*

(1) $\|\rho_n - q_n\| \to \mathrm{dist}(\mathcal{U}, \mathcal{V})$.
(2) $\|u_n - q_n\| \to \mathrm{dist}(\mathcal{U}, \mathcal{V})$.

Then $\|\rho_n - u_n\|$ converges to zero.

Lemma 3: [7] *Consider $\varphi : [0, \infty) \to [0, \infty)$ where $\varphi(0) = 0$ and φ strictly increasing. If $\{t_n\}$ in $[0, \infty)$ fulfills $\lim_{n\to\infty}\varphi(t_n) = 0$, then $\lim_{n\to\infty} t_n = 0$.*

Main results

Theorem 2: *Assume W is UCB-space and suppose $\mathcal{U}, \mathcal{V} \subseteq W$ where $\mathcal{U}, \mathcal{V}$ closed and convex. If $T : \mathcal{U} \cup \mathcal{V} \to \mathcal{U} \cup \mathcal{V}$ is MTCC-map, then there exists a unique BPP ρ in $\mathcal{U}$. Additionally, if $\rho_0 \in \mathcal{U}$ and $\rho_{n+1} = T\rho_n$, then $\{\rho_{2n}\}$ converges to the BPP.*

Proof: Suppose $dist(\mathcal{U}, \mathcal{V}) = 0$, then $\mathcal{U} \cap \mathcal{V} \neq \emptyset$. Assume $dist(\mathcal{U}, \mathcal{V}) \neq 0$. Since

$$\|\rho_{2n} - T\rho_{2n}\| \to dist(\mathcal{U}, \mathcal{V}),$$

and

$$\|T^2\rho_{2n} - T\rho_{2n}\| \to dist(\mathcal{U}, \mathcal{V}).$$

By Lemma 2, $\|\rho_{2n} - \rho_{2(n+1)}\| \to 0$. Similarly, it can show that $\|T\rho_{2n} - T\rho_{2(n+1)}\| \to 0$. Now to show that $\forall \varepsilon > 0$, $\exists N_0$ such that $\forall m > n \geq N_0$, $\|\rho_{2m} - T\rho_{2n}\| \leq dist(\mathcal{U}, \mathcal{V}) + \varepsilon$.

If that is not the case, then $\exists\ \varepsilon > 0$ such that $\forall k \in N$ there is $m_k > n_k \geq N_0$ for which $\|\rho_{2m_k} - T\rho_{2n_k}\| \geq dist(\mathcal{U}, \mathcal{V}) + \varepsilon$.

Now

$$dist(\mathcal{U}, \mathcal{V}) + \varepsilon \leq \|\rho_{2m_k} - T\rho_{2n_k}\| \leq \|\rho_{2m_k} - \rho_{2(m_k-1)}\| + \|\rho_{2(m_k-1)} - T\rho_{2n_k}\|$$

Hence $\lim_{k\to\infty}\|\rho_{2m_k} - T\rho_{2n_k}\| = dist(\mathcal{U}, \mathcal{V}) + \varepsilon$

Consequently

$$\|\rho_{2m_k} - T\rho_{2n_k}\| \leq \|\rho_{2m_k} - \rho_{2(m_k+1)}\| + \|\rho_{2(m_k+1)} - T\rho_{2(n_k+1)}\| + \|T\rho_{2(n_k+1)} - T\rho_{2n_k}\|$$

$$\begin{aligned}\|\rho_{2(m_k+1)} - T\rho_{2(n_k+1)}\| &= \|T\rho_{2m_k+1} - T\rho_{2(n_k+1)}\| \\ &\leq \phi\left(\|\rho_{2m_k+1} - \rho_{2(n_k+1)}\|\right)\|\rho_{2m_k+1} - \rho_{2(n_k+1)}\| \\ &\quad + \left(1 - \phi\left(\|\rho_{2m_k+1} - \rho_{2(n_k+1)}\|\right)\right) dist\,(\mathcal{U}, \mathcal{V}) \\ &\leq \phi\left(\|\rho_{2m_k+1} - \rho_{2(n_k+1)}\|\right)\|T\rho_{2m_k} - T\rho_{2n_k+1}\| \\ &\quad + \left(1 - \phi\left(\|\rho_{2m_k+1} - \rho_{2(n_k+1)}\|\right)\right) dist\,(\mathcal{U}, \mathcal{V}) \\ &\leq \phi\left(\|\rho_{2m_k+1} - \rho_{2(n_k+1)}\|\right)\left[\phi\left(\|\rho_{2m_k} - \rho_{2n_k+1}\|\right)\|\rho_{2m_k} - \rho_{2n_k+1}\| + \left(1 - \phi\left(\|\rho_{2m_k} - \rho_{2n_k+1}\|\right)\right) dist\,(\mathcal{U}, \mathcal{V})\right] \\ &\quad + (1 - \phi\,(\|\rho_{2m_k+1} - \rho_{2(n_k+1)}\|))\, dist(\mathcal{U}, \mathcal{V}).\end{aligned}$$

Let $\alpha = \phi(\|\rho_{2m_k+1} - \rho_{2(n_k+1)}\|)$ then one get, $\|\rho_{2(m_k+1)} - T\rho_{2(n_k+1)}\| \leq \alpha[\phi((dist(\mathcal{U}, \mathcal{V}) + \varepsilon))(dist(\mathcal{U}, \mathcal{V}) + \varepsilon) + (1 - \phi(dist(\mathcal{U}, \mathcal{V}) + \varepsilon))dist(\mathcal{U}, \mathcal{V})] + (1 - \alpha)dist(\mathcal{U}, \mathcal{V})$.

Now let $\beta = \phi((dist(\mathcal{U}, \mathcal{V}) + \varepsilon))$ then

$$\begin{aligned}\|\rho_{2(m_k+1)} - T\rho_{2(n_k+1)}\| &\leq \alpha\left[\beta\left(dist\,(\mathcal{U}, \mathcal{V}) + \varepsilon\right) + (1 - \beta)\, dist\,(\mathcal{U}, \mathcal{V})\right] + (1 - \alpha)\, dist\,(\mathcal{U}, \mathcal{V}) \\ &\leq \alpha\left[\beta dist\,(\mathcal{U}, \mathcal{V}) + \beta\varepsilon + dist\,(\mathcal{U}, \mathcal{V}) - \beta dist\,(\mathcal{U}, \mathcal{V})\right] + (1 - \alpha)\, dist\,(\mathcal{U}, \mathcal{V}) \\ &\leq \alpha\left[\beta\varepsilon + dist\,(\mathcal{U}, \mathcal{V})\right] + (1 - \alpha)\, dist\,(\mathcal{U}, \mathcal{V}) \\ &\leq \alpha\beta\varepsilon + dist(\mathcal{U}, \mathcal{V}).\end{aligned}$$

Hence

$$\begin{aligned}dist\,(\mathcal{U}, \mathcal{V}) + \varepsilon &\leq \alpha\left[\beta\varepsilon + dist\,(\mathcal{U}, \mathcal{V})\right] + (1 - \alpha)\, dist\,(\mathcal{U}, \mathcal{V}) \\ &= \alpha\beta\varepsilon + dist\,(\mathcal{U}, \mathcal{V})\end{aligned}$$

Now the Mann iterative technique for determining the BPP is discussed in the following.

Theorem 3: *Suppose that* W *is a UCB-space and $(\mathcal{U}, \mathcal{V})$ is a closed, disjoint, bounded, and convex pair in W and assume $T : \mathcal{U} \cup \mathcal{V} \to \mathcal{U} \cup \mathcal{V}$ be MTCC-map.*

Consider $k_\circ \in \mathcal{U}$ and

$$k_{n+1} = (1 - \alpha_n)\, k_n + \alpha_n T^2 k_n, \tag{2}$$

where $\alpha_n \in (\varepsilon, 1 - \varepsilon)$ and $\varepsilon \in (0, 1/2]$. Then $\{k_n\}$ strongly converges to the BPP of T.

Proof: From Theorem 2, $\exists\ q \in \mathcal{V}_\circ$ with $\|q - Tq\| = dist(\mathcal{U}, \mathcal{V})$. Since $\|T^2q - Tq\| = dist(\mathcal{U}, \mathcal{V})$ and since $(\mathcal{U}, \mathcal{V})$ possesses the P-p, conclude that $q = T^2q$, that is, q is a FP of $T^2 : \mathcal{V} \to \mathcal{V}$.

Now,

$$\begin{aligned}\|k_{n+1} - q\| &= \|\,(1 - \alpha_n)\, k_n + \alpha_n T^2 k_n - (1 - \alpha_n)\, q + \alpha_n T^2 q\| \\ &\leq (1 - \alpha_n)\,\|k_n - q\| + \alpha_n\|T^2k_n - T^2q\| \\ &\leq \|k_n - q\|.\end{aligned}$$

Thus $\{\|k_n - q\|\}_{n\geq1}$ is a decreasing sequence. Assuming that $\lim_{n\to\infty}\|k_n - q\| = l \geq dist(\mathcal{U}, \mathcal{V})$.

Based on Lemma 3, it can be deduced that $\varphi\colon [0, \infty) \to [0, \infty)$ is exists with $\varphi(0) = 0$. Then

$$\begin{aligned}\|k_{n+1} - q\|^2 &= \|\,(1 - \alpha_n)\, k_n + \alpha_n T^2 k_n - (1 - \alpha_n)\, q + \alpha_n T^2 q\|^2 \\ &= \|\,(1 - \alpha_n)\left(k_n - q\right) + \alpha_n\left(T^2k_n - T^2q\right)\|^2 \\ &\leq (1 - \alpha_n)\,\|k_n - q\|^2 + \alpha_n\|T^2k_n - T^2q\|^2 - \alpha_n\,(1 - \alpha_n)\,\varphi\left(\|k_n - T^2k_n\|\right) \\ &\leq \|k_n - q\|^2 - \alpha_n\,(1 - \alpha_n)\,\varphi\left(\|k_n - T^2k_n\|\right)\end{aligned}$$

Therefore

$$\begin{aligned}\varepsilon^2\varphi\left(\|k_n - T^2k_n\|\right) &< \alpha_n\,(1 - \alpha_n)\,\varphi\left(\|k_n - T^2k_n\|\right) \\ &\leq \|k_{n+1} - q\|^2 - \|k_n - q\|^2.\end{aligned}$$

If $n \to \infty$, deduce that:

$\lim_{n\to\infty}\varphi(\|k_n - T^2k_n\|) = 0$. we conclude that $\|k_n - T^2k_n\| \to 0$. Moreover, it is worth noting that each UCB-space is reflexive, which means that A is weakly compact. As a result, $\{k_n\}_{n\geq1}$ has a weak convergent subsequence $\{k_{n_k}\}_{k\geq1}$, converging to $\rho^* \in \mathcal{U}$.

$$\begin{aligned}\|k_n - Tk_n\| &= \|k_n - T^2k_n\| + \|T^2k_n - Tk_n\| \\ &\leq \|k_n - T^2k_n\| + \phi\left(\mathrm{d}\left(k_n, Tk_n\right)\right)\|k_n - Tk_n\| + \left[1 - \phi\left(\mathrm{d}\left(k_n, Tk_n\right)\right)\right] dist\,(\mathcal{U}, \mathcal{V})\end{aligned}$$

$$\begin{aligned}&\|k_n - Tk_n\| - \phi\left(\mathrm{d}\left(k_n, Tk_n\right)\right)\ \|k_n - Tk_n\| \\ &\leq \|k_n - T^2k_n\| + \left[1 - \phi\left(\mathrm{d}\left(k_n, Tk_n\right)\right)\right] dist\,(\mathcal{U}, \mathcal{V})\end{aligned}$$

$$\left[1-\phi\left(\|k_n-Tk_n\|\right)\right]\|k_n-Tk_n\|$$
$$\leq \|k_n-T^2k_n\|+\left[1-\phi\left(\|k_n-Tk_n\|\right)\right] dist\left(\mathcal{U},\mathcal{V}\right)$$
$$\|k_n-Tk_n\| \leq \frac{\|k_n-T^2k_n\|}{\left[1-\phi\left(\|k_n-Tk_n\|\right)\right]}+dist\left(\mathcal{U},\mathcal{V}\right),$$

and so $\|k_n-Tk_n\| \to dist(\mathcal{U},\mathcal{V})$. Since $\{k_{n_k}\}_{k\geq 1}$, converging to $\rho^* \in A$, then

$$\|\rho^*-T\rho^*\| \leq \lim_{k\to\infty} inf\|k_{n_k}-Tk_{n_k}\|$$
$$= \lim_{k\to\infty}\|k_{n_k}-Tk_{n_k}\|$$
$$= \lim_{n\to\infty}\|k_n-Tk_n\| = dist\left(\mathcal{U},\mathcal{V}\right).$$

Again since $(\mathcal{U},\mathcal{V})$ possesses P-p and uniqueness of BPP, it follows that $Tq=\rho^*$ and as a result $k_n\to\rho^*$.

The Ishikawa iterative method for determining the BPP is discussed in the following theorem.

Theorem 4: *Suppose that* W *is a UCB-space and* $(\mathcal{U},\mathcal{V})$ *is a closed, disjoint, bounded, and convex pair in W and assume* $T:\mathcal{U}\cup\mathcal{V}\to\mathcal{U}\cup\mathcal{V}$ *be MTCC-map. Consider* $k_\circ\in\mathcal{U}$ *and*

$$\kappa_{n+1}=(1-\alpha_n)\kappa_n+\alpha_nT^2\zeta_n, \tag{3}$$
$$\zeta_n=(1-\beta_n)\kappa_n+\beta_nT^2\kappa_n,\quad \forall n\in N\cup\{0\},$$

where $0<\varepsilon\leq\alpha_n\leq 1$ *and* $0<\varepsilon\leq\beta_n(1-\beta_n)$. *Then* $\|k_n-T^2k_n\|\to 0$ *and* $\{k_n\}$ *strongly converge to the BPP of* T.

Proof: Using comparable sense to the evidence of Theorem 3, Assume $q\in Fix(T^2|\mathcal{V}_\circ)$.

Now

$$\|k_{n+1}-q\| = \|(1-\alpha_n)\zeta_n+\alpha_nT^2\zeta_n$$
$$-(1-\alpha_n)q+\alpha_nT^2q\|$$
$$\leq (1-\alpha_n)\|\zeta_n-q\|+\alpha_n\|T^2\zeta_n-T^2q\|$$
$$\leq (1-\alpha_n)\|\zeta_n-q\|+\alpha_n\|\zeta_n-q\| = \|\zeta_n-q\|$$

also,

$$\|\zeta_n-q\| = \|(1-\beta_n)k_n+\beta_nT^2k_n$$
$$-(1-\beta_n)q+\beta_nT^2q\|$$
$$\leq (1-\beta_n)\|k_n-q\|+\beta_n\|T^2k_n-T^2q\|$$
$$=\|k_n-q\|.$$

Therefore, $\|k_{n+1}-q\|\leq\|k_n-q\|$ and this demonstrates that $\{\|k_n-q\|\}$ is decreasing.

Thus $\lim_{n\to\infty}\|k_n-q\|$ exists for any $q\in Fix(T^2|\mathcal{V}_\circ)$. Because W is UCB-space, there are functions $\varphi,\Upsilon:[0,\infty)\to[0,\infty)$ where $\varphi(0)=0$, $\Upsilon(0)=0$ such that

$$\|k_{n+1}-q\|^2 = \|(1-\alpha_n)\zeta_n+\alpha_nT^2\zeta_n$$
$$-(1-\alpha_n)q+\alpha_nT^2q\|^2$$
$$= \|(1-\alpha_n)(\zeta_n-q)+\alpha_n\left(T^2\zeta_n-T^2q\right)\|^2$$
$$\leq (1-\alpha_n)\|\zeta_n-q\|^2+\alpha_n\|T^2\zeta_n-T^2q\|^2$$
$$-\alpha_n(1-\alpha_n)\varphi\left(\|\zeta_n-T^2\zeta_n\|\right)$$
$$\leq (1-\alpha_n)\|\zeta_n-q\|^2+\alpha_n\|(1-\beta_n)k_n$$
$$+\beta_nT^2k_n-q\|^2-\alpha_n(1-\alpha_n)\varphi\left(\|\zeta_n-T^2\zeta_n\|\right)$$
$$\leq (1-\alpha_n)\|\zeta_n-q\|^2+\alpha_n(1-\beta_n)\|k_n-q\|^2$$
$$+\alpha_n\beta_n\|T^2k_n-T^2q\|^2$$
$$-\alpha_n\beta_n(1-\beta_n)\Upsilon\left(\|k_n-T^2k_n\|\right)$$
$$-\alpha_n(1-\alpha_n)\varphi\left(\|\zeta_n-T^2\zeta_n\|\right)$$
$$= \|k_n-q\|^2-\alpha_n\beta_n(1-\beta_n)\Upsilon\left(\|k_n-T^2k_n\|\right)$$
$$-\alpha_n(1-\alpha_n)\varphi\left(\|\zeta_n-T^2\zeta_n\|\right).$$

Therefore,

$$\alpha_n\beta_n(1-\beta_n)\Upsilon\left(\|k_n-T^2k_n\|\right)$$
$$\leq \alpha_n\beta_n(1-\beta_n)\Upsilon\left(\|k_n-T^2k_n\|\right)$$
$$+\alpha_n(1-\alpha_n)\varphi\left(\|\zeta_n-T^2\zeta_n\|\right)$$
$$\leq (1-\alpha_n)\|k_n-q\|^2+\alpha_n(1-\beta_n)\|k_n-q\|^2$$
$$\leq \|k_n-q\|^2-\|k_{n+1}-q\|^2.$$

Hence

$$\varepsilon^2\Upsilon\left(\|k_n-T^2k_n\|\right)\leq\|k_n-q\|^2-\|k_{n+1}-q\|^2$$

and this guarantees that $\lim_{n\to\infty}\|T^2k_n-k_n\|=0$. By a similar proof of Theorem 3. $\{k_n\}$ converges to the BPP of T in $\mathcal{U}$.

Results and discussion

Example 1: *Consider* $W=\mathbb{R}$ *be a Banach space and let* $\mathcal{U}=[0,\frac{1}{3}]$, $\mathcal{V}=[\frac{2}{3},1]$.

Suppose $T:\mathcal{U}\cup\mathcal{V}\to\mathcal{U}\cup\mathcal{V}$ *where:*

$$T(\rho)=\begin{cases}\frac{2}{3}, & if\ \rho\in\left[0,\frac{1}{3}\right],\\ \frac{1}{3}, & if\ \rho\in\left[\frac{2}{3},1\right].\end{cases}$$

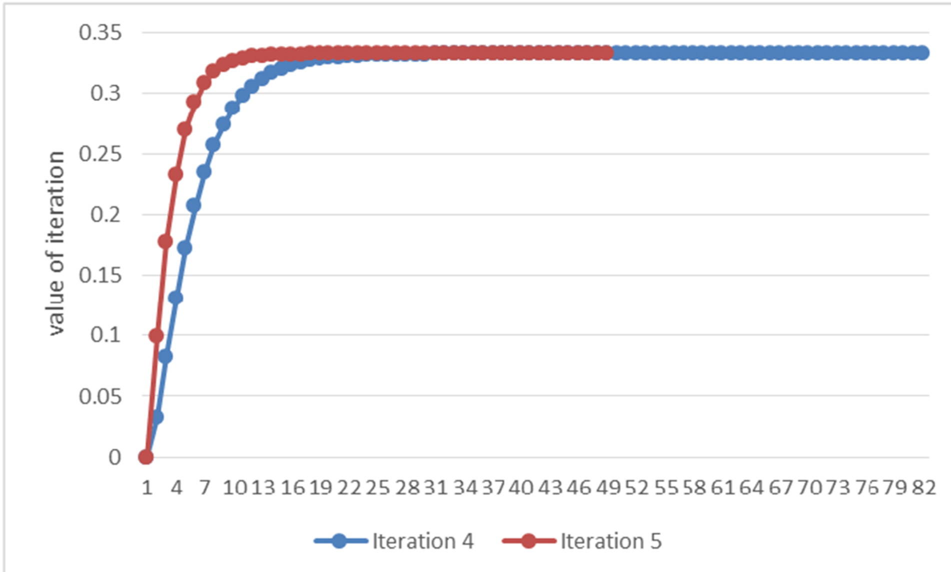

Fig. 1. The comparison between Iteration 4 and Iteration 5.

Let $\phi : [0, \infty) \to [0, 1)$ *be specified as:*

$\phi(t) = \frac{1}{t+1}$ for all $t \in [0, \infty)$.

T is MTCC-map since $d(T\rho, Tq) = |T(\rho) - T(q)| = |\frac{2}{3} - \frac{1}{3}| = \frac{1}{3} = dist(\mathcal{U}, \mathcal{V})$ *and,* $\phi(d(\rho, q))\, d(\rho, q) + (1 - \phi(d(\rho, q)))dist(\mathcal{U}, \mathcal{V}) = \frac{1}{d(\rho,q)+1}.d(\rho, q) + (1 - (\frac{1}{d(\rho,q)+1})).\frac{1}{3} >= d(T\rho, Tq)$

In this example, $\rho^* = 0.3333333333$ *is the BPP of* T. *For* $(k_\circ, \zeta_\circ) \in \mathcal{U} \times \mathcal{V}$ *and each* $n \in N \cup \{0\}$ *our iterative sequences are:*

$$k_{n+1} = (1 - \alpha_n) k_n + \alpha_n T^2 k_n, \quad (4)$$

$\forall n \in N \cup \{0\}$, *where* $\alpha_n \in (\varepsilon, 1 - \varepsilon)$ *and* $\varepsilon \in (0, 1/2]$

$$\kappa_{n+1} = (1 - \alpha_n) \kappa_n + \alpha_n T \zeta_n, \quad (5)$$

$$\zeta_{n+1} = (1 - \beta_n) \kappa_n + \beta_n T \kappa_n,\ \forall n \in N \cup \{0\}$$

where $\alpha_n \in (\varepsilon,\ 1 - \varepsilon)$ *and* $\varepsilon \in (0, 1/2]$.

Consider $\kappa_0 = 0$ *the initial point in our example and consider*

$$\alpha_n = \frac{2n+1}{8n+10}, \quad \beta_n = \frac{5n+1}{8n+10}, \quad \varepsilon = \frac{2}{100}.$$

It is evident from Table 1 that iteration 5 approaches convergence to the BPP at a more rapidly rate than iteration 4. Furthermore, the findings presented in Table 1 are depicted in Fig. 1.

Example 2: *Consider* $W = \mathbb{R}$ *and let* $\mathcal{U} = [\frac{1}{6}, \frac{1}{4}]$, $\mathcal{V} = [\frac{1}{2}, 1]$.

Table 1. The rate of convergence for new proposed iterations.

n	Iteration 4	n	Iteration 5
0	0	0	0
1	0.0333333333	1	0.0994152046
2	0.0833333333	2	0.1781376518
3	0.1314102564	3	0.2334060662
4	0.1729826546	4	0.2701805418
5	0.2073435143	5	0.2939259914
6	0.2350612745	6	0.3089691497
7	0.2570877704	7	0.3183749989
8	0.2744163074	8	0.3242000721
9	0.2879512998	9	0.3277814071
10	0.2984666491	10	0.3299707167
⋮	⋮	⋮	⋮
76	0.3333333330	43	0.3333333332
77	0.3333333331	44	0.3333333332
78	0.3333333331	45	0.3333333332
79	0.3333333332	46	0.3333333333
80	0.3333333333	47	0.3333333333

Suppose $T : \mathcal{U} \cup \mathcal{V} \to \mathcal{U} \cup \mathcal{V}$,

$$T(\rho) = \begin{cases} \frac{1}{2}, & if\ \rho \in \left[\frac{1}{6}, \frac{1}{4}\right], \\ \frac{1}{4}, & if\ \rho \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Consider $\phi : [0, \infty) \to [0, 1)$ *be specified as* $\phi(t) = \frac{1}{t+1}$ *for all* $t \in [0, \infty)$. *Consider* $(\rho, q) \in \mathcal{U} \times \mathcal{V}$, *then*

$$|\rho - q| = \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4},$$

$$d(T\rho,\ Tq) = |T(\rho) - T(q)| = \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4}$$

$$= dist(\mathcal{U}, \mathcal{V}).$$

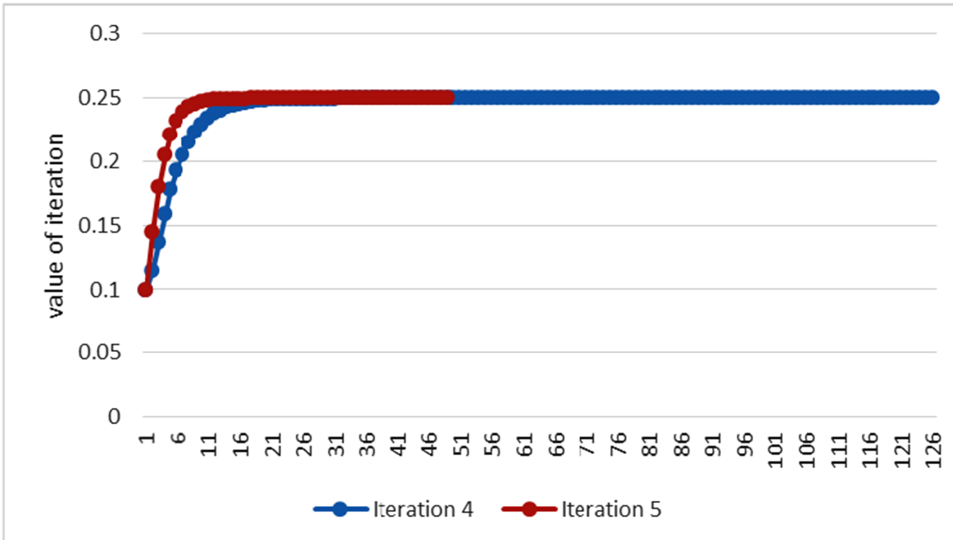

Fig. 2. The comparison between Iteration 4 and Iteration 5.

Table 2. The rate of convergence for new proposed iterations.

n	Iteration 4	n	Iteration 5
0	0.1	0	0.1
1	0.1150000000	1	0.1447368421
2	0.1375000000	2	0.1801619433
3	0.1591346153	3	0.2053272982
4	0.1778421945	4	0.2215812438
5	0.1933045814	5	0.2322666961
6	0.2057775735	6	0.2390361173
7	0.2156894967	7	0.2432687495
8	0.2234873383	8	0.2458900324
9	0.2295780849	9	0.2475016332
10	0.2343099920	10	0.2484868225
⋮	⋮	⋮	⋮
118	0.2499999999	62	0.2499999999
119	0.2499999999	63	0.2499999999
120	0.2499999999	64	0.2499999999
121	0.2500000000	65	0.2500000000
122	0.2500000000	66	0.2500000000

T is MTCC-map since

$$\phi\left(\mathrm{d}\left(\rho, \mathrm{q}\right)\right) \mathrm{d}\left(\rho, \mathrm{q}\right) + \left(1 - \phi\left(\mathrm{d}\left(\rho, \mathrm{q}\right)\right)\right) \mathrm{dist}\left(\mathcal{U}, \mathcal{V}\right)$$
$$= \frac{1}{\mathrm{d}\left(\rho, \mathrm{q}\right) + 1} . \mathrm{d}\left(\rho, \mathrm{q}\right) + \left(1 - \left(\frac{1}{\mathrm{d}\left(\rho, \mathrm{q}\right) + 1}\right)\right) . \frac{1}{4}$$
$$\geq \mathrm{d}\left(T\rho, T\mathrm{q}\right)$$

In this example $\rho^* = 0.25$ is the BPP of the mapping T.

Consider $\kappa_0 = 0.1$ the initial point in this example and consider

$$\alpha_n = \frac{2n+1}{8n+10}, \quad \beta_n = \frac{5n+1}{8n+10}, \quad \varepsilon = \frac{2}{100}.$$

It is evident from Table 2 that iteration 5 approaches convergence to the BPP at a more rapidly rate than iteration 4. Furthermore, the findings presented in Table 2 are depicted in Fig. 2.

Conclusion

This paper's main goal is to introduce novel iteration approaches for estimating the best proximity point (BPP). This work examines the Mann iteration scheme and Ishikawa iterative strategy to develop novel iteration techniques for approximating the BPP for MTCC-mapping. In addition illustrative numerical examples are provided to substantiate the primary outcome. The findings indicate that the second iteration is more efficient than the newly suggested first iteration.

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Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.

- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of University of Technology.

Authors' contribution statement

R. I. S. significantly advanced the study results' analysis, application, and design. The results were revised and checked in part by B. A. A. By use of a joint examination of the results, the two authors improved the finished work.

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التقريب التكراري لأفضل نقاط القرب لدوال الانكماش الدوري M-T

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المستخلص

في العديد من الحالات، ليس من الممكن الحصول على حل لمعادلة مشغل ما باستخدام التقنيات التحليلية التقليدية، حتى بعد إثبات وجود مثل هذا الحل. على المرء أن يعرف القيمة التقريبية لهذا الحل من أجل حل مثل هذه المواقف. وللقيام بذلك، يجب أولاً إعادة هيكلة معادلة المشغل بحيث تأخذ شكل معادلة النقطة الصامدة (FP). في معادلة النقطة الصامدة، يتم استخدام الطريقة التكرارية الأكثر ملائمة، وحدود التسلسل الذي تم إنشاؤه بواسطة هذه الخوارزمية هي في الواقع قيمة النقطة الصامدة المطلوبة لمعادلة النقطة الصامدة، وكذلك حلول لمعادلة المشغل. أصبح الحساب العددي لنقاط النقطة الصامدة للمشغلين غير الخطيين الآن موضوعًا بحثيًا مثيرًا للاهتمام في التحليل غير الخطي نظرًا لإمكانية تطبيقه في العديد من المجالات. لقد طور العديد من الباحثين مجموعة واسعة من التقنيات لتقدير النقطة الصامدة لأنواع مختلفة من التطبيقات. الهدف الأساسي من هذه الورقة هو تقديم مخططات تكرارية جديدة لتقريب أفضل نقطة قرب. لقد تم البحث في تقارب أفضل نقاط القرب (BPP) لتعيينات الانكماش الدوري M-T في سياق فضاءات بناخ المحدبة بشكل منتظم . لقد تم اخذ نهج التكرارات الذي اقترحه مان وإيشيكاوا في نظر الاعتبار ونتيجة لذلك، تم الحصول على بعض نتائج التقارب القوية لقيمة افضل نقطة قرب لدوال الانكماش الدوري M-T. علاوة على ذلك، تم تقديم أمثلة عددية تدعم الاستنتاج الاساسي، وتمت مقارنة سلوك التقارب للتكرارات.

الكلمات المفتاحية: أفضل نقطة تقارب، المتتابعات التكرارية، دالة الانكماش الدوري M-T، التقارب القوي، فضاءات باناخ المحدبة بشكل منتظم.