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Solving Undamped and Damped Fractional Oscillators via Integral Rohit Transform

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ABSTRACT: ***Background:*** *The dynamics of fractional oscillators are generally described by fractional differential equations, which include the fractional derivative of the Caputo or Riemann-Liouville type. These equations induce classical oscillator equations like the harmonic oscillator equation, to include fractional order derivatives. Solving fractional differential equations numerically can be challenging due to the non-local nature of fractional derivatives.* ***Objective:*** *In this paper, a recently developed integral Rohit transform is utilized for solving systems of undamped and damped fractional oscillators characterized by differential equations of fractional or non-integral order involving the Caputo-fractional derivative operator. The solutions of fractional systems which include undamped-simple fractional oscillators, undamped-driven fractional oscillators, damped-driven fractional oscillators, and damped-fractional oscillators are obtained.* ***Methods:*** *by applying the integral Rohit transform, also written as RT. Differential equations of fractional or non-integral order are generally solved by utilizing methods which include the fractional variational iteration approach, the homotopy-perturbation method, the equivalent linearized method, the Adomian decomposition method, etc.* ***Results:*** *This paper demonstrates the effectiveness, reliability, and efficiency of the integral Rohit transform in solving fractional systems, which include undamped-simple fractional oscillators, undamped-driven fractional oscillators, damped-driven fractional oscillators, and damped-fractional oscillators and are characterized by differential equations of fractional or non-integral order involving the Caputo-fractional derivative operator.* ***Conclusions:*** *The Rohit transform brought the progressive principles or methodologies that offer new insights or views on the problems examined in the paper, distinguishing itself from existing methods and doubtlessly beginning up new research instructions. It provided precise results for the specific problems discussed in the paper, surpassing the capabilities of other methods in terms of decision, constancy, or robustness to noise and disturbances.*

KEYWORDS: *Rohit transform; Fractional oscillators; Caputo-fractional derivative operator; Fractional differential equations*

INTRODUCTION

Fractional oscillators have become popular due to their ability to capture complex behaviors that are not adequately represented by classical integral-order differential equations. Fractional calculus extends the ideas of derivatives and integrals of non-integral order. Fractional differential equations involve fractional derivatives, which induce the concept of differentiation into non-integer orders. The unique characteristic of fractional oscillators is that they have a long memory, which means that their behavior can not only be controlled by the present moment but also by past events over numerous periods. Due to this characteristic, fractional oscillators are relevant for modeling systems with memory-related effects. The current state of fractional oscillators can be affected by events from the remote past. This non-local behavior of fractional oscillators differentiates them from classical oscillators. Fractional oscillators find applications in the modeling of viscoelastic materials, control systems with memory, modeling of neuron firing patterns, and modeling of long-range dependence in financial time series [1]–[3]. Fractional oscillators are enforced in control systems to design controllers for those systems that are used to remember past actions. Fractional oscillators are enforced in signal processing to filter, cut out noise, and resolve time series. These oscillators are used in modeling the electrical circuits with memory effects and non-local interactions. They are also used in designing circuits with upgraded stability, good frequency response, and noise elimination [4]–[6]. The fractional oscillators are described by fractional differential equations, which include the Caputo-type or Riemann-Liouville-type fractional derivative operator [7]–[9]. These fractional differential equations cannot be easily solved numerically due to the non-local behavior of fractional derivative operators. In this paper, the undamped-simple fractional oscillator, undamped-driven fractional oscillator, damped-driven fractional oscillator, and damped fractional oscillator are described by fractional differential equations, which include the Caputo-type fractional derivative operator. Fractional differential equations, which include the Caputo-type or Riemann-Liouville-type fractional derivative operator are generally solved by the equivalent linearized technique [10], the method of differential transform [11], [12], homotopy-perturbation technique [13], Adomian decomposition method [14], and the method of fractional variational iteration [15]. The paper explores the application of the Rohit transform technique to solve fractional systems, including undamped-simple fractional oscillators, undamped-driven fractional oscillators, damped-driven fractional oscillators, and damped fractional oscillators., which include the Caputo-type fractional derivative operator. The Rohit transform has not been sufficiently utilized to solve systems of fractional oscillators due to its recent appearance. The author Rohit Gupta has proffered the Rohit transform, also written as RT, in recent years to expedite the process of solving differential equations. This transform has been successfully applied to solve many initial value problems in the physical sciences and engineering [16]. The Rohit transform [17] is defined for a function of exponential order by the integral equations as

$$R\{h(t)\} = q^3 \int_0^\infty e^{-qt} h(t)\, dt, \; t \geq 0, \; q_1 \leq q \leq q_2.$$

The variable q is used to factor the variable t into the argument of the function h. The Rohit transforms of some unidentified functions [18] are given by

$$R\ \{t^n\} = \frac{n!}{q^{n-2}},$$

$$R\ \{sinbt\} = \frac{b\ q^3}{q^2+b^2},$$

$$R\ \{cosbt\} = \frac{q^4}{q^2+b^2},$$

$$R\ \{e^{bt}\} = \frac{q^3}{q-b}.$$

The Rohit transforms of some derivatives [19], [20] are given by

$$R\ \{g'(t)\} = qG(q) - q^3 g(0),$$

$$R\{g''(t)\} = q^2 G(q) - q^4 g(0) - q^3 g'(0),$$

$$R\{g'''(t)\} = q^3 G(q) - q^5 g(0) - q^4 g'(0) - q^3 g''(0).$$

In general,

$$R\{g^n(t)\} = q^n R\{g(t)\} - \sum_{k=1}^{n} q^{n-k+3} g^{k-1}(0).$$

The organization of the paper is as follows: Firstly, brief information on a Mittag-Leffler function, and fractional operators such as the Caputo fractional derivative operator and their attributes is provided. Secondly, the Rohit transform of the Mittag-Leffler function and the Caputo fractional derivative operator are obtained. Thirdly, the solutions of fractional systems such as undamped-simple fractional oscillators, undamped-driven fractional oscillators, damped-driven-fractional oscillators, and damped fractional oscillators, characterized by differential equations of non-integral orders entailing the Caputo fractional derivative operator, are obtained by applying Rohit transform (RT). Finally, the conclusions of the study are presented.

Rohit Transform of Convolution

The Rohit transform of convolution: $(f \circ g)(t)$ is given by

$$R(f \circ g)\,(t) = \frac{1}{q^3}\;F(q)G(q). \tag{1}$$

Proof: Since $(f \circ g)\,(t) = \int_0^t f\,(t-x)\,g\,(x)\,dx$, therefore,

$$\begin{aligned} R\left\{(f \circ g)\,(t)\right\} &= q^3 \int_0^\infty e^{-qt}\,(f \circ g)\,(t)\;\; dt \\ &= q^3 \int_0^\infty e^{-qt} \int_0^t f\,(t-x)\,g\,(x)\;\; dx\;\, dt \\ &= q^3 \int_0^\infty \int_0^t e^{-qt} f\,(t-x)\,g\,(x)\;\; dx\;\, dt. \end{aligned}$$

By altering the order of integration, the above integral becomes

$$R\left\{(f \circ g)\,(t)\right\} = q^3 \int_0^\infty \int_t^\infty e^{-qt} f\,(t-x)\,g\,(x)\;\; dt\;\, dx$$

$$R\left\{(f \circ g)\,(t)\right\} = q^3 \int_0^\infty e^{-qx} g\,(x)\,dx \int_x^\infty e^{-q(t-x)} f\,(t-x)\;\; dt.$$

Let $t - x = y$, then

$$R\left\{(f \circ g)\,(t)\right\} = q^3 \int_0^\infty e^{-qx} g\,(x)\,dx \int_0^\infty e^{-qy} f(y)\;\; dy$$

$$R\left\{(f \circ g)\,(t)\right\} = \frac{1}{q^3}\left[\,q^3 \int_0^\infty e^{-qx} g\,(x)\,dx\right]\left[\,q^3 \int_0^\infty e^{-qy} f\,(y)\,dy\right]$$

$$R\left\{(f \circ g)\,(t)\right\} = \frac{1}{q^3} G(q)F(q).$$

Special Functions and Their Properties

The Mittag-Leffler function, with two parameters, is defined as

$$E_{\alpha,\,b}\;\,(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma\,(\alpha n + b)}. \tag{2}$$

Here t belongs to the complex plane, $\alpha > 0, b$ belongs to real numbers, and Γ is the gamma function. For the particular values of the parameters α and b, we find well-known classical functions. For example,

$$E_{0,1}\;\,(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t},\;\; |t| < 1,$$

$$E_{1,1}\ (t)\ = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+1)} = e^t,$$

$$E_{1,2}\ (t)\ = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma\,(n+2)} = \frac{(e^t-1)}{t},$$

$$E_{2,1}\ (-t^2)\ = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{\Gamma(2n+1)} = cost,$$

$$E_{2,2}\ (-t^2)\ = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{\Gamma(2n+2)} = \frac{sint}{t},$$

$$E_{2,2}\ (t^2)\ = \sum_{n=0}^{\infty} \frac{t^{2n}}{\Gamma(2n+2)} = \frac{sinht}{t},$$

$$E_{1,1}\ (t^2)\ = \sum_{n=0}^{\infty} \frac{t^{2n}}{\Gamma(2n+1)} = cosht,$$

$$E_{\frac{1}{2},1}\ (t)\ = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\frac{1}{2}n+1)} =\ e^{t^2}\ erfc(-t).$$

The Riemann-Liouville fractional integral [4] of order α is put into words as

$$_{\alpha_0}I_x^{\alpha} f(x) = \frac{1}{\Gamma\,(\alpha)} \int_{\alpha_0}^{\alpha} (x-t)^{\alpha-1}\, f\,(t)\, dt, \quad \alpha > 0. \tag{3}$$

Some of the attributes of the Riemann-Liouville fractional integral [4] are given by

$$_{\alpha_0}I_x^{0} f(x) = f\,(x)\,,$$

$$_{\alpha_0}I_x^{\alpha}\left({}_{\alpha_0}I_x^{\beta} f(x)\right) = \alpha_0 I_x^{\alpha+\beta} f(x),$$

$$_{0}I_x^{\alpha}\,(C) = \frac{C}{\Gamma\,(\alpha+1)} x^{\alpha}, \quad \alpha > 0$$

$$_{0}I_x^{\alpha}\,(x^n) = \frac{\Gamma\,(n+1)}{\Gamma\,(n+\alpha+1)} x^{n+\alpha}, \quad \alpha, (n+1) > 0$$

$$_{-\infty}I_x^{\alpha}\left(e^{kx}\right) = \frac{e^{kx}}{k^{\alpha}}, \quad \alpha,\ k > 0$$

$$_{-\infty}I_x^{\alpha}\,(sinkx) = k^{\alpha} \sin\left(kx - \frac{\alpha\pi}{2}\right), \quad \alpha > 0$$

$$_{-\infty}I_x^{\alpha}\,(coskx) = k^{\alpha} \cos\left(kx - \frac{\alpha\pi}{2}\right), \quad \alpha, k > 0.$$

The Caputo fractional derivative [5] with order α is defined as follows:

$$^{C}_{\alpha_0}D_x^{\alpha} f\,(x) = \left(\frac{d}{dx}\right)^n {}_{\alpha_0}I_x^{n-\alpha} f\,(x) = \frac{1}{\Gamma\,(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_{\alpha_0}^{\alpha} (x-t)^{n-\alpha-1}\, f\,(t)\, dt, \tag{4}$$

where $\alpha > 0$ and $(n-1) < \alpha \leq n$. Some of the attributes of the Caputo fractional derivative [5] are given by

$$^{C}_{0}D_x^{\alpha}\,(C) = 0\ , \quad \alpha > 0$$

$$^{C}_{0}D_x^{\alpha}\,(x^n) = \frac{\Gamma\,(n+1)}{\Gamma\,(n-\alpha+1)} x^{n-\alpha}\ , \quad \alpha, (n+1) > 0$$

$$^{C}_{0}D_x^{\alpha}\left(e^{kx}\right) = k^n x^{n-\alpha} E_{1,n-\alpha+1}\ (kx),\ \alpha,\ k > 0$$

$$^{C}_{0}D_x^{\alpha}\,(sinkx) = -\frac{i}{2} (ik)^n x^{n-\alpha} [E_{1,n-\alpha+1}\ \ (ikx) - (-1)^n E_{1,n-\alpha+1}\ \ (-ikx)], \alpha > 0.$$

Rohit Transform of Special Functions

This section presents the Rohit transform applied to various special functions, including the Riemann-Liouville fractional integral, the Caputo fractional derivative, and the Mittag-Leffler function.

$$\begin{aligned} R\left\{{}_0I_x^{\alpha}f\left(x\right)\right\} &= R\left\{\frac{1}{\Gamma\left(\alpha\right)}\int_0^{\alpha}\left(x-t\right)^{\alpha-1}f\left(t\right)dt\right\} \\ &= R\left\{\frac{1}{\Gamma\left(\alpha\right)}\left(x\right)^{\alpha-1}\circ f\left(x\right)\right\} \\ &= \frac{1}{\Gamma\left(\alpha\right)}\frac{1}{q^3}R\left\{x^{\alpha-1}\right\}R\left\{f\left(x\right)\right\} \\ &= \frac{1}{\Gamma\left(\alpha\right)}\frac{1}{q^3}\frac{\Gamma\left(\alpha\right)}{q^{\alpha-1-2}}F\left(cq\right) \\ &= q^{-\alpha}F\left(q\right), \end{aligned}$$

Hence

$$R_0I_x^{\alpha}f(x) = q^{-\alpha}F\left(q\right). \tag{5}$$

Let $g\left(x\right) = \left(\frac{d}{dx}\right)^n f\left(x\right)$ and since $R_0I_x^{n-\alpha}f(x) = q^{-n+\alpha}F\left(q\right)$, therefore,

$$R_0I_x^{n-\alpha}g(x) = q^{-n+\alpha}G\left(q\right),$$

$$R_0I_x^{n-\alpha}g(x) = q^{-n+\alpha}R\left(\frac{d}{dx}\right)^n f\left(x\right),$$

$$R_0I_x^{n-\alpha}g(x) = q^{-n+\alpha}\left[q^nR\left\{f\left(t\right)\right\} - \sum_{k=0}^{n-1}q^{n-k+2}f^k\left(0\right)\right],$$

$$R_0I_x^{n-\alpha}g(x) = q^{\alpha}R\left\{f\left(t\right)\right\} - \sum_{k=0}^{n-1}q^{\alpha-k+2}f^k\left(0\right). \tag{6}$$

Now,

$$R_0^CD_x^{\alpha}f\left(x\right) = R\left(\frac{d}{dx}\right)^n {}_0I_x^{n-\alpha}f\left(x\right),$$

$$R_0^CD_x^{\alpha}f\left(x\right) = R\left\{{}_0I_x^{n-\alpha}g\left(x\right)\right\}. \tag{7}$$

From equations (6) and (7), we have

$$R_0^CD_x^{\alpha}f\left(x\right) = q^{\alpha}F\left(q\right) - \sum_{k=0}^{n-1}q^{\alpha-k+2}f^k\left(0\right).$$

Hence

$$R_0^CD_x^{\alpha}f\left(x\right) = q^{\alpha}F\left(q\right) - \sum_{k=0}^{n-1}q^{\alpha-k+2}f^k\left(0\right). \tag{8}$$

As $R\left\{t^{\alpha n+b-1}\right\} = q^3\int_0^{\infty}e^{-qt}t^{\alpha n+b-1}dt$. By putting $x = qt$, we have

$$R\left\{t^{\alpha n+b-1}\right\} = q^2\int_0^{\infty}e^{-x}\left(\frac{x}{q}\right)^{\alpha n+b-1}dx,$$

$$R\left\{t^{\alpha n+b-1}\right\} = q^{-\alpha n-b+3}\int_0^{\infty}e^{-x}x^{\alpha n+b-1}dx,$$

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$$R\left\{t^{\alpha n+b-1}\right\} = q^{-\alpha n-b+3}\,\Gamma(\alpha n+b). \tag{9}$$

Since $\sum_{n=0}^{\infty} \sigma^n\, q^{-(n+1)\alpha} = (q^{\alpha}-\sigma)^{-1}$, therefore, (9) becomes

$$R\left\{t^{b-1}E_{\alpha,\,b}\,(\sigma t^{\alpha})\right\} = R\left\{t^{b-1}\sum_{n=0}^{\infty}\frac{(\sigma t^{\alpha})^{n}}{\Gamma(\alpha n+b)}\right\},$$

$$R\left\{t^{b-1}E_{\alpha,\,b}\,(\sigma t^{\alpha})\right\} = \sum_{n=0}^{\infty}\frac{\sigma^{n}R\left\{t^{\alpha n+b-1}\right\}}{\Gamma(\alpha n+b)},$$

$$R\left\{t^{b-1}E_{\alpha,\,b}\,(\sigma t^{\alpha})\right\} = \sum_{n=0}^{\infty}\frac{\sigma^{n}\,q^{-\alpha n-b+3}\,\Gamma(\alpha n+b)}{\Gamma(\alpha n+b)},$$

$$R\left\{t^{b-1}E_{\alpha,\,b}\,(\sigma t^{\alpha})\right\} = \sum_{n=0}^{\infty}\sigma^{n}\,q^{-\alpha n-b+3}\,,$$

$$R\left\{t^{b-1}E_{\alpha,\,b}\,(\sigma t^{\alpha})\right\} = q^{\alpha-b+3}\sum_{n=0}^{\infty}\sigma^{n}\,q^{-\alpha n-\alpha}\,,$$

$$R\left\{t^{b-1}E_{\alpha,\,b}\,(\sigma t^{\alpha})\right\} = q^{\alpha-b+3}\sum_{n=0}^{\infty}\sigma^{n}\,q^{-(n+1)\alpha}\,,$$

$$R\left\{t^{b-1}E_{\alpha,\,b}\,(\sigma t^{\alpha})\right\} = q^{\alpha-b+3}\,(q^{\alpha}-\sigma)^{-1},$$

Hence,

$$R\left\{t^{b-1}E_{\alpha,\,b}\,(\sigma t^{\alpha})\right\} = \frac{q^{\alpha-b+3}}{q^{\alpha}-\sigma}. \tag{10}$$

MATERIALS AND METHODS

This section applies the Rohit transform to solve the following systems of fractional oscillators involving the Caputo fractional derivative: undamped simple fractional oscillators, undamped driven fractional oscillators, damped driven fractional oscillators, and damped fractional oscillators.

Example 1: Consider the undamped-simple fractional oscillator system involving the Caputo fractional derivative of the form:

$${}_{0}^{C}D_{t}^{\alpha}f(t)+\omega^{\alpha}f(t)=0. \tag{11}$$

where $t>0$, and $1<\alpha<2$, subjected to the initial conditions [21]: $f(0)=c$, and $f'(0)=0$.
Solution: Taking the RT of ${}_{0}^{C}D_{t}^{\alpha}f(t)+\omega^{\alpha}f(t)=0$,we get

$$R\left\{{}_{0}^{C}D_{t}^{\alpha}f(t)\right\}+\omega^{\alpha}R\left\{f(t)\right\}=0. \tag{12}$$

Since $R{}_{0}^{C}D_{t}^{\alpha}f(t)=q^{\alpha}F(q)-\sum_{k=0}^{n-1}q^{\alpha-k+2}f^{k}(0)$ and $(n-1)<\alpha<n$, therefore, (12) becomes

$$q^{\alpha}F(q)-\sum_{k=0}^{1}q^{\alpha-k+2}f^{k}(0)\;+\omega^{\alpha}\;F(q)=0,$$

$$q^{\alpha}F(q)-q^{\alpha+2}c-q^{\alpha+1}f'(0)+\omega^{\alpha}\;F(q)=0,$$

$$q^{\alpha}F(q)-q^{\alpha+2}c+\omega^{\alpha}\;F(q)=0,$$

$$F(q)=\frac{q^{\alpha+2}c}{q^{\alpha}+\omega^{\alpha}}. \tag{13}$$

As $R^{-1}\left\{\frac{q^{\alpha-b+3}}{q^{\alpha}-\sigma}\right\} = t^{b-1}E_{\alpha,\ b}\ (\sigma t^{a})$, applying inverse RT to (13), we have

$$f\left(t\right) = C\ t^{1-1}E_{\alpha,\ 1}\ \left(-\omega^{\alpha}t^{\alpha}\right),$$
$$f\left(t\right) = C\ E_{\alpha,\ 1}\ \left(-\omega^{\alpha}t^{\alpha}\right). \tag{14}$$

Equation (14) illustrates the behavior of an undamped-simple fractional oscillator. For a classical non-fractional oscillator, consider: $\alpha = 2$, then from (14), we have

$$f\left(t\right) = C\ E_{2,\ 1}\ \left(-\omega^{2}t^{2}\right) = C\ cos\omega t. \tag{15}$$

For $C = 1$ and $\omega = 314$, the graph of (15) is shown in Figure 1.

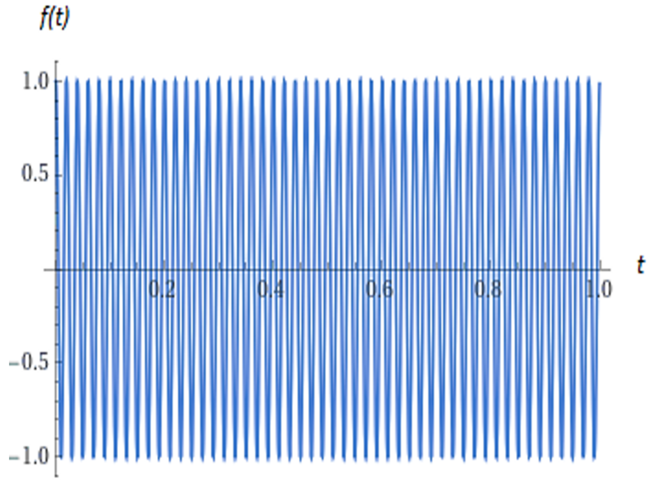

Figure 1. Numerical solution of an undamped-simple non-fractional oscillator

Example 2: Consider the undamped-driven fractional oscillator system involving the Caputo fractional derivative of the form: ${}_{0}^{C}D_{t}^{\alpha}f\left(t\right) + \omega^{\alpha}$

$$f\left(t\right) = h\left(t\right). \tag{16}$$

where $t > 0$, and $1 < \alpha < 2$, with the initial conditions: $f\left(0\right) = c, and\ f'\left(0\right) = 0$.
Solution: Taking the RT of ${}_{0}^{C}D_{t}^{\alpha}f\left(t\right) + \omega^{\alpha}f\left(t\right) = h(t)$, we get

$$R\left\{{}_{0}^{C}D_{t}^{\alpha}f\left(t\right)\right\} + \omega^{\alpha}R\left\{f\left(t\right)\right\} = Rh\left(x\right) \tag{17}$$

Since $R_{0}^{C}D_{t}^{\alpha}f(x) = q^{\alpha}F\left(q\right) - \sum_{k=0}^{n-1}q^{\alpha-k+2}f^{k}\left(0\right)$ and $\left(n-1\right) < \ \alpha < n$, therefore, (17) becomes

$$q^{\alpha}F\left(q\right) - \sum_{k=0}^{1}q^{\alpha-k+2}f^{k}\left(0\right)\ + \omega^{\alpha}\ F\left(q\right) = H(q),$$

$$q^{\alpha}F\left(q\right) - q^{\alpha+2}c - q^{\alpha+1}f'\left(0\right) + \omega^{\alpha}\ F\left(q\right) = H\left(q\right),$$
$$q^{\alpha}F\left(q\right) - q^{\alpha+2}c + \omega^{\alpha}\ F\left(q\right) = H(q),$$
$$F\left(q\right) = \frac{q^{\alpha+2}c}{q^{\alpha}+\omega^{\alpha}} + \frac{H\left(q\right)}{q^{a}+\omega^{\alpha}},$$
$$F\left(q\right) = \left\{\frac{q^{\alpha+2}c}{q^{\alpha}+\omega^{\alpha}}\right\} + \left\{\frac{q^{3}H\left(q\right)}{q^{3}(q^{\alpha}+\omega^{\alpha})}\right\}. \tag{18}$$

Applying inverse RT to (18), we get

$$f(t) = R^{-1}\left\{\frac{q^{\alpha+2}c}{q^{\alpha}+\omega^{\alpha}}\right\} + R^{-1}\left\{\frac{H(q)\,G(q)}{q^{3}}\right\}, \tag{19}$$

where $G(q) = \frac{q^3}{q^{\alpha}+\omega^{\alpha}}$ As $R^{-1}\left\{\frac{q^{\alpha-b+3}}{q^{\alpha}-\sigma}\right\} = t^{b-1}E_{\alpha,\,b}\ (\sigma t^{\alpha})$, therefore,

$$R^{-1}\{G(q)\} = g(t) = R^{-1}\left\{\frac{q^3}{q^{\alpha}+\omega^{\alpha}}\right\} = t^{a-1}E_{\alpha,\,\alpha}\ (-\omega^{\alpha}t^{\alpha}),$$

where $b = \alpha$ And $R^{-1}\left\{\frac{q^{\alpha+2}c}{q^{a}+\omega^{\alpha}}\right\} = c\ t^{1-1}E_{\alpha,\,1}\ (-\omega^{\alpha}t^{\alpha}) = cE_{\alpha,\,1}\ (-\omega^{\alpha}t^{\alpha})$, where $b = 1$. Hence, (19) becomes

$$f(t) = cE_{\alpha,\,1}\ (-\omega^{\alpha}t^{\alpha}) + R^{-1}\left\{\frac{H(q)\ G(q)}{q^{3}}\right\},$$

$$f(t) = cE_{\alpha,\,1}\ (-\omega^{\alpha}t^{\alpha}) + g(t) * h(t). \tag{20}$$

Since $g(t) * h(t) = \int_0^t g(t-\tau)^{h}(\tau)\,d\tau = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\,\alpha}\ (-\omega^{\alpha}(t-\tau)^{\alpha})\,h(\tau)\,d\tau$, therefore, (20) becomes

$$f(t) = cE_{\alpha,\,1}\ (-\omega^{\alpha}t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\,\alpha}\ (-\omega^{\alpha}(t-\tau)^{\alpha})\,h(\tau)\,d\tau. \tag{21}$$

Equation (21) illustrates the behavior of an undamped-driven fractional oscillator. For a classical non-fractional oscillator, consider:$\alpha = 2$, and let $h(t) = sin\omega t$, then from (21), we have

$$f(t) = cE_{2,\,1}\ \left(-\omega^2t^2\right) + \int_0^t (t-\tau)^{2-1} E_{\alpha,\,\alpha}\ \left(-\omega^2(t-\tau)^2\right)(sin\omega\tau)\,d\tau,$$

$$f(t) = c\ cos\omega t + \int_0^t (t-\tau)\,sin\omega(t-\tau)\,(sin\omega\tau)\,d\tau,$$

$$f(t) = c\ cos\omega t + \frac{1}{2}0tt - [cost - 2 - cost]d,$$

$$f(t) = c\ cos\omega t + \frac{1}{2}\int_0^t (t-\tau)\cos(\omega t - 2\omega\tau)d\tau - \frac{1}{2}\int_0^t (t-\tau)\cos(\omega t)d\tau. \tag{22}$$

Since $\int_0^t (t-\tau)\cos(\omega t - 2\omega\tau)d\tau = \frac{sin\omega t}{2\omega}$ and $\int_0^t (t-\tau)\cos(\omega t)d\tau = t^2\frac{cos\omega t}{2}$, therefore, from (22), we have

$$f(t) = c\ cos\omega t + \frac{sin\omega t}{4\omega} - \frac{t^2}{4}\ cos\omega t. \tag{23}$$

For $C = 1000$ and $\omega = 3.14$, the graph of (23) is shown in Figure 2.

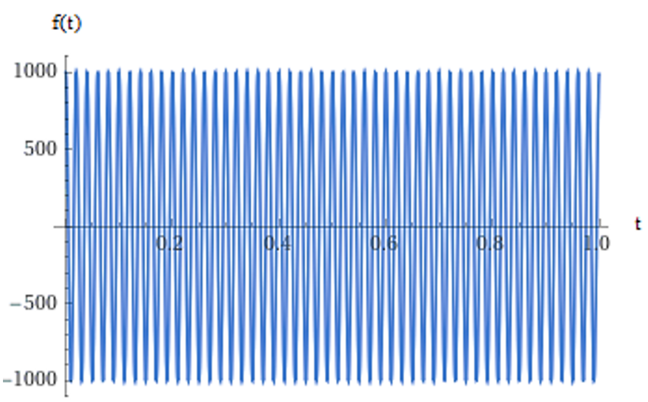

Figure 2. Numerical solution of an undamped-driven non-fractional oscillator

Example 3: Consider the damped-fractional oscillator involving the Caputo fractional derivative of the form:

$$ {}_0^C D_t^{\alpha} f(t) + A {}_0^C D_t^{\beta} f(t) + B f(t) = 0. \tag{24} $$

where $t > 0,\ f(0) = C$ and

$$ 1 < \alpha < 2,\ 0 < \beta < 1,\ f'(0) = D. $$

Solution: Taking the RT of ${}_0^C D_t^{\alpha} f(t) + A {}_0^C D_t^{\beta} f(t) + B f(t) = 0$, we get

$$ R\left\{{}_0^C D_t^{\alpha} f(t)\right\} + AR\left\{{}_0^C D_t^{\beta} f(t)\right\} + BRf(t) = 0. \tag{25} $$

Since $R {}_0^C D_t^{\alpha} f(t) = q^{\alpha} F(q) - \sum_{k=0}^{n-1} q^{\alpha-k+2} f^k(0)$ *and* $(n-1) < \alpha < n$, therefore, (25) becomes

$$ q^{\alpha} F(q) - \sum_{k=0}^{n-1} q^{\alpha-k+2} f^k(0) + Aq^{\beta} F(q) - A \sum_{k=0}^{n-1} q^{\beta-k+2} f^k(0) + BF(q) = 0, $$

$$ q^{\alpha} F(q) - \sum_{k=0}^{1} q^{\alpha-k+2} f^k(0) + Aq^{\beta} F(q) - A \sum_{k=0}^{0} q^{\beta-k+2} f^k(0) + BF(q) = 0, $$

$$ q^{\alpha} F(q) - q^{\alpha+2} C - q^{\alpha+1} D + Aq^{\beta} F(q) - Aq^{\beta+2} C + BF(q) = 0, $$

$$ F(q) = \frac{q^{\alpha+2} C + q^{\alpha+1} D + Aq^{\beta+2} C}{q^{\alpha} + Aq^{\beta} + B}. \tag{26} $$

Now, let us simplify the term: $\frac{1}{(q^{\alpha} + dq^{\beta} + b)}$ as follows:

$$ \frac{1}{(q^{\alpha} + dq^{\beta} + b)} = \frac{q^{-\beta}}{(q^{\alpha-\beta} + d)\left(1 + \frac{bq^{-\beta}}{q^{\alpha-\beta}+d}\right)}, $$

$$ \frac{1}{(q^{\alpha} + dq^{\beta} + b)} = \frac{q^{-\beta}}{(q^{\alpha-\beta} + d)} \frac{1}{\left(1 - \frac{-bq^{-\beta}}{q^{\alpha-\beta}+d}\right)}, $$

$$ \frac{1}{(q^{\alpha} + dq^{\beta} + b)} = \frac{q^{-\beta}}{(q^{\alpha-\beta} + d)} \sum_{k=0}^{\infty} \left(\frac{-bq^{-\beta}}{q^{\alpha-\beta} + d}\right)^k, $$

$$ \frac{1}{(q^{\alpha} + dq^{\beta} + b)} = \sum_{k=0}^{\infty} (-b)^k \frac{q^{-\beta k-\beta}}{(q^{\alpha-\beta} + d)^{k+1}}, $$

$$ \frac{1}{(q^{\alpha} + dq^{\beta} + b)} = \sum_{k=0}^{\infty} (-b)^k \frac{q^{-\beta k-\beta}}{(q^{\alpha-\beta})^{k+1} (1 + dq^{\beta-\alpha})^{k+1}}, $$

$$ \frac{1}{(q^{\alpha} + dq^{\beta} + b)} = \sum_{k=0}^{\infty} (-b)^k \frac{q^{-\beta k-\beta} (q^{\alpha-\beta})^{-k-1}}{(1 + dq^{\beta-\alpha})^{k+1}}, $$

$$ \frac{1}{(q^{\alpha} + dq^{\beta} + b)} = \sum_{k=0}^{\infty} (-b)^k \frac{q^{-\beta k-\beta} q^{-\alpha k+\beta k-\alpha+\beta}}{(1 + dq^{\beta-\alpha})^{k+1}}, $$

$$ \frac{1}{(q^{\alpha} + dq^{\beta} + b)} = \sum_{k=0}^{\infty} (-b)^k \frac{q^{-\alpha k-\alpha}}{(1 + dq^{\beta-\alpha})^{k+1}}, $$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\,q^{-\alpha k-\alpha}\sum_{r=0}^{\infty}\binom{k+1+r-1}{r}\left(-dq^{\beta-\alpha}\right)^{r},$$

$$\left[\because\ \frac{1}{(1\ -\ x)^{n}}=\sum_{r=0}^{\infty}\binom{n+r-1}{r}x^{r}, where\binom{n+r\ \ -1}{r}=(n+r\ \ -1)_{C_r}=\frac{(n\ +\ r\ -\ 1)!}{r!\ (n-1)!}\right]$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r}(q^{\beta r-\alpha r-\alpha k-\alpha}). \tag{27}$$

Using (27) in (26), we have

$$F\left(q\right)=\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r}(q^{\beta r-\alpha r-\alpha k-\alpha})\left[q^{\alpha+2}C+q^{\alpha+1}D+Aq^{\beta+2}C\right],$$

$$F\left(q\right)=\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r})(q^{\beta r-\alpha r-\alpha k+2})C+(q^{\beta r-\alpha r-\alpha k+1})D+A(q^{\beta r+\beta-\alpha r-\alpha k-\alpha+2})C. \tag{28}$$

As $R^{-1}\left(\frac{1}{q^{n-2}}\right)=R^{-1}\left(q^{2-n}\right)=\frac{t^{n}}{\Gamma(n+1)}$, or $R^{-1}\left(q^{z}\right)=\frac{t^{2-z}}{\Gamma(3-z)}$, applying inverse RT to (28), we have

$$f\left(t\right)=\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r}$$
$$\left(\frac{t^{\alpha r-\beta r+\alpha k}}{\Gamma\left(\alpha r-\beta r+\alpha k+1\right)}C+\frac{t^{\alpha r-\beta r+\alpha k+1}}{\Gamma\left(\alpha r-\beta r+\alpha k+2\right)}D+A\frac{t^{\alpha r-\beta r-\beta+\alpha k+\alpha}}{\Gamma\left(\alpha r-\beta r-\beta+\alpha k+\alpha+1\right)}C\right),$$

$$f(t)=\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\frac{(k+r)!}{r!k!}(-d)^{r}$$
$$\left(\frac{t^{\alpha r-\beta r+\alpha k}}{\Gamma(\alpha r-\beta r+\alpha k+1)}C+\frac{t^{\alpha r-\beta r+\alpha k+1}}{\Gamma(\alpha r-\beta r+\alpha k+2)}D+A\frac{t^{\alpha r-\beta r-\beta+\alpha k+\alpha}}{\Gamma(\alpha r-\beta r-\beta+\alpha k+\alpha+1)}C\right). \tag{29}$$

Equation (29) illustrates the behavior of a damped-fractional oscillator.

Example 4: Consider the damped-driven fractional oscillator involving the Caputo fractional derivative of the form:

$${}_{0}^{C}D_{t}^{\alpha}f\left(t\right)+A{}_{0}^{C}D_{t}^{\beta}f\left(t\right)+Bf\left(t\right)=h\left(t\right). \tag{30}$$

where $t>0$, $1<\alpha<2$, $0<\beta<1$, $f'\left(0\right)=D$ and $f\left(0\right)=C$.

Solution: Taking the RT of ${}_{0}^{C}D_{t}^{\alpha}f\left(t\right)+A{}_{0}^{C}D_{t}^{\beta}f\left(t\right)+Bf\left(t\right)=h(t)$, we get

$$R\left\{{}_{0}^{C}D_{t}^{\alpha}f\left(t\right)\right\}+AR\left\{{}_{0}^{C}D_{t}^{\beta}f\left(t\right)\right\}+BRf\left(t\right)=Rh\left(t\right). \tag{31}$$

Since $R{}_{0}^{C}D_{t}^{\alpha}f(t)=q^{\alpha}F\left(q\right)-\sum_{k=0}^{n-1}q^{\alpha-k+2}f^{k}\left(0\right)$ and $(n-1)<\ \alpha<n$, therefore, (31) becomes

$$q^{\alpha}F\left(q\right)-\sum_{k=0}^{n-1}q^{\alpha-k+2}f^{k}\left(0\right)+Aq^{\beta}F\left(q\right)-A\sum_{k=0}^{n-1}q^{\beta-k+2}f^{k}\left(0\right)+BF(q)=H(q),$$

$$q^{\alpha}F\left(q\right)-\sum_{k=0}^{1}q^{\alpha-k+2}f^{k}\left(0\right)+Aq^{\beta}F\left(q\right)-A\sum_{k=0}^{0}q^{\beta-k+2}f^{k}\left(0\right)+BF(q)=H(q),$$

$$q^{\alpha}F\left(q\right)-q^{\alpha+2}C-q^{\alpha+1}D+Aq^{\beta}F\left(q\right)-Aq^{\beta+2}C+BF(q)=H(q),$$

$$F\left(q\right)=\frac{q^{\alpha+2}C+q^{\alpha+1}D+Aq^{\beta+2}C}{q^{\alpha}+Aq^{\beta}+B}+\frac{H(q)}{q^{\alpha}+Aq^{\beta}+B}. \tag{32}$$

Now, let us simplify the term: $\frac{1}{(q^{\alpha}+dq^{\beta}+b)}$ as follows:

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\frac{q^{-\beta}}{(q^{\alpha-\beta}+d)\left(1+\frac{bq^{-\beta}}{q^{\alpha-\beta}+d}\right)},$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\frac{q^{-\beta}}{(q^{\alpha-\beta}+d)}\frac{1}{\left(1-\frac{-bq^{-\beta}}{q^{\alpha-\beta}+d}\right)},$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\frac{q^{-\beta}}{(q^{\alpha-\beta}+d)}\sum_{k=0}^{\infty}\left(\frac{-bq^{-\beta}}{q^{\alpha-\beta}+d}\right)^{k},$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\,\frac{q^{-\beta k-\beta}}{\left(q^{\alpha-\beta}+d\right)^{k+1}},$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\,\frac{q^{-\beta k-\beta}}{\left(q^{\alpha-\beta}\right)^{k+1}\left(1+dq^{\beta-\alpha}\right)^{k+1}},$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\,\frac{q^{-\beta k-\beta}\left(q^{\alpha-\beta}\right)^{-k-1}}{\left(1+dq^{\beta-\alpha}\right)^{k+1}},$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\,\frac{q^{-\beta k-\beta}q^{-\alpha k+\beta k-\alpha+\beta}}{\left(1+dq^{\beta-\alpha}\right)^{k+1}},$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\,\frac{q^{-\alpha k-\alpha}}{\left(1+dq^{\beta-\alpha}\right)^{k+1}},$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\,q^{-\alpha k-\alpha}\sum_{r=0}^{\infty}\binom{k+1+r-1}{r}\left(-dq^{\beta-\alpha}\right)^{r},$$

$$\left[\because\ \frac{1}{\left(1-x\right)^{n}}=\sum_{r=0}^{\infty}\binom{n+r-1}{r}x^{r},\text{where}\binom{n+r\ \ -1}{r}=(n+r-1)_{C_r}=\frac{(n+r-1)!}{r!(n-1)!}\right]$$

$$\frac{1}{(q^{\alpha}+dq^{\beta}+b)}=\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r}(q^{\beta r-\alpha r-\alpha k-\alpha}). \tag{33}$$

Using (33) in (32), we have

$$F\left(q\right)=\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r}(q^{\beta r-\alpha r-\alpha k-\alpha})\left[q^{\alpha+2}C+q^{\alpha+1}D+Aq^{\beta+2}C\right]+$$
$$\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r}(q^{\beta r-\alpha r-\alpha k-\alpha})H(q),$$

$$F\left(q\right)=\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r})(q^{\beta r-\alpha r-\alpha k+2})C+(q^{\beta r-\alpha r-\alpha k+1})D+$$
$$A(q^{\beta r+\beta-\alpha r-\alpha k-\alpha+2})C+\sum_{k=0}^{\infty}(-b)^{k}\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^{r}(q^{\beta r-\alpha r-\alpha k-\alpha})H(q),$$

$$F(q)=\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^r$$
$$\left(q^{\beta r-\alpha r-\alpha k+2}C+q^{\beta r-\alpha r-\alpha k+1}D+Aq^{\beta r+\beta-\alpha r-\alpha k-\alpha+2}C\right)$$
$$+\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^r\,\frac{G(q)H(q)}{q^3}. \tag{34}$$

where $G(q)=(q^{\beta r-\alpha r-\alpha k-\alpha+3})$ As $R^{-1}\left(\frac{1}{q^{n-2}}\right)=R^{-1}\left(q^{2-n}\right)=\frac{t^n}{\Gamma(n+1)}$, or $R^{-1}\left(q^z\right)=\frac{t^{2-z}}{\Gamma(3-z)}$, applying inverse RT to (34), we have

$$f(t)=\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^r$$
$$\left(\frac{t^{\alpha r-\beta r+\alpha k}}{\Gamma(\alpha r-\beta r+\alpha k+1)}C+\frac{t^{\alpha r-\beta r+\alpha k+1}}{\Gamma(\alpha r-\beta r+\alpha k+2)}D+A\frac{t^{\alpha r-\beta r-\beta+\alpha k+\alpha}}{\Gamma(\alpha r-\beta r-\beta+\alpha k+\alpha+1)}C\right)+$$
$$\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^r\;R^{-1}\left\{\frac{G(q)\;H(q)}{q^3}\right\},$$

$$f(t)=\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\frac{(k+r)!}{r!k!}(-d)^r$$
$$\left(\frac{t^{\alpha r-\beta r+\alpha k}}{\Gamma(\alpha r-\beta r+\alpha k+1)}C+\frac{t^{\alpha r-\beta r+\alpha k+1}}{\Gamma(\alpha r-\beta r+\alpha k+2)}D+\right.$$
$$\left.A\frac{t^{\alpha r-\beta r-\beta+\alpha k+\alpha}}{\Gamma(\alpha r-\beta r-\beta+\alpha k+\alpha+1)}C\right)+$$
$$\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^r\,g(t)*h(t). \tag{35}$$

Since $R^{-1}\{G(q)\}=g(t)=R^{-1}(q^{\beta r-\alpha r-\alpha k-\alpha+3})=\frac{t^{-\beta r+\alpha r+\alpha k+\alpha-1}}{\Gamma(-\beta r+\alpha r+\alpha k+\alpha)}$ and $g(t)*h(t)=\int_0^t g(t-\tau)^h(\tau)\,d\tau=\int_0^t\frac{(t-\tau)^{-\beta r+\alpha r+\alpha k+\alpha-1}}{\Gamma(-\beta r+\alpha r+\alpha k+\alpha)}h(\tau)\,d\tau$, therefore, (35) becomes

$$f(t)=\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\frac{(k+r)!}{r!k!}(-d)^r$$
$$\left(\frac{t^{\alpha r-\beta r+\alpha k}}{\Gamma(\alpha r-\beta r+\alpha k+1)}C+\frac{t^{\alpha r-\beta r+\alpha k+1}}{\Gamma(\alpha r-\beta r+\alpha k+2)}D+\right.$$
$$\left.A\frac{t^{\alpha r-\beta r-\beta+\alpha k+\alpha}}{\Gamma(\alpha r-\beta r-\beta+\alpha k+\alpha+1)}C\right)+$$
$$\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-d)^r\int_0^t\frac{(t-\tau)^{-\beta r+\alpha r+\alpha k+\alpha-1}}{\Gamma(-\beta r+\alpha r+\alpha k+\alpha)}h(\tau)d\tau. \tag{36}$$

Equation (36) illustrates the behavior of a damped-driven fractional oscillator.

CONCLUSION

The paper explores the application of the Rohit transform technique to solve fractional systems, including undamped-simple fractional oscillators, undamped-driven fractional oscillators, damped-driven fractional oscillators, and damped fractional oscillators. These systems are described by fractional differential equations, which include the Caputo-type fractional derivative operator. The behavior of these systems is expressed in terms of the Mittag-Leffler function. The solutions obtained by the integral Rohit transform technique are the same as those obtained by methods available in the literature [10]–[15]. The advantage of the integral Rohit transform is that it involves a simple formulation and less calculation than the methods [10]–[15].

The integral Rohit transform technique has solved many initial value problems in the areas of science, engineering, geology, and economics. In future, it will be applied in the area of cryptography to increase security in communication systems.

SUPPLEMENTARY MATERIAL

None.

AUTHOR CONTRIBUTIONS

In the manuscript, the conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation and original draft preparation have been done together by the authors: Rohit Gupta and Rahul Gupta. The review and editing, visualization, supervision and project administration have been done by the author Dinesh Verma. All authors have read and agreed to the published version of the manuscript.

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The authors declare no conflicts of interest.

REFERENCES

[1] S. Bayin, *Mathematical Methods in Science and Engineering*. United States: John Wiley & Sons, Inc., 2006, ISBN: 978-0-470-04142-0.

[2] R. Hilfer, *Applications of Fractional Calculus in Physics*. Singapore: World Scientific, 2000. doi: 10.1142/3779.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Netherlands: Elsevier, 2006, ISBN: 978-0-444-51832-3.

[4] T. Kisela, *Fractional Differential Equations and Their Applications*. BRNO University of Technology, 2008.

[5] S. Liang, R. Wu, and L. Chen, "Laplace transform of fractional order differential equations," *Electronic Journal of Differential Equations*, vol. 2015, no. 139, pp. 1–15, 2015.

[6] P. L. Butzer and U. Westphal, *An Introduction to Fractional Calculus*. Singapore: World Scientific Press, 2000. doi: 10.1142/9789812817747_0001.

[7] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Yverdon: Gordon and Breach, 1993, ISBN: 9782881248641.

[8] J. L. Schi, *The Laplace Transform: Theory and Applications*. New York: Springer, 1999.

[9] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier, 2006, ISBN: 978-0-444-51832-3.

[10] Y. O. El-Dib, "Immediate solution for fractional nonlinear oscillators using the equivalent linearized method," *Journal of Low Frequency Noise, Vibration and Active Control*, vol. 41, no. 4, pp. 1411–1425, 2022. doi: 10.1177/14613484221098788.

[11] A. Al-Rabtah, V. S. Erturk, and S. Momani, "Solutions of a fractional oscillator using differential transform method," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1356–1362, 2010. doi: 10.1016/j.camwa.2009.06.036.

[12] A. Arikoglu and I. Ozkol, "Solution of fractional differential equations using differential transform method," *Chaos, Solitons & Fractals*, vol. 34, no. 5, pp. 1473–1481, 2007. doi: 10.1016/j.chaos.2006.09.004.

[13] O. Abdulaziz, I. Hashim, and S. Momani, "Application of homotopy-perturbation method to fractional initial value problems," *Journal of Computational and Applied Mathematics*, vol. 216, no. 2, pp. 574–584, 2008. doi: 10.1016/j.cam.2007.06.010.

[14] S. A. El-Wakil, A. Elhanbaly, and M. A. Abdou, "Adomian decomposition method for solving fractional nonlinear differential equations," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 313–324, 2006. doi: 10.1016/j.amc.2006.02.055.

[15] G.-C. Wu, "A fractional variational iteration method for solving fractional nonlinear differential equations," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 2186–2190, 2011. doi: 10.1016/j.camwa.2010.09.010.

[16] R. Gupta, "On novel integral transform: Rohit transform and its application to boundary value problems," *ASIO Journal of Chemistry, Physics, Mathematics and Applied Sciences (ASIO-JCPMAS)*, vol. 4, no. 1, pp. 08–13, 2020. [Online]. Available: http://doi-ds.org/doilink/06.2020-62339259/.

[17] R. Gupta, I. Singh, and A. Sharma, "Response of an undamped forced oscillator via rohit transform," *International Journal of Emerging Trends in Engineering Research*, vol. 10, no. 8, pp. 396–400, 2022. doi: 10.30534/ijeter/2022/031082022.

[18] R. Gupta, "Mechanically persistent oscillator supplied with ramp signal," *Al-Salam Journal for Engineering and Technology*, vol. 2, no. 2, pp. 112–115, 2023. doi: 10.55145/ajest.2023.02.02.014.

[19] R. Gupta, R. Gupta, and D. Verma, "Response of rlc network circuit with steady source via rohit transform," *International Journal of Engineering, Science and Technology*, vol. 14, no. 1, pp. 21–27, 2022. doi: 10.4314/ijest.v14i1.3.

[20] L. Talwar and R. Gupta, "Analysis of electric network circuits with sinusoidal potential sources via rohit transform," *International Journal of Advanced Research in Electrical, Electronics and Instrumentation Engineering*, vol. 9, no. 11, pp. 3929–3023, 2020.

[21] Y. E. Ryabov and A. Puzenko, "Damped oscillations in view of the fractional oscillator equation," *Physical Review B*, vol. 66, p. 184 201, 2002. doi: 10.1103/PhysRevB.66.184201.