Baghdad Science Journal

Volume 22 | Issue 9 Article 24

9-23-2025

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How to Cite this Article
Raja, Sarathy and Jeyaraj, Ravi Sankar (2025) "Laplacian Energy and Eccentricity Based Energy for Cross Monic Zero Divisor Graph Associated With Commutative Rings," *Baghdad Science Journal*: Vol. 22: Iss. 9, Article 24.
DOI: https://doi.org/10.21123/2411-7986.5065

BAGHDAD SCIENCE JOURNAL 2025;22(9):3080–3093

RESEARCH ARTICLE

Laplacian Energy and Eccentricity Based Energy for Cross Monic Zero Divisor Graph Associated With Commutative Rings

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ABSTRACT

For a commutative ring, a cross monic zero divisor graph is discussed, whose vertices are nonzero zero divisors of the commutative ring, then the two vertices x and y are adjacent if and only if xy = 0. Sum of absolute auxiliary eigenvalues of Laplacian matrix of a simple graph is known as Laplacian energy of a simple graph. Eccentricity sum matrix of a graph is a positive matrix, whose entries are sum of the eccentricities of two vertices and 0 for diagonals of the eccentricity sum matrix. Then the sum of absolute eigenvalues of eccentricity sum matrix is known as eccentricity sum energy. Laplacian sum eccentricity energy is the absolute difference between the eigenvalues of Laplacian sum eccentricity matrix and twice the edge cardinality by vertex cardinality. Maximum eccentricity energy is the sum of absolute eigenvalues of maximum eccentricity matrix. For a simple graph, the entries of average degree eccentricity matrix, if the two vertices are adjacent, are average sum of degree and eccentricity of their two vertices, otherwise 0. Then the sum of absolute eigenvalues of average degree eccentricity matrix is known as average degree eccentricity energy. This paper discusses the degree squence, vertex cardinality, and edge cardinality of the cross monic zero divisor graph for commutative rings. For a certain family of cross monic zero divisor graphs for commutative ring, moveover, Laplacian energy, eccentricity sum energy, Laplacian sum eccentricity energy, maximum eccentricity energy, and average degree-eccentricity energy are also briefly covered.

Keywords: Average degree-eccentricity energy, Eccentricity sum energy, Laplacian energy, Laplacian sum eccentricity energy, Maximum eccentricity energy, Zero divisor graphs

Introduction

Let $\mathcal{R}$ be a commutative ring with multiplicative identity $1 \neq 0$. If there exists $x_2 \in \mathcal{R}(x_2 \neq 0)$ such that $x_1x_2 = 0$, then $x_1 \in \mathcal{R}(x_1 \in 0)$ is referred to as a zero divisor of $\mathcal{R}$. The collection of zero divisors is symbolized by $\mathbb{Z}(\mathscr{R})$, while $\mathbb{Z}(\mathscr{R})/\{0\} = \mathbb{Z}(\mathscr{R})^*$ is the collection of nonzero zero divisors of $\mathcal{R}$. The zero divisor graph $\Gamma(\mathscr{R})$ of $\mathcal{R}$ is a graph, where $\mathbb{Z}(\mathscr{R})$ is its node set and two different nodes $y, z \in \mathbb{Z}(\mathscr{R})$ are connected if $yz = zy = 0$. Beck[1] established such graphs over commutative rings; in his concept, he incorporated the identity and was primarily concerned with the coloring of commutative rings. The order of $\Gamma(\mathbb{Z}_n)$ is $n - 1 - \phi(n)$, whereas ϕ is Euler's phi function. Quotient energy,[2] Laplacian energy,[3–5] eccentricity energy,[6–8] dominant metric,[9]

Received 13 November 2023; revised 17 December 2024; accepted 19 December 2024.
Available online 23 September 2025

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https://doi.org/10.21123/2411-7986.5065

eccentricity sum energy,[10] Laplacian sum eccentricity energy,[11] maximum eccentricity energy,[12] and average degree-eccentricity energy[13] help us to study some more properties of graphs associated to commutative ring.

Definition 1: Cross monic zero divisor graph of a commutative ring, denoted $\mathcal{CMZG}(\mathbb{Z}_n \times \mathbb{Z}_m[x]/\langle f(x)\rangle)$, whose vertices are the nonzero zero divisors of the commutative ring, and whose two vertices $\mathcal{X}_a, \mathcal{Y}_b$ are connected by an edge if and only if $\mathcal{X}_a\mathcal{Y}_b = 0$. For example, cross monic zero divisor graphs of $\mathbb{Z}_4 \times \mathbb{Z}_2[x]/\langle x^2 + 2x\rangle$ is shown in Fig. 1.

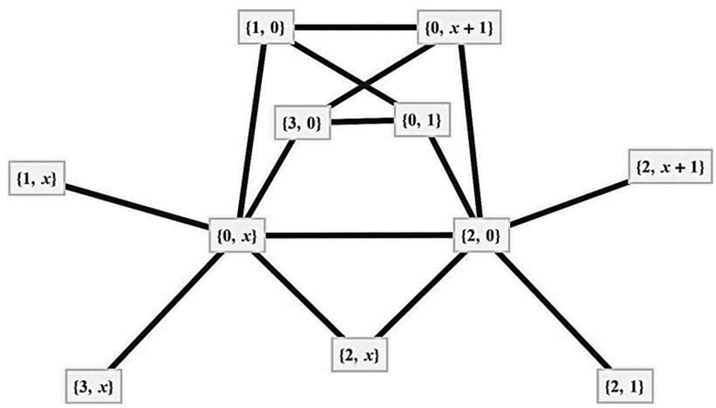

Fig. 1. $\mathcal{CMZG}(\mathbf{Z}_4 \times \mathbf{Z}_2[\mathbf{x}]/\langle \mathbf{x}^2 + 2\mathbf{x}\rangle)$.

The vertex and edge counts for the cross monic zero divisor graph $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 + px\rangle$, $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 + (p-1)x\rangle$, $\mathbb{Z}_4 \times \mathbb{Z}_p[x]/\langle x^2 + px\rangle$ and $\mathbb{Z}_4 \times \mathbb{Z}_p[x]/\langle x^2 + (p-1)x\rangle$ are presented in Table 1, while the degree sequence is provided in Table 2.

Table 1. Vertex and Edge cardinality of cross monic zero divisor graph.

$\mathcal{CMZD}$	Vertex Cardinality	Edge Cardinality
$\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 + px\rangle$	$p^2 + p - 1$	$\frac{1}{2}(5p^2 - 7p + 2)$
$\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 + (p-1)x\rangle$	$p^2 + 2p - 2$	$4p^2 - 6p + 2$
$\mathbb{Z}_4 \times \mathbb{Z}_p[x]/\langle x^2 + px\rangle$	$2p^2 + 2p - 1$	$8p^2 - 9p + 1$
$\mathbb{Z}_4 \times \mathbb{Z}_p[x]/\langle x^2 + (p-1)x\rangle$	$2p^2 + 4p - 3$	$4(3p^2 - 4p + 1)$

Table 2. Degree Squence of cross monic zero divisor graph.

$\mathcal{CMZD}$	Degree Squence
$\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 + px\rangle$	$\underbrace{p^2-1}_{1}, \underbrace{2(p-1)}_{p-1}, \underbrace{p-1}_{p-1}, \underbrace{1}_{p(p-1)}$
$\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 + (p-1)x\rangle$	$\underbrace{p^2-1}_{1}, \underbrace{2p-1}_{2(p-1)}, \underbrace{p-1}_{2(p-1)}, \underbrace{1}_{(p-1)^2}$
$\mathbb{Z}_4 \times \mathbb{Z}_p[x]/\langle x^2 + px\rangle$	$\underbrace{2(p^2-1)}_{1}, \underbrace{p^2-1}_{2}, \underbrace{4p-2}_{p-1}, \underbrace{2(p-1)}_{p-1}, \underbrace{p-1}_{2(p-1)}, \underbrace{3}_{p(p-1)}, \underbrace{1}_{p(p-1)}$
$\mathbb{Z}_4 \times \mathbb{Z}_p[x]/\langle x^2 + (p-1)x\rangle$	$\underbrace{2(p^2-1)}_{1}, \underbrace{p^2-1}_{2}, \underbrace{2(4p-1)}_{2p-2}, \underbrace{2(2p-1)}_{2(p-1)}, \underbrace{2(p-1)}_{4(p-1)}, \underbrace{3}_{(p-1)^2}, \underbrace{1}_{(p-1)^2}$

Definition 2: For a simple graph G, μ_α be the eigenvalues of the Laplacian matrix, then Laplacian energy is

$$\mathcal{LE}(\mathcal{G}) = \sum_{\alpha} \left| \mu_\alpha - \frac{2|E|}{|V|} \right|.$$

Definition 3: Let $e_\alpha = ecc(v_\alpha)$ be the eccentricity of v_α. Then $\mathcal{ES}(\mathcal{G})$ is called the eccentricity sum matrix of a graph,

$$a_{\alpha\beta} = \begin{cases} e_\alpha + e_\beta, & if \quad \alpha \neq \beta \\ 0 & otherwise \end{cases}$$

Eccentricity sum energy of a graph is defined as

$$\mathcal{E}_{\mathcal{ES}}(\mathcal{G}) = \sum_{\alpha=1}^{|V|} |\zeta_\alpha|$$

Definition 4: Laplacian sum-eccentricity spectrum of graph, consisting of $\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_{|V|}$. This leads us to define the Laplacian sum eccentricity energy of a graph as

$$\mathcal{LE}(G) = \sum_{\alpha=1}^{|V|} \left| \zeta_\alpha - \frac{2|E|}{|V|} \right|$$

Definition 5: Maximum eccentricity matrix of graph defining as,

$$e_{\alpha\beta} = \begin{cases} Max\left\{e(v_\alpha), e\left(v_\beta\right)\right\}, & if \quad v_\alpha v_\beta \in E(G) \\ 0 & otherwise \end{cases}$$

The maximum eccentricity energy of graph is defined as

$$\mathcal{EM}_e(\mathcal{G}) = \sum_{\alpha=1}^{|V|} |\lambda_\alpha|.$$

Definition 6: Average degree-eccentricity matrix of graph is $|V| \times |V|$ matrix whose elements are given by

$$\mathcal{M}_{\alpha\beta} = \begin{cases} \frac{1}{2}\left[d(v_\alpha) + d\left(v_\beta\right) + e(v_\alpha) + e\left(v_\beta\right)\right], & if \quad v_\alpha v_\beta \in E(G) \\ 0 & otherwise \end{cases}$$

Then average degree-eccentricity energy $\mathcal{E}_{ade}(G)$ of graph is

$$\mathcal{E}_{ade}(G) = \sum_{\alpha=1}^{|V|} |\lambda_\alpha|.$$

Laplacian energy for cross monic zero divisor graphs

Theorem 1: *Let $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2 + px\rangle$ be a simple graph, then*

$$\mathcal{LE} \leq \sqrt{\frac{(-5p+2)^2\left(p^7 + 2p^6 - 2p^5 - 4p^4 - 2p^3 + 2p^2 + 11p - 8\right)}{4\left(p^2 + p - 1\right)(p+2)}}$$

Proof. It is known that,

$$\sum_{k=1}^{p^2+p-1} (\lambda_k)^2 = 2M$$

Here

$$M = \frac{1}{2}\sum_{k=1}^{p^2+p-1}\left(\delta_k - \frac{5p^2-7p+2}{p^2+p-1}\right)^2 + \frac{1}{2}\left(5p^2-7p+2\right)$$

$$= \frac{1}{2}\left(\sum_{k=1}^{p^2+p-1}\delta_k^2 - \frac{2\left(5p^2-7p+2\right)}{p^2+p-1}\sum_{k=1}^{p^2+p-1}\delta_k + \left(\frac{5p^2-7p+2}{p^2+p-1}\right)^2\right) + \frac{1}{2}\left(5p^2-7p+2\right)$$

$$= \frac{1}{2}\sum_{k=1}^{p^2+p-1}\delta_k^2 - \frac{\left(4p^2-8p+3\right)\left(5p^2-7p+2\right)}{2\left(p^2+p-1\right)}$$

Using Cauchy-Schwartz inequality,

$$\mathcal{LE} = \sum_{k=1}^{p^2+p-1} |\gamma_k|$$

$$\leq \sqrt{\left(p^2+p-1\right)\sum_{k=1}^{p^2+p-1}\gamma_k^2}$$

$$= M\left(2p^2+2p-2\right)$$

Hence

$$\mathcal{LE} \leq \sqrt{\frac{(-5p+2)^2\left(p^7+2p^6-2p^5-4p^4-2p^3+2p^2+11p-8\right)}{4\left(p^2+p-1\right)(p+2)}}$$

Theorem 2: *Laplacian energy of cross monic zero divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2+(p-1)x\rangle$ is*

$$\mathcal{LE} = \frac{p^{\frac{5}{2}}\sqrt{\mathcal{M}} + 2p^{\frac{3}{2}}\sqrt{\mathcal{M}} - 2\sqrt{p}\sqrt{\mathcal{M}} + 10p^{\frac{9}{2}} - 24p^{\frac{7}{2}} + 22p^{\frac{5}{2}} - p^2 - 12p^{\frac{3}{2}} - 2p + 4\sqrt{p} + 2}{\sqrt{p}\left(p^2+2p-2\right)}$$

where $\mathcal{M} = 8p^2 - 12p + 5$

Proof. Cross monic zero divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2+(p-1)x\rangle$ be a simple graph, then Laplacian matrix is

$$\begin{bmatrix} \mathcal{I} & -\mathcal{J} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ -\mathcal{J} & p^2-1 & \mathcal{O} & -\mathcal{J} & -\mathcal{J} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & (p-1)\mathcal{I} & -\mathcal{J} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & -\mathcal{J} & -\mathcal{J} & (2p-1)\mathcal{I} & -\mathcal{J} & \mathcal{O} \\ \mathcal{O} & -\mathcal{J} & \mathcal{O} & -\mathcal{J} & (2p-1)\mathcal{I} & -\mathcal{J} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & -\mathcal{J} & (p-1)\mathcal{I} \end{bmatrix}$$

then spectrum of graph is shown in Table 3,

Table 3. Spectrum of Graph.

Spectrum	
0	1
$\frac{1}{\sqrt{p}}$	1
1	$(p-1)^2-1$
$\frac{1}{2}(4(p-1)-\sqrt{8p^2-12p+5})$	1
$p-1$	$2(p-2)$
$2(p-1)$	1
$2p-1$	$2(p-2)$
$\frac{1}{2}(4(p-1)+\sqrt{8p^2-12p+5})$	1
p^2	1

Laplacia energy of graph is

$$\mathcal{LE} = \sum_{i=1}^{p^2+2p-2} \left|\lambda_i - \frac{4\left(2p^2-3p+1\right)}{p^2+2p-2}\right|$$

$$\mathcal{LE} = \left|0 - \frac{4\left(2p^2-3p+1\right)}{p^2+2p-2}\right| + \left|\frac{1}{\sqrt{p}} - \frac{4\left(2p^2-3p+1\right)}{p^2+2p-2}\right| + \left((p-1)^2-1\right)\left|1 - \frac{4\left(2p^2-3p+1\right)}{p^2+2p-2}\right|$$

$$+\left|\frac{1}{2}\left(4\left(p-1\right)-\sqrt{8p^2-12p+5}\right)\right| + \left(2\left(p-2\right)\right)\left|\left(p-1\right) - \frac{4\left(2p^2-3p+1\right)}{p^2+2p-2}\right|$$

$$+\left|\left(2\left(p-1\right)\right) - \frac{4\left(2p^2-3p+1\right)}{p^2+2p-2}\right| + \left(2\left(p-2\right)\right)\left|\left(2p-1\right) - \frac{4\left(2p^2-3p+1\right)}{p^2+2p-2}\right|$$

$$+\left|\frac{1}{2}\left(4\left(p-1\right)+\sqrt{8p^2-12p+5}\right)\right| + \left|p^2 - \frac{4\left(2p^2-3p+1\right)}{p^2+2p-2}\right|$$

$$= \frac{4\left(-1+p\right)\left(-1+2p\right)}{-2+2p+p^2} + (-1)\left(-\frac{2+4\sqrt{p}-2p-12p^{\left(\frac{3}{2}\right)}-p^2+8p^{\left(\frac{5}{2}\right)}}{\sqrt{p}\left(-2+2p+p^2\right)}\right)$$

$$+\left((-1)\left((p-1)^2-1\right)\left(\frac{-7p^2+14p-6}{-2+2p+p^2}\right)\right) + \frac{2\left(-2+p\right)\left(-1+p\right)p}{-2+2p+p^2}$$

$$+(-1)\left(\frac{1}{2}\left(4\left(-1+p\right) - \frac{8\left(-1+p\right)\left(-1+2p\right)}{-2+2p+p^2} - \sqrt{5-12p+8p^2}\right)\right) + \cdots + \frac{-4+12p-10p^2+2p^3+p^4}{-2+2p+p^2}$$

$$= \frac{p^{\frac{5}{2}}\sqrt{\mathcal{M}} + 2p^{\frac{3}{2}}\sqrt{\mathcal{M}} - 2\sqrt{p}\sqrt{\mathcal{M}} + 10p^{\frac{9}{2}} - 24p^{\frac{7}{2}} + 22p^{\frac{5}{2}} - p^2 - 12p^{\frac{3}{2}} - 2p + 4\sqrt{p} + 2}{\sqrt{p}\left(p^2+2p-2\right)}.$$

Eccentricity based energy for cross monic zero divisor graphs

Theorem 3: *Let $\mathcal{CMZG}$ be simple graph with order p^2+p-1 and $\eta_\triangle$ be a absolute value of the determinant of $\mathcal{ES}$(eccentricity sum matrix). Then*

$$\sqrt{2\mathscr{M}+\left(p^2+p-1\right)(p+2)(p-1)\,\eta_\triangle{}^{\frac{2}{p^2+p-1}}}\le \mathcal{E}_{\mathcal{ES}}$$

$$\le \sqrt{2\left(p^2+p-2\right)\mathscr{M}+\left(p^2+p-1\right)\eta_\triangle{}^{\frac{2}{p^2+p-1}}}$$

Proof. First, the lower bound is examined

$$[\mathcal{E}_{\mathcal{ES}}]^2=\left(\sum_{\alpha=1}^{p^2+p-1}(\zeta_\alpha)\right)^2$$

$$=\sum_{\alpha=1}^{p^2+p-1}(\zeta_\alpha)^2+2\sum_{\alpha<\beta}|\zeta_\alpha|\left|\zeta_\beta\right|$$

$$=2\mathscr{M}+\sum_{\alpha\neq\beta}|\zeta_\alpha|\left|\zeta_\beta\right|$$

Since the geometric mean is greater than the arithmetic mean for non-negative numbers,

$$\frac{1}{\left(p^2+p-1\right)(p+2)(p-1)}\sum_{\alpha\neq\beta}|\zeta_\alpha|\left|\zeta_\beta\right|\ge\left(\prod_{\alpha\neq\beta}|\zeta_\alpha|\left|\zeta_\beta\right|\right)^{\frac{1}{\left(p^2+p-1\right)(p+2)(p-1)}}$$

$$=\left(\prod_{\alpha=1}|\zeta_\alpha|^{2p^2+2p-4}\right)^{\frac{1}{\left(p^2+p-1\right)(p+2)(p-1)}}$$

$$=\prod_{\alpha=1}^{p^2+p-1}|\zeta_\alpha|^{\frac{2}{p^2+p-1}}$$

$$=\eta_\triangle{}^{\frac{2}{p^2+p-1}}$$

Therefore

$$\sum_{\alpha\neq\beta}|\zeta_\alpha|\left|\zeta_\beta\right|\ge\left(p^2+p-1\right)(p+2)(p-1)\,\eta_\triangle{}^{\frac{2}{p^2+p-1}}$$

Now obtained,

$$[\mathcal{E}_{\mathcal{ES}}]^2\ge 2\mathscr{M}+\left(p^2+p-1\right)(p+2)(p-1)\,\eta_\triangle{}^{\frac{2}{p^2+p-1}}$$

$$\mathcal{E}_{\mathcal{ES}}\ge\sqrt{2\mathscr{M}+\left(p^2+p-1\right)(p+2)(p-1)\,\eta_\triangle{}^{\frac{2}{p^2+p-1}}}$$

For upper bound, put $\sqrt{a'_\alpha} = |\zeta_\alpha|$, $\alpha = 1, 2, 3, \ldots, p^2 + p - 1$. Then

$$p^2 + p - 1 \left(\frac{1}{p^2 + p - 1} \sum_{\alpha=1}^{p^2+p-1} \zeta_\alpha^2 - \left(\prod_{\alpha=1}^{p^2+p-1} \zeta_\alpha^2 \right)^{\frac{1}{p^2+p-1}} \right)$$

$$\leq \left(p^2 + p - 1\right) \sum_{\alpha=1}^{p^2+p-1} \zeta_\alpha^2 - \left(\sum_{\alpha=1}^{p^2+p-1} |\zeta_\alpha| \right)^2$$

$$= \left(p^2 + p - 1\right)(p+2)(p-1) \left(\frac{1}{p^2 + p - 1} \sum_{\alpha=1}^{p^2+p-1} \zeta_\alpha^2 - \left(\prod_{\alpha=1}^{p^2+p-1} \zeta_\alpha^2 \right)^{\frac{1}{p^2+p-1}} \right)$$

$$2\mathscr{M} - \left(p^2 + p - 1\right)(p+2)(p-1)\,\eta_\triangle^{\frac{2}{p^2+p-1}} \leq 2\mathscr{M}\left(p^2 + p - 1\right)(p+2)(p-1) - [\mathcal{E}_{\mathcal{ES}}]^2$$

$$[\mathcal{E}_{\mathcal{ES}}]^2 \leq 2\left(p^2 + p - 2\right)\mathscr{M} + \left(p^2 + p - 1\right)(p+2)(p-1)\,\eta_\triangle^{\frac{2}{p^2+p-1}}$$

where $\mathscr{M} = \sum_{1 \leq \alpha < \beta \leq p^2+p-1} (e_\alpha + e_\beta)^2$.

Theorem 4: *Let $\mathcal{CMZG}$ be a $(p^2 + 2p - 2, 4p^2 - 6p + 2)$ graph, then*

$$\mathcal{LS}_e\mathcal{E} \leq \sqrt{p^2 + 2p - 2 \left(\sum_{\alpha=1}^{p^2+2p-2} \sum_{\beta=1}^{p^2+2p-2\beta} l_{\alpha\beta}^2 - \frac{4(4p^2 - 6p + 2)^2}{p^2 + 2p - 2} \right)}$$

Proof. It follows that

$$\sum_{\alpha=1}^{p^2+2p-2} \eta_\alpha^2 = \sum_{\alpha=1}^{p^2+2p-2} \left(\zeta_\alpha - \frac{2\left(4p^2 - 6p + 2\right)}{p^2 + 2p - 2} \right)^2$$

$$= \sum_{\alpha=1}^{p^2+2p-2} \left(\zeta_\alpha^2 - \frac{4\left(4p^2 - 6p + 2\right)}{p^2 + 2p - 2} \zeta_\alpha + \frac{4(4p^2 - 6p + 2)^2}{(p^2 + 2p - 2)^2} \right)$$

$$= \sum_{\alpha=1}^{p^2+2p-2} \zeta_\alpha^2 - \frac{4\left(4p^2 - 6p + 2\right)}{p^2 + 2p - 2} \sum_{\alpha=1}^{p^2+2p-2} \zeta_\alpha + \frac{4(4p^2 - 6p + 2)^2}{p^2 + 2p - 2}$$

$$= \sum_{\alpha=1}^{p^2+2p-2} \zeta_\alpha^2 - \frac{8(4p^2 - 6p + 2)^2}{p^2 + 2p - 2} + \frac{4(4p^2 - 6p + 2)^2}{p^2 + 2p - 2}$$

$$= \sum_{\alpha=1}^{p^2+2p-2} \zeta_\alpha^2 - \frac{4(4p^2 - 6p + 2)^2}{p^2 + 2p - 2}$$

Using the Cauchy-Schwarz inequality,

$$\sum_{\alpha=1}^{p^2+2p-2} |\eta_\alpha| \leq \sqrt{p^2+2p-2\left(\sum_{\alpha=1}^{p^2+2p-2} \eta_\alpha^2\right)}$$

$$\sum_{\alpha=1}^{p^2+2p-2} |\eta_\alpha| \leq \sqrt{p^2+2p-2\left(\sum_{\alpha=1}^{p^2+2p-2} \zeta_\alpha^2 - \frac{4(4p^2-6p+2)^2}{p^2+2p-2}\right)}.$$

Theorem 5: *Let $\mathcal{CMZG}$ be a $(2p^2+2p-1, 8p^2-9p+1)$ graph with maximal vertex degree $2(p^2-1)$. Then*

$$\mathcal{LS}_e\mathcal{E} \leq \mathcal{ES}_e + t\left(2\left(p^2-1\right)\right) + \frac{2\left(8p^2-9p+1\right)}{2p^2+2p-1}\left(2p+1\right)$$

where $t = |\{\zeta_\alpha : \zeta_\alpha \geq \frac{2(8p^2-9p+1)}{2p^2+2p-1}\}|$.
Proof. Let $\zeta_1 \geq \zeta_2 \cdots \geq \zeta_\alpha$ be the Laplacian sum eccentricity eigenvalues, $\mu_1 \geq \mu_2 \cdots \geq \mu_\alpha$ be the sum eccentricity eigenvalues and $\rho_1 \geq \rho_2 \cdots \geq \rho_\alpha$ be the eigenvalues of the degree matrix. It is necessary to distinguish between two cases.

Case (i): If $\zeta_\alpha - \frac{2(8p^2-9p+1)}{2p^2+2p-1} \geq 0$, then $|\zeta_\alpha - \frac{2(8p^2-9p+1)}{2p^2+2p-1}| \leq \mu_\alpha + \rho_1$. If $t\zeta_{\alpha'}s$, satisfy this condition, then

$$\sum_{\alpha=1}^{t}\left|\zeta_\alpha - \frac{2\left(8p^2-9p+1\right)}{2p^2+2p-1}\right| \leq \sum_{\alpha=1}^{t}\left(\mu_\alpha+\rho_1\right) \leq \sum_{\alpha=1}^{t}\left|\mu_\alpha\right| + t\rho_1.$$

Case (ii): If $\zeta_\alpha - \frac{2(8p^2-9p+1)}{2p^2+2p-1} \leq 0$, then $|\zeta_\alpha - \frac{2(8p^2-9p+1)}{2p^2+2p-1}| \leq |\mu_\alpha - -\frac{2(8p^2-9p+1)}{2p^2+2p-1}|$. If there is, $\mu_\alpha \leq 0$ for $\alpha = t+1, \ldots, t_r$ and $\mu_\alpha \geq 0$ for $\alpha = t_r+1, \ldots, 2p^2+2p-1$. Then

$$\sum_{\alpha=t+1}^{2p^2+2p-1}\left|\zeta_\alpha - \frac{2\left(8p^2-9p+1\right)}{2p^2+2p-1}\right| \leq \sum_{\alpha=t+1}^{t_r}\left|\mu_\alpha - \frac{2\left(8p^2-9p+1\right)}{2p^2+2p-1}\right|$$

$$+\sum_{\alpha=t_r+1}^{2p^2+2p-1}\left|\mu_\alpha - \frac{2\left(8p^2-9p+1\right)}{2p^2+2p-1}\right|$$

$$\sum_{\alpha=1}^{2p^2+2p-1}\left|\zeta_\alpha - \frac{2\left(8p^2-9p+1\right)}{2p^2+2p-1}\right| \leq \sum_{\alpha=1}^{t}\left|\mu_\alpha\right| + t\rho_1 + \sum_{\alpha=t+1}^{t_r}\left|\mu_\alpha\right|$$

$$-\sum_{\alpha=t_r+1}^{2p^2+2p-1}|\mu_\alpha| + \frac{2\left(8p^2-9p+1\right)}{2p^2+2p-1}\left(2p^2+2p-1-t\right)$$

$$\leq \mathcal{ES}_e + t\rho_1 + \frac{2\left(8p^2-9p+1\right)}{2p^2+2p-1}\left(2p^2+2p-1-t\right).$$

Theorem 6: *Let $\mathcal{CMZG}$ be a connected graph of order $2p^2+4p-3$ and size $4(3p^2-4p+1)$; and $av_1(av_{2p^2+4p-3})$ are maximum (minimum) absolute values of $v_{\alpha'}s$, then*

$$\mathcal{EL}_e \geq \sqrt{\left(2p^2+4p-3\right)\mathcal{E}_1-\zeta^2+4\left(2p^2+4p-3\right)(3p-1)(p-1)-\frac{\left(2p^2+4p-3\right)^2}{4}\left(av_1-av_{2p^2+4p-3}\right)^2}.$$

Proof. Let t_α and s_α, $1\leq\alpha\leq 2p^2+4p-3$ are non negative real numbers, then it follows that,

$$\sum_{\alpha=1}^{2p^2+4p-3} t_\alpha^2 \sum_{\alpha=1}^{2p^2+4p-3} s_\alpha^2-\left(\sum_{\alpha=1}^{2p^2+4p-3} t_\alpha s_\alpha\right)^2 \leq \frac{\left(2p^2+4p-3\right)^2}{4}\left(\tau_1\tau_2-\tau'_1\tau'_2\right)^2$$

where $\tau_1=max(t_\alpha)$, $\tau'_1=min(t_\alpha)$, $\tau_2=max(s_\alpha)$ and $\tau'_2=min(s_\alpha)$.

$$\sum_{\alpha=1}^{2p^2+4p-3} |v'_\alpha|^2 \sum_{\alpha=1}^{2p^2+4p-3} (1)^2-\left(\sum_{\alpha=1}^{2p^2+4p-3} \left|v'_\alpha\right|\right)^2$$

$$\leq \frac{\left(2p^2+4p-3\right)^2}{4}\left(v'_1-v'_2p^2+4p-3\right)^2.$$

Thus, it can be written

$$(\mathcal{EL}_e)^2 \geq \left(2p^2+4p-3\right)\sum_{\alpha=1}^{2p^2+4p-3} |v'_\alpha|^2-\frac{\left(2p^2+4p-3\right)^2}{4}\left(v'_1-v'_{2p^2+4p-3}\right)^2$$

$$=2p^2+4p-3\left(\mathcal{E}_1-\frac{\zeta^2}{2p^2+4p-3}+8\left(3p^2-4p+1\right)\right)-\frac{\left(2p^2+4p-3\right)^2}{4}\left(v'_1-v'_{2p^2+4p-3}\right)^2.$$

Theorem 7: *Let $\mathcal{CMZG}$ be a connected graph of order $2p^2+2p-1$ and size $8p^2-9p+1$. Let η_r and $\eta_{\mathcal{D}}$ be the radius and diameter of cross monic zero divisor graph respectively. Then*

$$\eta_r\sqrt{2\left(8p^2-9p+1\right)} \leq \mathcal{EM}_e\left(\mathcal{CMZG}\right) \leq \eta_{\mathcal{D}}\sqrt{2\left(2p^2+2p-1\right)\left(8p^2-9p+1\right)}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{\alpha=1}^{2p^2+2p-1} \beta_\alpha\beta'_\alpha\right)^2 \leq \left(\sum_{\alpha=1}^{2p^2+2p-1} \beta_\alpha^2\right)\left(\sum_{\alpha=1}^{2p^2+2p-1} \beta'^2_\alpha\right)$$

By takeing $\beta_\alpha=1$ and $\beta'_\alpha=|\lambda_\alpha|$, the result is

$$\left(\mathcal{EM}_e\left(\mathcal{CMZG}\right)\right)^2=\left(\sum_{\alpha=1}^{2p^2+2p-1} |\lambda_\alpha|\right)^2$$

$$\leq \left(\sum_{\alpha=1}^{2p^2+2p-1} 1\right)\left(\sum_{\alpha=1}^{2p^2+2p-1} \lambda_\alpha^2\right)$$

$$\leq 2p^2+2p-1\left(2\sum_{\alpha=1}^{2p^2+2p-1}\left(x_\alpha+y_\alpha\right)e^2\left(v_\alpha\right)\right).$$

Since $\sum_{\alpha=1}^{2p^2+2p-1}(x_\alpha+y_\alpha)=8p^2-9p+1$, it follows that

$$\mathcal{EM}_e\left(\mathcal{CMZG}\right)\leq\eta_{\mathcal{D}}\sqrt{2\left(2p^2+2p-1\right)\left(8p^2-9p+1\right)}$$

Now, since $\left(\sum_{\alpha=1}^{2p^2+2p-1}|\lambda_\alpha|\right)^2\geq\sum_{\alpha=1}^{2p^2+2p-1}\lambda_\alpha^2$, it follows that $(\mathcal{EM}_e(\mathcal{CMZG}))^2\geq 2\sum_{\alpha=1}^{2p^2+2p-1}(x_\alpha+y_\alpha)e^2(v_\alpha)$. Since, $e(v)\geq\eta_r$, for every vertices in graphs, then

$$\mathcal{EM}_e\left(\mathcal{CMZG}\right)\geq\eta_r\sqrt{2\left(8p^2-9p+1\right)}$$

Theorem 8: *Let $\mathcal{CMZG}$ be a connected graph of order $2p^2+4p-3$, size $4(3p^2-4p+1)$ and η_r be radius. Then*

$$\mathcal{EM}_e\left(\mathcal{CMZG}\right)\leq\frac{2\eta_r\left(4\left(3p^2-4p+1\right)\right)}{2p^2+4p-3}$$

$$+\frac{4\left(3p^2-4p+1\right)}{2p^2+4p-3}\sqrt{2(2p^2+4p-3)^2\eta_{\mathcal{D}}^2-4\eta_r^2\left(4\left(3p^2-4p+1\right)\right)}$$

Proof. It is known that,

$$(\mathcal{EM}_e-|\lambda_1|)^2\leq p^2+4p-4\left(2\sum_{\alpha=1}^{2p^2+2p-1}\left(x_\alpha+y_\alpha\right)e^2\left(v_\alpha\right)-\lambda_1\right)$$

$$\leq p^2+4p-4\left(2\eta_{\mathcal{D}}^2\left(12p^2-16p+4\right)\right)-\lambda_1^2)$$

Therefore,
$\mathcal{EM}_e(\mathcal{CMZG})\leq\lambda_1+\sqrt{2p^2+4p-4(2\eta_{\mathcal{D}}^2(12p^2-16p+4))-\lambda_1^2)}$ implies that,

$$1-\frac{n\left(2p^2+4p-4\right)}{\sqrt{2p^2+4p-4\left(2\eta_{\mathcal{D}}^2\left(12p^2-16p+4\right)\right)-x^2)}}\leq 0$$

and hence
Let $f(n)=n+\sqrt{2p^2+4p-4(8(3p^2-4p+1)+\eta_{\mathcal{D}}^2-n^2)}$. For decreasing function $f(n)$, $f'(n)\leq 0$

$$n\geq\sqrt{\frac{2\eta_{\mathcal{D}}^2\left(12p^2-16p+4\right)}{2p^2+4p-3}}.$$

If $\lambda_1\geq\frac{2\eta_r(12p^2-16p+4)}{2p^2+4p-3}$, it follows that

$$\sqrt{\frac{2\eta_{\mathcal{D}}^2\left(12p^2-16p+4\right)}{2p^2+4p-3}}\leq\frac{2\eta_{\mathcal{D}}^2\left(12p^2-16p+4\right)}{2p^2+4p-3}\leq\lambda_1.$$

Thus $f(\lambda_1) \leq f(\frac{2\eta_{\mathcal{D}}^2(12p^2-16p+4)}{2p^2+4p-3})$. That means,

$$\mathcal{E}\mathcal{M}_e \leq f(\lambda_1) \leq f\left(\frac{2\eta_{\mathcal{D}}^2\left(12p^2-16p+4\right)}{2p^2+4p-3}\right)$$

Therefore,

$$\mathcal{E}\mathcal{M}_e(\mathcal{CMZG}) \leq \lambda_1 + \sqrt{2p^2+4p-4\left(2\eta_{\mathcal{D}}^2\left(12p^2-16p+4\right)\right)-\lambda_1^2)}$$

$$\leq \frac{2\eta_r\left(4\left(3p^2-4p+1\right)\right)}{2p^2+4p-3} + \sqrt{12p^2-16p+4\left(\left(24p^2-32p+8\right)(\eta_{\mathcal{D}}) - \left(\frac{24p^2-32p+8\left(\eta_{\mathcal{D}}\right)}{2p^2+4p-3}\right)^2\right)}$$

$$\leq \frac{2\eta_r\left(4\left(3p^2-4p+1\right)\right)}{2p^2+4p-3} + \frac{4\left(3p^2-4p+1\right)}{2p^2+4p-3}\sqrt{2(2p^2+4p-3)^2\eta_{\mathcal{D}}^2 - 4\eta_r^2\left(4\left(3p^2-4p+1\right)\right)}.$$

Theorem 9: *Let $\mathcal{CMZG}$ be a $(2p^2+4p-3, 4(3p^2-4p+1))$-graph. Then*

$$\mathcal{E}_{ade} \leq \frac{2p^2+4p-4(2p^2+4p-3)^2}{2}$$

Proof. By Gershgirin's theorem,

$$\mathcal{E}_{ade} = \sum_{\alpha=1}^{2p^2+4p-3} |\lambda_\alpha|$$

$$\leq \sum_{\alpha=1}^{2p^2+4p-3} \sum_{\alpha_0=1,\alpha_0\neq\alpha}^{2p^2+4p-3} t_{\alpha_0\alpha}$$

$$\leq \sum_{\alpha=1}^{2p^2+4p-3} \sum_{\alpha_0=1,\alpha_0\neq\alpha}^{2p^2+4p-3} \frac{2p^2+4p-3}{2}$$

$$= \frac{2p^2+4p-4(2p^2+4p-3)^2}{2}$$

Theorem 10: *Let $\mathcal{CMZG}$ be a $(2p^2+2p-1, 8p^2-9p+1)$-graph. Then*

$$\mathcal{E}_{ade} \geq \sqrt{2(\left(8p^2-9p+1\right)|det\,(\mathbf{ADE})|^{\frac{2}{2p^2+2p-1}})}$$

Proof.

$$(\mathcal{E}_{ade}(\mathcal{CMZG}))^2 = \left(\sum_{\alpha=1}^{2p^2+2p-1} |\lambda_\alpha|\right)^2$$

$$= \sum_{\alpha=1}^{2p^2+2p-1} \lambda_\alpha^2 + \sum_{\alpha=1,\alpha\neq\alpha_0}^{2p^2+2p-1} |\lambda_\alpha| \left|\lambda_{\alpha_0}\right|$$

$$= -2c_2 + \sum_{\alpha=1,\alpha\neq\alpha_0}^{2p^2+2p-1} |\lambda_\alpha| \left|\lambda_{\alpha_0}\right|$$

Then,

$$\sum_{\alpha=1,\alpha\neq\alpha_0}^{2p^2+2p-1} |\lambda_\alpha| \left|\lambda_{\alpha_0}\right| = 2\left(2p^2+2p-1\right)\left(p^2+p-1\right)$$

$$\left(\prod_{\alpha=1,\alpha\neq\alpha_0}^{2p^2+2p-1} |\lambda_\alpha| \left|\lambda_{\alpha_0}\right|\right)^{\frac{1}{2\left(2p^2+2p-1\right)\left(p^2+p-1\right)}}$$

$$= 2\left(2p^2+2p-1\right)\left(p^2+p-1\right)\left(\prod_{\alpha=1,\alpha\neq\alpha_0}^{2p^2+2p-1} |\lambda_\alpha|^{2\left(2p^2+2p-2\right)}\right)^{\frac{1}{2\left(2p^2+2p-1\right)\left(p^2+p-1\right)}}$$

$$= 2\left(2p^2+2p-1\right)\left(p^2+p-1\right)\left(\prod_{\alpha=1,\alpha\neq\alpha_0}^{2p^2+2p-1} |\lambda_\alpha|\right)^{\frac{2}{2p^2+2p-1}}$$

$$= 2\left(8p^2-9p+1\right)\left(\prod_{\alpha=1,\alpha\neq\alpha_0}^{2p^2+2p-1} |\lambda_\alpha|\right)^{\frac{2}{2p^2+2p-1}}$$

$$= 2\left(8p^2-9p+1\right)|det\,(\mathbf{ADE})|^{\frac{2}{2p^2+2p-1}}.$$

$$\mathcal{E}_{ade} \geq \sqrt{2\left((8p^2-9p+1)|det(\mathbf{ADE})|^{\frac{2}{2p^2+2p-1}}\right)}$$

Conclusion

This study focuses on a specific family of cross monic zero divisor graphs constructed from commutative rings, such as $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2+px\rangle$, $\mathbb{Z}_2 \times \mathbb{Z}_p[x]/\langle x^2+(p-1)x\rangle$, $\mathbb{Z}_4 \times \mathbb{Z}_p[x]/\langle x^2+px\rangle$ and $\mathbb{Z}_4 \times \mathbb{Z}_p[x]/\langle x^2+(p-1)x\rangle$. The research delves into the computation of various energy metrics associated with these graphs. Specifically, it provides the Laplacian energy, which is related to the graph's Laplacian matrix; the eccentricity sum energy, which aggregates the eccentricities of vertices; and the Laplacian sum eccentricity energy, which combines Laplacian and eccentricity aspects. Additionally, the work explores the maximum eccentricity energy, which highlights the extremal eccentricity values, and the average degree-eccentricity energy, which reflects the interplay between vertex degrees and their eccentricities. This comprehensive analysis offers deeper insights into the structural properties of these algebraically defined graphs.

Authors' declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Vellore Institute of Technology University, India.

Author's contribution statement

S. R. proposed and presented the idea and developed the paper. R. S. J. proposed corrections, modifications of proofs, presentation style of the draft paper. The final copy was prepared through discussion by both the authors.

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طاقة لابلاس والطاقة القائمة على الانحراف لرسم بياني لمقسوم صفر أحادي متقاطع مرتبط بحلقات تبديلية

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المستخلص

بالنسبة لحلقة تبديلية، تتم مناقشة رسم بياني لقواسم صفرية أحادية متقاطعة، حيث تكون رؤوسه قواسم صفرية غير صفرية لحلقة تبديلية، ثم تكون الرأسان x وy متجاورتين إذا وفقط إذا كانت $xy = 0$. يُعرف مجموع القيم الذاتية المساعدة المطلقة لمصفوفة لابلاس للرسم البياني البسيط باسم طاقة لابلاس للرسم البياني البسيط. مصفوفة مجموع الانحراف للرسم البياني هي مصفوفة موجبة، تكون مدخلاتها هي مجموع انحراف رأسين و0 لقطري مصفوفة مجموع الانحراف. ثم يُعرف مجموع القيم الذاتية المطلقة لمصفوفة مجموع الانحراف باسم طاقة مجموع الانحراف. طاقة مجموع الانحراف لمجموع لابلاس هي الفرق المطلق بين القيم الذاتية لمصفوفة مجموع الانحراف لمجموع لابلاس وضعف عدد الحواف بعدد الرؤوس. طاقة الانحراف الأقصى هي مجموع القيم الذاتية المطلقة لمصفوفة الانحراف الأقصى. بالنسبة للرسم البياني البسيط، تكون مدخلات مصفوفة الانحراف المتوسط للدرجة، إذا كانت الرأسان متجاورتين، هي متوسط مجموع الدرجة والانحراف لرأسيهما، وإلا فهي 0. عندئذٍ يُعرف مجموع القيم الذاتية المطلقة لمصفوفة الانحراف المتوسط للدرجة باسم متوسط طاقة الانحراف للدرجة. تناقش هذه الورقة تسلسل الدرجات، وعدد الرؤوس، وعدد الحواف لمخطط القاسم الصفري المتقاطع أحادي الاتجاه للحلقات التبديلية. بالنسبة لعائلة معينة من الرسوم البيانية المقسومة الصفرية المتقاطعة أحادية الاتجاه للحلقة التبديلية، يتم أيضًا تغطية طاقة لابلاس، وطاقة مجموع الانحراف، وطاقة مجموع الانحراف لابلاس، وطاقة الحد الأقصى للانحراف، وطاقة الانحراف المتوسط للدرجة.

الكلمات المفتاحية: متوسط طاقة درجة الانحراف، مجموع طاقة الانحراف، طاقة لابلاس، طاقة مجموع الانحراف اللابلاس، طاقة الانحراف الأقصى، رسوم بيانية للمقسوم الصفري.