2025,Volume 2 , Issue 7 , PSIJK (16-22)

Pure Sciences International Journal of Kerbala

Use the q-Integral Operator that Implicitly Contain the q-Ruscheweyh Derivative in a New Class of Analytic Univalent Functions Described by Some of their Finite Negative Invariant Coefficients

Hazha Zirar Hussain [a*]

[a] *Department of Mathematics, College of Sciences, Salahaddin University-Erbil, Iraq*

P A P E R   I N F O

Received: 09.04.2025
Accepted: 23.06.2025
Published: 30.09.2025

Keywords:
Analytic Functions, Univalent Functions, q-Ruscheweyh Derivative Operator, q-Integral Operator,Convolution.

A B S T R A C T

In this article, we study some of the basic geometric properties involved in finding an estimate or determining the value of the coefficients on the basis that the function is characterized by the starlike and convexity of the order ϑ, respectively. In addition to other properties, all this is done by defining a new class of analytic univalent functions by applying a quantum integral that implicitly contains a Ruscheweyh's quantum derivative to this special class, described by some of its non-variable coefficients.

DOI: 10.53851/psijk.v2.i7.16 -22

1. INTRODUCTION

The research starts by assuming the class $\mathfrak{A}$, which is the set of all the analytic univalent functions that have the following natural form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \qquad (1.1)$$

By fulfilling the normalized conditions $f(0) = 0, f'(0) = 1$, all of this is in the famous unit disc $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$.

Now we will take a subclass $\mathfrak{A}^*$of our original class such that each function has the following form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, (a_k \geq 0). \qquad (1.2)$$

Since the quantum derivative operator or Jackson derivative operator plays an important and effective role in this field of complex analysis, especially recently, many researchers have used this operator in their research and obtained important and interesting results such as (Illafe, Mohd, Yousef, & Supramaniam, 2024) (Bangi, 2024) (Alsoboh & Oros, 2024) (Al-Rawashdeh, 2025) (Abubaker, Matarneh, Khan, Suha, & Kamal, 2024) (Delphi, 2024)

Let us not forget that the first person who defined this operator was Jackson in his researches (Jackson, 1905) (Jackson, 1909) (Jackson, 1910a) (Jackson, Fukuda, Dunn, et al., 1910b) where he studied the q calculation and its applications in his researches.

Let us remember the definition of the involutional product of two functions f defined by (1.1) and j defined as follows:

$$j(z) = z + \sum_{k=2}^{\infty} b_k z^k, \qquad (1.3)$$

in class $\mathfrak{A}$, it is symbolized by the symbol $f * j$ and is known as follows:

$$(f * j)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \qquad (1.4)$$

Now I need to recall the concept of the q-number $[\zeta]_q$ as follows:

$$[\zeta]_q = \frac{1 - q^{\zeta}}{1 - q}, \qquad (1.5)$$

*Corresponding Author Institutional Email:
hazha.hussain@su.edu.krd (Hazha Zirar Hussain)

if ζ is complex number, and

$$[\zeta]_{\mathrm{q}} = 1 + \sum_{k=1}^{m-1} \mathrm{q}^k, \qquad (1.6)$$

if $\zeta = m$ is natural number.

Based on what we have mentioned $[0]_{\mathrm{q}} = 0$, on condition $0 < \mathrm{q} < 1$.

Under the same condition we take the definition of the q-factorial $[k]_{\mathrm{q}}!$ as

$$[k]_{\mathrm{q}}! = \begin{cases} 1 & if\ k = 0 \\ \prod_{m=1}^{k} [m]_{\mathrm{q}} & if\ k \in \mathbb{N} \end{cases}$$

$[s]_{\mathrm{q},k}!$ is also known as

$$[s]_{\mathrm{q},k}! = \begin{cases} 1 & if\ k = 0 \\ \prod_{m=s}^{s+k-1} [m]_{\mathrm{q}} & if\ k \in \mathbb{N} \end{cases}$$

and that is when $s \in \{0, 1, 2, \dots.\}$.

Likewise, Jackson's quantum derivative for a function belonging to the class $\mathfrak{A}$ under the same condition is usually written in the following form:

$$\mathfrak{D}_{\mathrm{q}} f(z) = \begin{cases} \dfrac{f(z) - f(\mathrm{q}z)}{(1-\mathrm{q})z} & if\ z \neq 0 \\ f'(z) & if\ z = 0 \end{cases}$$

it is certainly true when $f'(z)$ exists.In addition we have:

$$(\mathfrak{D}_{\mathrm{q}}^2 f)(z) = \big(\mathfrak{D}_{\mathrm{q}}(\mathfrak{D}_{\mathrm{q}} f)\big)(z). \qquad (1.7)$$

In Source No. [11], Ruscheweyh's q-derivative of a function in the class $\mathfrak{A}$ is defined in the following way:

$$\mathcal{R}_{\mathrm{q},\zeta} f(z) = f(z) * \left(z + \sum_{k=2}^{\infty} \frac{[\zeta+1]_{\mathrm{q},k-1}}{[k-1]_{\mathrm{q}}!} z^k \right), \quad \zeta > -1. \qquad (1.8)$$

More over, in the research in sources No. (Srivastava, Wanas, & Srivastava, 2021) (Srivastava, Ahmad, Khan, Khan, & Khan, 2019) the q integration operator for a function in the class $\mathfrak{A}$ was defined by:

$$\mathcal{R}_{\mathrm{q},\zeta}^{-1} f(z) + \mathcal{R}_{\mathrm{q},\zeta}(z) = z\big(\mathfrak{D}_{\mathrm{q}} f(z)\big), \qquad (1.9)$$

$$\mathcal{R}_{\mathrm{q},\zeta}^{-1} f(z) = z + \sum_{k=2}^{\infty} \frac{[k]_{\mathrm{q}}!}{[\zeta+1]_{\mathrm{q},k-1}} z^k, k \geq 2. \qquad (1.10)$$

A special case of it was studied in source No. (Noor, 1999) when $\mathrm{q} \to 1^-$.

The q integral operator $\mathfrak{I}_{\mathrm{q}}^{\zeta}$ in source [13] was defined by the following formula:

$$\mathfrak{I}_{\mathrm{q}}^{\zeta} f(z) = f(z) * \mathcal{R}_{\mathrm{q},\zeta}^{-1} f(z) = z + \sum_{k=2}^{\infty} \frac{[k]_{\mathrm{q}}!}{[\zeta+1]_{\mathrm{q},k-1}} a_k z^k. \qquad (1.11)$$

The following relations are also true.

$$\mathfrak{I}_{\mathrm{q}}^{\zeta} f(z) = z\big(\mathfrak{D}_{\mathrm{q}} f(z)\big), \qquad (1.12)$$

$$\mathfrak{I}_{\mathrm{q}}^{1} f(z) = f(z). \qquad (1.13)$$

Now we come to the heart of the article, which is defining our class as a first step, as follows:

Definition 1.1: If we determine or prove the conditions , $\zeta > -1, 0 < \mathrm{q} < 1,\ \chi \in \mathbb{C}\backslash\{0\}, 0 < \upsilon \leq 1,\ \phi \leq 1$, in the light of this, the function$f \in \mathfrak{A}^*$, is an element of the class $\mathfrak{H}_{\mathrm{q}}^{\zeta}(\chi, \phi, \nu)$ if the following inequality is satisfied:

$$\left| \frac{1}{\chi} \left(\frac{z\mathfrak{D}_{\mathrm{q}}\left(\mathfrak{I}_{\mathrm{q}}^{\zeta} f(z)\right)}{\mathfrak{I}_{\mathrm{q}}^{\zeta} f(z)} - \phi \right) \right| < \nu, \qquad (1.14)$$

and don't forget that z is in the open unit disk.

2. DISCUSSION OF THE EMERGING RESULTS

In this part of the article, we present and discuss the most important results and facts that have been reached and proven by first studying the characteristics and determining the values of the coefficients in the following theorem:

Theorem 2.1: The function f, which is formulated in the form (1.2), is an element of the distinct class $\mathfrak{H}_{\mathrm{q}}^{\zeta}(\chi, \phi, \nu)$ if and only if the following condition is met:

$$\sum_{k=2}^{\infty} \frac{\big([k]_{\mathrm{q}} - \phi + \upsilon|\chi|\big)\, [k]_{\mathrm{q}}!}{[\zeta+1]_{\mathrm{q},k-1}} a_k \leq (1 - \phi + \upsilon|\chi|), \qquad (2.1)$$

that's when $\zeta > -1, 0 < \mathrm{q} < 1,\ \chi \in \mathbb{C}\backslash\{0\}, 0 < \upsilon \leq 1,\ \phi \leq 1$.

Proof: The first step to start the proof is to take f in the defined class $\mathfrak{H}_{\mathrm{q}}^{\zeta}(\chi, \phi, \nu)$. So

$$\mathcal{R}e\left(\frac{z\mathfrak{D}_{\mathrm{q}}\left(\mathfrak{I}_{\mathrm{q}}^{\zeta} f(z)\right)}{\mathfrak{I}_{\mathrm{q}}^{\zeta} f(z)} - \phi \right) > -\upsilon|\chi|. \qquad (2.2)$$

Let's take the quantity on the left side of the previous equation and simplify it as follows:

$$\frac{z\mathfrak{D}_{\mathrm{q}}\left(\mathfrak{I}_{\mathrm{q}}^{\zeta} f(z)\right)}{\mathfrak{I}_{\mathrm{q}}^{\zeta} f(z)} = \frac{z - \sum_{k=2}^{\infty}[k]_{\mathrm{q}} \frac{[k]_{\mathrm{q}}!}{[\zeta+1]_{\mathrm{q},k-1}} a_k z^k}{z - \sum_{k=2}^{\infty} \frac{[k]_{\mathrm{q}}!}{[\zeta+1]_{\mathrm{q},k-1}} a_k z^k}. \qquad (2.3)$$

Consequently,

$$\mathcal{R}e\left(\frac{(1-\phi)z - \sum_{k=2}^{\infty}\big([k]_{\mathrm{q}} - \phi\big) \frac{[k]_{\mathrm{q}}!}{[\zeta+1]_{\mathrm{q},k-1}} a_k z^k}{z - \sum_{k=2}^{\infty} \frac{[k]_{\mathrm{q}}!}{[\zeta+1]_{\mathrm{q},k-1}} a_k z^k} \right) > -\upsilon|\chi|. \qquad (2.4)$$

Now we give the variable z only the real value, i.e. on the real axis and approaching one from the left, and thus we complete the proof of the first step.

The second step of the proof is to prove the fact that inequality (2.1) is true and we start by taking

$$\left|\frac{z\mathfrak{D}_{q}\left(\mathfrak{I}_{q}^{\zeta}f(z)\right)}{\mathfrak{I}_{q}^{\zeta}f(z)}-\phi\right|$$
$$=\left|\frac{(1-\phi)z-\sum_{k=2}^{\infty}([k]_{q}-\phi)\frac{[k]_{q}!}{[\zeta+1]_{q,k-1}}a_{k}z^{k}}{z-\sum_{k=2}^{\infty}\frac{[k]_{q}!}{[\zeta+1]_{q,k-1}}a_{k}z^{k}}\right|$$
$$\leq\frac{(1-\phi+\upsilon|\chi|)\left(1-\sum_{k=2}^{\infty}[k]_{q}\frac{[k]_{q}!}{[\zeta+1]_{q,k-1}}a_{k}\right)}{1-\sum_{k=2}^{\infty}\frac{[k]_{q}!}{[\zeta+1]_{q,k-1}}a_{k}}$$
$$\leq 1-\phi+\upsilon|\chi|. \qquad (2.5)$$

Here lies the importance of the maximum modulus theorem in its application and obtaining that f is an element of the distinct class $\mathfrak{H}_{q}^{\zeta}(\chi,\phi,\nu)$.

Thus, we have successfully demonstrated the proof of the theorem.

We note that for the extreme functions in the special class $\mathfrak{H}_{q}^{\zeta}(\chi,\phi,\nu)$, it can be written in the form:

$$f(z)=z-\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}}{([k]_{q}-\phi+\upsilon|\chi|)\,[k]_{q}!}z^{k},k\geq 2. \qquad (2.6)$$

That is, the theorem is sharp for them, and thus we obtain the following clear result.

Corollary: If we take the function f, which has the formula (1.2), in the special class $\mathfrak{H}_{q}^{\zeta}(\chi,\phi,\nu)$, its coefficients satisfy the following condition:

$$a_{k}\leq\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}}{([k]_{q}-\phi+\upsilon|\chi|)\,[k]_{q}!},k\geq 2. \qquad (2.7)$$

As we have previously noted, the extreme functions achieve the equality condition for inequality (2.7).

Now we come to the main goal of this article, which is to define a basic subclass of the class $\mathfrak{H}_{q}^{\zeta}(\chi,\phi,\nu)$, by fixing a finite number of coefficients of its functions defined in the form (1.2), i.e. with a finite number of fixed coefficients in the following form:

Definition 2.3: Let $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_{m})$ be a subclass of the distinct class $\mathfrak{H}_{q}^{\zeta}(\chi,\phi,\nu)$, each of its elements or rather its functions, is written in the following form:

$$f(z)=z-\sum_{l=2}^{m}\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}\tau_{l}}{([l]_{q}-\phi+\upsilon|\chi|)\,[l]_{q}!}z^{l}-\sum_{k=m+1}^{\infty}a_{k}z^{k}. \qquad (2.8)$$

The class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_{m})$ is equivalent to the class $\mathfrak{H}_{q}^{\zeta}(\chi,\phi,\nu)$ when$\tau_{m}=1$.

As we see in sources (Lashin, Badghaish, & Alshehri, 2024) (Maran, Juma, & Al-Saphory, 2025) (Al-Saleem, 2025) (Akyar, 2025) (Al-Hawary, Frasin, & Salah, 2025) various classes of univalent functions that have the property of negative coefficients and fixing a finite number of them have been studied.This is an important step in serving the development of geometric field in mathematics, the queen of sciences.

From the series of results, we have the following theorem.

Theorem 2.4: The function f of the form (1.2) is an element of the class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_{m})$ if and only if the following condition is met:

$$\sum_{k=m+1}^{\infty}\frac{([k]_{q}-\phi+\upsilon|\chi|)\,[k]_{q}!}{[\zeta+1]_{q,k-1}}a_{k}\leq(1-\phi+\upsilon|\chi|)\left(1-\sum_{l=2}^{m}\tau_{l}\right). \qquad (2.9)$$

Proof: We use the important fact in Theorem 2.1 . In addition, we take

$$a_{k}=\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}\tau_{l}}{([k]_{q}-\phi+\upsilon|\chi|)\,[k]_{q}!},k=2,3,\dots m \qquad (2.10)$$

to arrive at

$$\sum_{l=2}^{m}\tau_{l}-\sum_{k=m+1}^{\infty}\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}}{([k]_{q}-\phi+\upsilon|\chi|)\,[k]_{q}!}a_{k}\leq 1, \qquad (2.11)$$

which leads us to the proof of the theorem easily and simply.

The result in the inequality (2.9) is strict for a function of the form

$$f(z)=z-\sum_{l=2}^{m}\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}\tau_{l}}{([l]_{q}-\phi+\upsilon|\chi|)\,[l]_{q}!}z^{l}-\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_{l})}{([k]_{q}-\phi+\upsilon|\chi|)\,[k]_{q}!}z^{k}, \qquad (2.12)$$

while the value of k starts from $m+1$ and larger.

Corollary 2.5: If f is an element of the class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_{m})$, then its coefficients are met:

$$a_{k}\leq\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_{l})}{([k]_{q}-\phi+\upsilon|\chi|)\,[k]_{q}!}, \qquad (2.13)$$

as we mentioned when $k\geq m+1$.

Let us pause for a moment and observe and test the convexity of the class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_{m})$ by studying and applying the convexity theories and thus stating and proving the validity and truth of the opposite of the case, all within proposal of the following theorem.

Theorem 2.6: A function f is an element of the class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_m)$ if and only if it can be written in the form

$$f(z) = \sum_{k=m}^{\infty} \eta_k f_k, \quad (2.14)$$

under the condition

$$\sum_{k=m}^{\infty} \eta_k = 1, \ \eta_k \geq 0 \quad (2.15)$$

and take the functions

$$f_m(z) = z - \sum_{l=2}^{m} \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\big([l]_q-\phi+\upsilon|\chi|\big)\,[l]_q!} z^l \ . \quad (2.16)$$

$$f_k(z) = z - \sum_{l=2}^{m} \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\big([l]_q-\phi+\upsilon|\chi|\big)\,[l]_q!} z^l - \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)}{\big([k]_q-\phi+\upsilon|\chi|\big)\,[k]_q!} z^k, \quad (2.17)$$

for all values of k starting from $m+1$ onwards.

Proof: First, we prove the first part by taking the function f, which can be written as follows:

$$f(z) = \sum_{k=m}^{\infty} \eta_k f_k. \quad (2.18)$$

So, in other words, it can be written as

$$f(z) = z - \sum_{l=2}^{m} \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}\tau_l}{\big([l]_q-\phi+\upsilon|\chi|\big)\,[l]_q!} z^l - \sum_{k=m+1}^{\infty} \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)}{\big([k]_q-\phi+\upsilon|\chi|\big)\,[k]_q!} z^k \ . \quad (2.19)$$

We call Theorem 2.4 and conclude the following

$$\sum_{k=m+1}^{\infty} \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)\,\eta_k}{\big([k]_q-\phi+\upsilon|\chi|\big)\,[k]_q!} \cdot \frac{\big([k]_q-\phi+\upsilon|\chi|\big)\,[k]_q!}{[\zeta+1]_{q,k-1}}$$
$$= \left(1-\sum_{l=2}^{m}\tau_l\right)(1-\phi+\upsilon|\chi|)(1-\eta_m) \leq \left(1-\sum_{l=2}^{m}\tau_l\right)(1-\phi+\upsilon|\chi|), \quad (2.20)$$

This gives us our desired goal, which is the truth that f is an element of the class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_m)$.

To complete the proof, we come to the second part and take $f \in \mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_m)$. Now since we are in Corollary 2.6, we proved that:

$$a_k \leq \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)}{\big([k]_q-\phi+\upsilon|\chi|\big)\,[k]_q!}, \quad (2.21)$$

when $k \geq m+1$.

$$\eta_k = \frac{\big([k]_q-\phi+\upsilon|\chi|\big)\,[k]_q!}{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)}. \quad (2.22)$$

We also put

$$\eta_m = \left(1-\sum_{k=m+1}^{\infty}\eta_k\right).$$

Consequently, it can be written

$$f(z) = \sum_{k=m}^{\infty} \eta_k f_k,$$

and once we complete the proof of this part, we have completed the proof of the theorem.

The proof of the previous theorem lies in our ability to determine the extreme points of the wonderful class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_m)$ and thus study the properties of these distinctive points. In fact , they are the same functions shown in equations (2.16}) and (2.17) in the statement of the theorem, which we will clearly shown in the following result.

Theorem 2.7: The functions

$$f_m(z) = z - \sum_{l=2}^{m} \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\big([l]_q-\phi+\upsilon|\chi|\big)\,[l]_q!} z^l \ . \quad (2.23)$$

$$f_k(z) = z - \sum_{l=2}^{m} \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\big([l]_q-\phi+\upsilon|\chi|\big)\,[l]_q!} z^l - \frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)}{\big([k]_q-\phi+\upsilon|\chi|\big)\,[k]_q!} z^k, \quad (2.24)$$

for all values of k starting from $m+1$ onwards, are exactly the extreme points of the wonderful class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_m)$.

I would like to point out that I have noticed through my review of previous research that some of them take the summation from m to ∞ when determining the extreme points $f(k)$, and this is wrong. At first I decided to point out some of this research, but I eventually backed down and wrote it as note only.

3. IDENTIFICATION OF Q-STARLIKE AND Q-CONVEX REGIONS OF ORDER $\psi(0 \leq \psi < 1)$ FOR THE CHARACTERISTIC CLASS $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_m)$ WITHIN THE UNIT DISK $\mathcal{U}$

In this part of the article we show the regions where the functions of the defined class $\mathfrak{H}\mathfrak{B}_{q}^{\zeta}(\chi,\phi,\nu,\tau_m)$ are q-starlike and q-curved in our attempt to derive and determine the radius of these regions shown in the following two theorems.

Theorem 3.1: If we take the function f in the class $\mathfrak{HB}_q^{\zeta}(\chi,\phi,\nu,\tau_m)$, it has a q-starlike property of order $\psi(0\le\psi<1)$ in the disk $|z|<\mathfrak{r}^*(\chi,\phi,\nu,\tau_m,\psi)$, and $\mathfrak{r}^*(\chi,\phi,\nu,\tau_m,\psi)$ is the largest value that satisfies the property

$$\sum_{l=2}^{m}\left([l-1]_q+1-\psi\right)\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\left([l]_q-\phi+\upsilon|\chi|\right)[l]_q!}\,\mathfrak{r}^{l-1}+\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)\left([k]_q-\psi+1\right)}{\left([k]_q-\phi+\upsilon|\chi|\right)[k]_q!}\,\mathfrak{r}^{k-1}\le 1-\psi. \quad (3.1)$$

The extreme functions is

$$f(z)=z-\sum_{l=2}^{m}\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\left([l]_q-\phi+\upsilon|\chi|\right)[l]_q!}\,z^l-\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)}{\left([k]_q-\phi+\upsilon|\chi|\right)[k]_q!}\,z^k. \quad (3.2)$$

The theorem acquires the limiting property in it.

Proof: In order for the function $f\in\mathfrak{HB}_q^{\zeta}(\chi,\phi,\nu,\tau_m)$ have the q-starlike property of order $\psi(0\le\psi<1)$, we start by taking the quantity

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le\frac{\sum_{l=2}^{m}[l-1]_q\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{([l]_q-\phi+\upsilon|\chi|)\,[l]_q!}\mathfrak{r}^{k-1}+\sum_{k=m+1}^{\infty}[k-1]_q a_k\mathfrak{r}^{k-1}}{1-\sum_{l=2}^{m}\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{([l]_q-\phi+\upsilon|\chi|)\,[l]_q!}\mathfrak{r}^{k-1}-\sum_{k=m+1}^{\infty}a_k\mathfrak{r}^{k-1}}. \quad (3.3)$$

It must be

$$\le 1-\psi,$$

by setting $|z|\le\mathfrak{r}$, so we have

$$\sum_{l=2}^{m}\left([l-1]_q+1-\psi\right)\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\left([l]_q-\phi+\upsilon|\chi|\right)[l]_q!}\,\mathfrak{r}^{l-1}+\sum_{k=m+1}^{\infty}\left([k-1]_q-\psi+1\right)a_k\mathfrak{r}^{k-1}\le 1-\psi. \quad (3.4)$$

By referring to Theorem 2.4, we have the important fact

$$a_k=\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)\eta_k}{\left([k]_q-\phi+\upsilon|\chi|\right)[k]_q!}, \quad (3.5)$$

that's when $\eta_k\ge 0,\sum_{k=3}^{\infty}\eta_k\le 1, k\ge 3$.

We determine the value of $k^*=k^*(\mathfrak{r})$ so that the quantity

$$\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k^*-1}\left([k^*]_q-\psi+1\right)}{\left([k^*]_q-\phi+\upsilon|\chi|\right)[k^*]_q!}\,\mathfrak{r}^{k^*-1}, \quad (3.6)$$

has its highest value by fixing the value of $\mathfrak{r}$ in it.

After that we have

$$\sum_{k=m+1}^{\infty}\left([k-1]_q-\psi+1\right)a_k\mathfrak{r}^{k-1}\le\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)\left([k^*]_q-\psi+1\right)}{\left([k^*]_q-\phi+\upsilon|\chi|\right)[k^*]_q!}\,\mathfrak{r}^{k^*-1}. \quad (3.7)$$

Finally, we have determined the region where the functions in the class $\mathfrak{HB}_q^{\zeta}(\chi,\phi,\nu,\tau_m)$ are q-starlike of order $\psi(0\le\psi<1)$ which is $|z|<r\,(\chi,\phi,\nu,\tau_m,\psi)$, provided that

$$\sum_{l=2}^{m}\left([l-1]_q+1-\psi\right)\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\left([l]_q-\phi+\upsilon|\chi|\right)[l]_q!}\,\mathfrak{r}^{l-1}+\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k^*-1}(1-\sum_{l=2}^{m}\tau_l)\left([k^*]_q-\psi+1\right)}{\left([k^*]_q-\phi+\upsilon|\chi|\right)[k^*]_q!}\,\mathfrak{r}^{k^*-1}\le 1-\psi. \quad (3.8)$$

We can now determine the value of $\mathfrak{r}^*=\mathfrak{r}^*(\chi,\phi,\nu,\tau_m,\psi)$ and the value of $k^*(\mathfrak{r}^*)$ associated with it by

$$\sum_{l=2}^{m}\left([l-1]_q+1-\psi\right)\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{([l]_q-\phi+\upsilon|\chi|)\,[l]_q!}\,\mathfrak{r}^{*l-1}+\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k^*-1}(1-\sum_{l=2}^{m}\tau_l)\left([k^*]_q-\psi+1\right)}{([k^*]_q-\phi+\upsilon|\chi|)\,[k^*]_q!}\,\mathfrak{r}^{*k^*-1}\le 1-\psi. \quad (3.9)$$

Which is the q-starlike radius of the functions in the class $\mathfrak{HB}_q^{\zeta}(\chi,\phi,\nu,\tau_m)$.

Since the concepts of ψ-starlike and ψ-curvature radii are related to each other.As we have found or defined the first, it is natural to think about defining the region of the ψ -curvature in the unit disk $\mathcal{U}$ for the functions in our special class $\mathfrak{HB}_q^{\zeta}(\chi,\phi,\nu,\tau_m)$ by proposing the following theorem:

Theorem 3.2: If we take the function f in the class $\mathfrak{HB}_q^{\zeta}(\chi,\phi,\nu,\tau_m)$, it has a q-convex property of order $\psi(0\le\psi<1)$ in the disk $|z|<\mathfrak{r}^{**}(\chi,\phi,\nu,\tau_m,\psi)$, and $\mathfrak{r}^{**}(\chi,\phi,\nu,\tau_m,\psi)$ is the largest value that satisfies the property

$$\sum_{l=2}^{m}[l]_q\left([l-1]_q+1-\psi\right)\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\left([l]_q-\phi+\upsilon|\chi|\right)[l]_q!}\,\mathfrak{r}^{l-1}+\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)\left([k]_q-\psi+1\right)}{\left([k]_q-\phi+\upsilon|\chi|\right)[k]_q!}\,\mathfrak{r}^{k-1}\le 1-\psi. \quad (3.10)$$

The extreme functions is

$$f(z)=z-\sum_{l=2}^{m}\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\left([l]_q-\phi+\upsilon|\chi|\right)[l]_q!}\,z^l-\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)}{\left([k]_q-\phi+\upsilon|\chi|\right)[k]_q!}\,z^k. \quad (3.11)$$

The theorem acquires the limiting property

Proof: To prove the theorem, we take

$$\left|\frac{zf''(z)}{f'(z)}\right|\le\left[\sum_{l=2}^{m}[l]_q\left([l-1]_q+1-\psi\right)\frac{(1-\phi+\upsilon|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{\left([l]_q-\phi+\upsilon|\chi|\right)[l]_q!}\,\mathfrak{r}^{k-1}+\sum_{k=m+1}^{\infty}[k]_q[k-1]_q a_k\mathfrak{r}^{k-1}\right]/$$

$$\left[1-\sum_{l=2}^{m}[l]_q \frac{(1-\phi+v|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{([l]_q-\phi+v|\chi|)\,[l]_q!}\mathfrak{r}^{k-1} - \sum_{k=m+1}^{\infty}[k]_q a_k \mathfrak{r}^{k-1}\right]. \quad (3.12)$$

$$\sum_{l=2}^{m}[l]_q([l-1]_q+1-\psi)\frac{(1-\phi+v|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{([l]_q-\phi+v|\chi|)\,[l]_q!}\mathfrak{r}^{l-1} + \sum_{k=m+1}^{\infty}[k]_q([k-1]_q-\psi+1)a_k\mathfrak{r}^{k-1} \le 1-\psi. \quad (3.13)$$

We determine the value of $k^* = k^*(\mathfrak{r})$ so that the quantity

$$\frac{(1-\phi+v|\chi|)\,[\zeta+1]_{q,k^*-1}([k^*]_q-\psi+1)}{([k^*]_q-\phi+v|\chi|)\,[k^*]_q!}\mathfrak{r}^{k^*-1}, \quad (3.14)$$

has its highest value by fixing the value of $\mathfrak{r}$ in it.

After that we have

$$\sum_{k=m+1}^{\infty}[k]_q([k-1]_q-\psi+1)a_k\mathfrak{r}^{k-1} \le \frac{(1-\phi+v|\chi|)\,[\zeta+1]_{q,k-1}(1-\sum_{l=2}^{m}\tau_l)([k^*]_q-\psi+1)}{([k^*]_q-\phi+v|\chi|)\,[k^*]_q!}\mathfrak{r}^{k^*-1}. \quad (3.15)$$

Finally, we have determined the region where the functions in the class $\mathfrak{H}\mathfrak{B}_q^{\zeta}(\chi,\phi,\nu,\tau_m)$ are q-convex of order $\psi(0\le\psi<1)$ which is $|z| < r\,(\chi,\phi,\nu,\tau_m,\psi)$, provided that

$$\sum_{l=2}^{m}[l]_q([l-1]_q+1-\psi)\frac{(1-\phi+v|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{([l]_q-\phi+v|\chi|)\,[l]_q!}\mathfrak{r}^{l-1} + \frac{(1-\phi+v|\chi|)\,[\zeta+1]_{q,k^*-1}(1-\sum_{l=2}^{m}\tau_l)([k^*]_q-\psi+1)}{([k^*]_q-\phi+v|\chi|)\,[k^*]_q!}\mathfrak{r}^{k^*-1} \le 1-\psi. \quad (3.16)$$

We can now determine the value of $\mathfrak{r}^{**} = \mathfrak{r}^{**}(\chi,\phi,\nu,\tau_m,\psi)$ and the value of $k^*(\mathfrak{r}^{**})$ associated with it by

$$\sum_{l=2}^{m}[l]_q([l-1]_q+1-\psi)\frac{(1-\phi+v|\chi|)\,[\zeta+1]_{q,l-1}\tau_l}{([l]_q-\phi+v|\chi|)\,[l]_q!}\mathfrak{r}^{**l-1} + \frac{(1-\phi+v|\chi|)\,[\zeta+1]_{q,k^*-1}(1-\sum_{l=2}^{m}\tau_l)([k^*]_q-\psi+1)}{([k^*]_q-\phi+v|\chi|)\,[k^*]_q!}\mathfrak{r}^{**k^*-1} \le 1-\psi, \quad (3.17)$$

which is the q-convex radius of the functions in the class $\mathfrak{H}\mathfrak{B}_q^{\zeta}(\chi,\phi,\nu,\tau_m)$.

4. CONCLUSION

Using the q-integral operator that implicitly contain the q-Ruscheweyh derivative, a new class of analytic univalent functions has been defined.They are distinguished by some of their finite negative invariant coefficients, and their most interesting fundumental geometric properties have been studied. In the future, the study of geometric properties can be extended to include other interesting properties. Researchers can also draw inspiration from this work to study other distinct classes, serving the field of geometry and complex analysis. The results we have reached can also be studied and analyzed on classes of multivalent functions.

REFERENCES

Abubaker, A. A., Matarneh, K., Khan, M. F. A., Suha, B., & Kamal, M. (2024). Study of quantum calculus for a new subclass of q-starlike bi-univalent functions connected with vertical strip domain. *AIMS Mathematics, 9*, 11789–11804. https://doi.org/10.3934/math.2024577

Akyar, A. (2025). On the sufficient conditions for the univalence of definite integral operators involving certain functions in S class. *Fundamentals of Contemporary Mathematical Sciences, 6*(1), 34–46. https://doi.org/10.54974/fcmathsci.1409461

Al-Hawary, T., Frasin, B. A., & Salah, J. (2025). Necessary and sufficient conditions for functions defined by binomial distribution to be in a general class of analytic functions. *Results in Nonlinear Analysis, 8*(1), 131–140. https://doi.org/10.31838/rna/2025.08.01.013

Al-Rawashdeh, W. (2025). On certain classes of bi-Bazilevic functions defined by q-Ruscheweyh differential operator. *WSEAS Transactions on Mathematics, 24*, 144–156. https://doi.org/10.37394/23206.2025.24.15

AL-Saleem, A. B. I. (2025). Certain class of analytic and univalent function with respect to symmetric. *Nonlinear Functional Analysis and Applications*, 99–111. https://doi.org/10.22771/nfaa.2025.30.01.07

Alsoboh, A., & Oros, G. I. (2024). A class of bi-univalent functions in a leaf-like domain defined through subordination via q-calculus. *Mathematics, 12*(10), 1594. https://doi.org/10.3390/math12101594

Delphi, A. M. (2024). A new subclass of analytic functions on unit disk defined by using integral operator. *Pure Sciences International Journal of Kerbala, 1*(4), 70–77.

Illafe, M., Mohd, M. H., Yousef, F., & Supramaniam, S. (2024). A subclass of bi-univalent functions defined by a symmetric q-derivative operator and Gegenbauer polynomials. *European Journal of Pure and Applied Mathematics, 17*(4), 2467–2480. https://doi.org/10.29020/nybg.ejpam.v17i4.5408

Jackson, F. H. (1905). The application of basic numbers to Bessel's and Legendre's functions (second paper). *Proceedings of the London Mathematical Society, 2*(1), 1–23. https://doi.org/10.1112/plms/s2-3.1.1

Jackson, F. H. (1909). On q-functions and a certain difference operator. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh, 46*(2), 253–281. http://dx.doi.org/10.1017/S0080456800002751

Jackson, F. H. (1910a). q-difference equations. *American Journal of Mathematics, 32*(4), 305–314. https://doi.org/10.2307/2370183

Jackson, F. H., Fukuda, T., Dunn, O., et al. (1910b). On q-definite integrals. *Quarterly Journal of Pure and Applied Mathematics, 41*(15), 193–203.

Kanas, S., & Răducanu, D. (2014). Some class of analytic functions related to conic domains. *Mathematica Slovaca, 64*(5), 1183–1196. https://doi.org/10.2478/s12175-014-0268-9

Lashin, A. M. Y., Badghaish, A. O. A., & Alshehri, F. A. (2024). A study on two subclasses of analytic and univalent functions with negative coefficients involving the Poisson distribution series. *Kyungpook Mathematical Journal, 64*(1), 47–55. https://doi.org/10.5666/KMJ.2024.64.1.47

Maran, A. H., Juma, A. R. S., & Al-Saphory, R. A. (2025). On a subclass of univalent functions with negative coefficients associated to linear Srivastava–Attiya operator. *AIP Conference Proceedings, 3264*(1). https://doi.org/10.1063/5.0258978

Noor, K. I. (1999). On new classes of integral operators. *Journal of Natural Geometry, 16*(1–2), 71–80.

Srivastava, H. M., Ahmad, Q. Z., Khan, N., Kahn, N., & Khan, B. (2019). Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain. *Mathematics, 7*(2), 181. https://doi.org/10.3390/math7020181

Srivastava, H. M., Wanas, A. K., & Srivastava, R. (2021). Applications of the q-Srivastava–Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials. *Symmetry, 13*(7), 1230. https://doi.org/10.3390/sym13071230

Universiti Kebangsaan Malaysia (UKM), Bangi. (2024). Properties for certain class of p-valent functions related to Jackson's operator. *Sains Malaysiana, 55*(1), 231–238. http://doi.org/10.17576/jsm-2024-5301