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Some Results on W-regular Rings

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ABSTRACT

Von Neumann regular rings, introduced in 1936, form a cornerstone of abstract algebra and were later extended to weakly regular rings. This work considers another extension, the W-regular rings defined by Wei [6], and explores their properties and distinctions from related notions.The present paper looks into the characterizations of a few fundamental qualities of W-regular rings. We prove that if $H/\Upsilon(\alpha)$ is a W-regular ring and $\Upsilon(\alpha^2) = \Upsilon(\alpha)$ for all $\alpha \in N_2(H)$, then H is a W-regular. If H is an NI-ring and $H/N(H)$ be a regular ring, then H be a W-regular ring if H is n-regular. A ring H is a W-regular ring if and only if $H\alpha$ is a direct summand for all $\alpha \in N_2(H)$. If H is a ring with every right simple singular H-module is np-injective, H is semiprime ring and an N-duo, then H is a W-regular.

Introduction

By H we will always an associative ring have identity and any the module unitary right H-module. For $y \in H$, $\Upsilon(y)$ and $l(y)$ denoted the right and left annihilators of y, respectively. $N_2(H)$, $N(H)$, and $J(H)$, denote the set of all non-nilpotent elements of H, the set of nilpotent elements of H, and the Jacobson radical of H, respectively.The ring H is regular (strongly regular) when every $\alpha \in H$ there exists a $b \in H$ such that $\alpha = \alpha b\alpha$ ($\alpha = \alpha^2 b$) [1] and [2]. Generalizations of regularity have been discussed in many papers ([3], [4], [5] and [6]). A ring H is n-regular if and only if for all $\alpha \in N(H), \alpha \in \alpha H\alpha$ [7]. Studies of n-regular rings include ([8],[10]and [11]). A ring H is an abelian when $\alpha \in H$, α is central. A ring H is semiprime if $\alpha H\alpha = 0$ implies $\alpha = 0$ for $\alpha \in H$. A ring H is reduced when $N(H) = \varphi$. Any reduced ring is semiprime, but the converse is not true.

Von Neumann regular rings, introduced in 1936, are considered a cornerstone of abstract algebra and were later generalized to weakly regular rings. This study adopts another generalization known as W-regular rings, following Wei's definition see [6], which differs significantly from the weakly regular generalization. Our aim in this work is to explore additional properties of this class of rings and to highlight its distinctions from other related concepts. This paper aims to study W-rings, investigate their properties, and explore their relationship with other rings.

Methodology

In this paper we use the Mathematical logic, alongside other theories, was used as the foundation for the proofs of problems and theorems

W-regular rings

In [6], ring H is called W-regular when, for every $\alpha \in N_2(H)$, there exist $b \in H$ such that $\alpha = \alpha b\alpha$. Moreover, simple examples of W-regular rings include Z_6, Z_8, Z_{10}. Also, H is a regular ring whenever H is n-regular and W-regular.

Z_8 is a W-regular ring which neither regular nor n-regular. On the other hand, Z is n-regular but neither regular nor W-regular.

Proposition 2.1:

Let $\alpha \in N_2(H)$ and suppose that $\alpha x\alpha - \alpha$ is W-regular for some $x \in H$. Then α is W-regular.

Proof:

Since $\alpha \in N_2(H)$ and $\alpha x\alpha - \alpha$ is W-regular, then $\exists\ y \in H$ such that

$$\alpha x\alpha - \alpha = (\alpha x\alpha - \alpha)y(\alpha x\alpha - \alpha)$$
$$= \alpha x\alpha y\alpha x\alpha - \alpha x\alpha y\alpha - \alpha y\alpha x\alpha + \alpha y\alpha$$
$$\alpha = \alpha(x - x\alpha y\alpha x + x\alpha y + y\alpha x + y)\alpha.$$

Set $b = x - x\alpha y\alpha x + x\alpha y + y\alpha x + y$. Then $\alpha = \alpha b\alpha$, so it is a W-regular-element.■

According to Wei and Chen (2011) in [6], H is a left (right) zero divisor power ring when any $0 \neq \alpha \in H$, $l(\alpha^n) = l(\alpha)$ $[\Upsilon(\alpha^n) = \Upsilon(\alpha)]$ for all n a positive integer such that $\alpha^n \neq 0$.

Theorem 2.2:

If $H/\Upsilon(\alpha)$ is W-regular and $\Upsilon(\alpha^2) = \Upsilon(\alpha)$ for all $\alpha \in N_2(H)$, then H is W-regular.

Proof:

for $H/\Upsilon(\alpha)$ is W-regular for all $\alpha \in N_2(H)$. Then $\exists\ b + \Upsilon(\alpha) \in H/\Upsilon(\alpha)$ such that

$$\alpha + \Upsilon(\alpha) = (\alpha + \Upsilon(\alpha))(b + \Upsilon(\alpha)(\alpha + \Upsilon(\alpha),$$
$$\alpha + \Upsilon(\alpha) = \alpha b\alpha + \Upsilon(\alpha).$$

Hence $\alpha - \alpha b\alpha \in \Upsilon(\alpha)$ and this implies that $(1 - b\alpha) \in \Upsilon(\alpha^2) = \Upsilon(\alpha)$ (H is a zero-divisor power), $\alpha = \alpha b\alpha$. Therefore, H is a W-regular-ring. ■

Corollary 2.3:

If $H/\Upsilon(\alpha)$ is W-regular for all $\alpha \in N_2(H)$ and H is a right zero divisor power, then H is W-regular.

Proposition 2.4:

Let $\alpha = \alpha x\alpha$ be a W-regular element for some $x \in H$, if α or x is central. Then $b^2\alpha = b$ where $b = x\alpha x$.

Proof:

Assume $\alpha = \alpha x\alpha$, for some, $x \in H$. Now, $b^2\alpha = b(b\alpha) = b(x\alpha x)\alpha = bx(\alpha x\alpha) = (bx)\alpha = b(x\alpha) = (x\alpha x)(\alpha x) = x(\alpha x\alpha)x = x\alpha x = b$. ■

Proposition 2.5:

Let H be abelian. Then H is a W-regular ring if and only if for every $\alpha, b \in N_2(H)$ $\exists$ $x \in H$ such that $\alpha b = b\alpha x\alpha b$.

Proof:

Assume that H is a W-regular ring and $\alpha, b \in N_2(H)$. Then $\alpha = \alpha y\alpha, b = bzb$ for some $y, z \in H$. Now set $x = yz$ and by (Theorem(4.2), [6]) $e = \alpha y, g = bz, \alpha = e\alpha$ and $b = gb$, where $e, g \in E(H)$. Then

$$\alpha b = \alpha gb = g\alpha b = bz\alpha b = bze\alpha b = bez\alpha b = b\alpha yz\alpha b = b\alpha(yz)\alpha b = b\alpha x\alpha b.$$

Conversely, assume that for any $\alpha, b \in N_2(H)$, $\exists$ $x \in H$ such that $\alpha b = b\alpha x\alpha b$. If we take b=1, then $\alpha = \alpha x\alpha$. So, H is a W-regular ring. ■

Proposition 2.6:

Let H be a W-regular ring. Suppose that $\alpha H \subseteq I$, for $\alpha \in N_2(H)$ and I being a right or left ideal. Then $\alpha HI = \alpha H$.

Proof:

It is clear that $bHI \subseteq bH$ for any $b \in H$. Now if $\alpha \in N_2(H)$ and $x \in \alpha H$, then there exists $r \in H$ with $x = \alpha r$. $\because$ H is a W-regular ring, there exists $y \in H$ such that $\alpha = \alpha y\alpha$, $x = \alpha y\alpha r$, hence $\alpha r \in \alpha H \subseteq I$. So $\alpha H \subseteq \alpha HI$. Therefore $\alpha HI = \alpha H$. ■

Corollary 2.7:

Let H be a ring such that for all $\alpha \in N_2(H)$ and any I right or left ideal of H, $\alpha H \subseteq I$. Then the two conditions are comparable:

1. H be a W-regular.
2. For all $\alpha \in N_2(H)$, $\alpha HI = \alpha H$.

Proof:

$1 \to 2$ by Proposition 2.6.

$2 \to 1$ Let $I = aH$ and then use Theorem 4.2 in [6]. ■

A ring H is called NI if $N(H)$ forms an ideal [13].

Theorem 2.8:

Let S be an NI-ring that is W-regular ring, then $S/N(S)$ be a regular ring.

Proof:

if S is a W-regular, and $\alpha + N(S) \in S/N(S)$. Now if $\alpha \in S$ and $\alpha \notin N(S)$ there exists a $y \in S$ such that $\alpha = \alpha y\alpha$. $\alpha + N(S) = \alpha y\alpha + N(S) = (\alpha + N(S))(y + N(S))(\alpha + N(S))$ for all $\alpha + N(S) \in S/N(S)$, so $S/N(S)$ it well be regular. ■

For example, Z_8 is W-regular and $N(Z_8) = \{0,2,4\}$, so $Z_8/N(Z_8)$ is regular. We see that Z_8 is not regular but the quotient ring $Z_8/N(Z_8)$ is.

In general, we don't know if the converse is always true, because there are many examples where the converse holds without an additional condition. However, to prove that the converse is true, we need to add the following condition.

Theorem 2.9:

Let S be an NI-ring such that $S/N(S)$ is a regular ring. Then S is a W-regular-ring if S is n-regular.

Proof:

Since $S/N(S)$ is regular and $\alpha + N(S) \in S/N(S)$, there exists $z + N(S) \in S/N(S)$, such that $\alpha + N(S) = \alpha z\alpha + N(S)$, so $\alpha z\alpha - \alpha \in N(S)$. Since S is n-regular and using then the fact that $N(S)$ is regular using the same strategies as in Proposition 2.1, α is regular and $\alpha \notin N(S)$ so $\alpha \in N_2(S)$, so S is W-regular. ■

The ring H be a right quasi-duo ring when any maximal right ideal of H also left ideal [12]. The ring H is called a J -regular ring when any $\alpha \in J(H)$, α is a regular element [16].

Lemma 2.10:

for H that is left or right quasi-duo ring, then $N(H) \subseteq J(H)$ [16].

Corollary 2.11:

When S be right or left quasi-duo ring that is W-regular. Then $S/J(S)$ be a regular ring.

Proof:

Use both Theorem 2.7 and Lemma 2.8. ■

Remark 2.12:

The converse does not hold if H is n-regular. Because it might happen that $N(H) \subset J(H)$. So there exists a non-nilpotent element which is not regular, but a converse may well be true, as in the next corollary.

Corollary 2.13:

When H a right or left quasi-duo ring that is W-regular. Then $H/J(H)$ is regular. The converse holds if H is J-regular.

Theorem 2.14:

The ring H is a W-regular ring if and only if $H\alpha$ is a direct summand for all $\alpha \in N_2(H)$.

Proof:

Let H be a W-regular ring, so for all $\alpha \in N_2(H)$, there exists a $b \in H$, such that $\alpha = \alpha b\alpha$, $e = b\alpha$, $e^2 = b(\alpha b\alpha) = b\alpha = e$, $\alpha = \alpha e$. Let $x \in H\alpha$, then there exists an $r \in H$ such that $x = r\alpha = r\alpha e \in He$, which implies that $H\alpha \subseteq He$. Now if $y \in He$, exists an $s \in H$ with $y = se$, since $e = b\alpha$, we have $y = se = sb\alpha \in H\alpha$. This implies that $He \subseteq H\alpha$, so $He = H\alpha$. So, we get that $H\alpha \oplus H(1-e) = H$.

Conversely, suppose $H\alpha$ is a direct summand, for every $\alpha \in N_2(H)$. Then, there is a left ideal K wiht $H\alpha \oplus K = H$, since $1 \in H$, there exists $h \in H$ and $k \in K$ such that $h\alpha + k = 1$, so $\alpha h\alpha + \alpha k = \alpha$, $\alpha k \in K$ and $\alpha k = \alpha - \alpha h\alpha = (1 - \alpha h)\alpha \in H\alpha$, so $\alpha k \in H\alpha \cap K = \{0\}$, therefore $\alpha = \alpha h\alpha$, for every $\alpha \in N_2(H)$. Hence H is a W-regular ring. ■

N is a right np-injective module, when any non-nilpotent element α of H, and any right N-homomorphism $g: \alpha N \to N$, there exists a $y \in N$, such that $g(\alpha n) = y\alpha n$ for all $n \in N$ [16]. A ring H is said to be N-duo if $\alpha H = H\alpha$, for all $\alpha \in H$ [17].

Theorem 2.15:

Suppose H is an N-duo ring that is semiprime. Suppose further that any right simple singular H-module it will be np-injective. Then H is a W-regular ring.

Proof:

We have to show that $\alpha H + r(\alpha) = H$ holds for every $\alpha \in N_2(H)$. Suppose not ture. exists a maximal ideal M of H with $\alpha H + r(\alpha) \subseteq M$. If a maximal ideal is not an essential ideal, then it is a direct summand ideal. There exists $0 \neq e = e^2 \in H$, such that $M = r(e)$, $H\alpha e \subseteq H\alpha H \subseteq \alpha H \subseteq M = r(e)$, since H is right N-duo ring, we get $eH\alpha e = 0$, now $(\alpha eH)^2 = \alpha(eH\alpha e)H = 0$, since H is semiprime, $\alpha eH = 0$, $\alpha e = 0$, so $e \in$ r$(\alpha) \subseteq M = r(e)$, $e \in r(e)$, $e^2 = 0$, which is a contradiction with $0 \neq e$. Therefore, M is an essential maximal, H/M is a right simple singular np-injective H -module.

Define a mapping $f: \alpha H \to H/M$, $f(\alpha r) = r + M$, $r \in H$. To show that f is well defined, let $\alpha x = \alpha y$, $\alpha(x - y) = 0$, $(x - y) + M = f\big(\alpha(x - y)\big) = f(0) = M$, $(x - y) + M = M$, $x + M = y + M$, so $f(\alpha x) = x + M = y + M = f(\alpha y)$, so f is well defined. Since H/M is np-injective, there exists $b + M \in H/M$, such that $1 + M = f(\alpha) = (b + M)(\alpha + M) = b\alpha + M$, $1 + M = b\alpha + M$, which implies that $1 - b\alpha \in M$, since $b\alpha \in H\alpha = \alpha H$, since H is duo ring, therefore $1 \in M$, it is a contradiction, so $\alpha H + r(\alpha) = H$, and as a special case, there exist $z \in H$ and $v \in r(\alpha)$ such that $\alpha y + z = 1, \alpha y a + z\alpha = \alpha$, $\alpha = \alpha y\alpha$ for all $\alpha \in N_2(H)$, so H is a W-regular ring. ■

Conclusion:

The ring H it will be W-regular from the quotient ring $H/\Upsilon(\alpha)$ which is W-regular for right annihlator $\Upsilon(\alpha)$ if $\Upsilon(\alpha^2) = \Upsilon(\alpha)$ for all $\alpha \in N_2(H)$. Also we get the W-regular ring from the quotient $H/N(H)$ where H is an NI-ring. The ring H be W-regular ring whenever $H\alpha$ be direct summand for all $\alpha \in N_2(H)$.

References

[1] Ming, R. Y. C. (1986). On von Neumann regular rings, XI. Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie, 30(4), 371-379.
[2] Jule, Z., & Xianneng, D. (1993). Von Neumann regularity of sf-rings. Communications in Algebra, 21(7), 2445-2451.
[3] Mahmood, R. D., & Mohammed, R. S. (2021). QR and JCPI–Rings. In Journal of Physics: Conference Series (Vol. 1879, No. 2, p. 022117). IOP Publishing.
[4] Mahmood, R. D. and Khider, K. J. (2020) "SJF-rings" International Mathematical Forum, Vol. 15, 2020, no. 1, 41-46.
[5] Ahmad, S. H., Mahmood, R. D., Mohammed, F. H., & Ibraheem, Z. M. (2023). On simple generalized m-flat rings. Journal of Discrete Mathematical Sciences and Cryptography, 26(4), 585-590.
[6] Wei, J., & Che, J. (2011). Weakly np-Injective Rings and Weakly C2 Rings. Kyungpook Mathematical Journal, 51(1), 93-108.
[7] Wei, J. C., & Chen, J. H. (2007). Nil-injective rings. International Electronic Journal of Algebra, 2(2), 1-21.
[8] Mahmood, R., & Youns, M. (2011). On n-Regular Rings. AL-Rafidain Journal of Computer Sciences and Mathematics, 8(2), 53-59.
[9] Mahammod, R., & Al-Neimi, M. (2012). On n-weakly regular rings. Al-Rafidain Journal of Computer Sciences and Mathematics, 9(2), 53-59.
[10] Hussain, E. A., Al-Shallawi, A. N., & Saied, H. A. (2022). Using maximum likelihood method to estimate parameters of the linear regression t truncated model. *NTU Journal of Pure Sciences*, *1*(4), 26-34
[11] Talat, H. (2024). Some Results on AWN-injective Rings. *NTU Journal of Pure Sciences*, *3*(4).
[12] G. Marks, On 2-primal Ore extensions, Comm. Alg., 29 No. 5 (2001) 2113-2123.
[13] Brown, S. H. (1973). Rings over which every simple module is rationally complete. Canadian Journal of Mathematics, 25(4), 693-701.
[14] Yu-e, Z., & Shujuan, Z. (2011). On JPP rings, JPF rings and J-regular rings. In International Mathematical Forum (Vol. 6, No. 34, pp. 1691-1696).
[15] Yu, H. P. (1995). On quasi-duo rings. Glasgow Mathematical Journal, 37(1), 21-31.
[16] Ming, R. Y. C. (1983). On quasi-Frobeniusean and Artinian rings. Publications de L'Institut Mathematique, 33(47), 239-245.
[17] Wei, J., & Li, L. (2011). Nilpotent elements and reduced rings. Turkish Journal of Mathematics, 35(2), 341-353.