DOI: 10.24996/ijs.2025.66.9.30

ISSN: 0067-2904

Solving Coupled Parabolic System by Using Galerkin - Implicit Differences Methods

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Received: 22/4/2024 Accepted: 1/9/2024 Published: 30/9/2025

Abstract

This paper deal with a technique of approximation which is insert to find the solution of a new type of couple system of constant coefficients parabolic equations (CPS) from mixing the method of finite element of Galerkin (GFEM) "in the variable of space" with the finite differences method of implicit type (IFDM) in the variable of time. Therefore, the abbreviation IGFEM is utilized to express on the method named by implicit Galerkin finite element method (IGFEM). In this method at each discrete value of time t_j, the CPS convert into a Galerkin system of algebraic equations (GLAS) and later it is resolved by utilizing the method of Cholesky Decomposition (ChDeM). Examples of explanations are awarded and the illustration for the results are presented through tables and figures, and they exhibit that the method which is proposed her is accurate and efficient.

Keywords: Coupled Parabolic System, Galerkin Finite Element Method, Implicit Finite Differences Method, Cholesky Method.

حل زوج نظام مكافيء باستخدام طريقتي كاليركن – الفروقات الضمنية

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الخلاصة

يتناول هذا البحث تقنية تقريبية ادخلت لايجاد الحل لنوع جديد لزوج من نظام الزوجين من المعادلات التفاضلية ذات المعاملات الثابته من القطع المكافيء من خلال مزج طريقة الفروقات المنتهية لكاليركن "بالنسبة لمتغير الفضاء " مع طريقة الفروقات المنتهية الضمنية لمتغير الزمن لذلك تم احتصار الطريقة بطريقة كاليركن للعناصر المنتهية الضمنية (IGFEM) . في هذه الطريقة وعند اية قيمة متقطعة الازمن t_j يتحول زوج نظام المكافيء ذات المعاملات الثابته الى نزام كاليركن للمعادلات الجبرية وتحل الاخيرة باستخدام طريقة جولسكي . عرضت امثلة توضيحبة وبينت النتائج ان الطريقة المقترحة هذه دقيقة وكفوءة .

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1. Introduction

Many implementations have large range as in; the science of natural technology and engineering which they are described generally by modules in mathematics [1] and by differential equations particularly [2]. More precisely many of these modules in mathematics are described by parabolic PDEs (PPDEs), and these equations usually need to find their solutions numerically. As is known, there are numerous numerical methods that are utilized to find the solution of PPDEs; like in 2021 the mixed Homotopy perturbation method (HPM) with Crank- Nicolson method [3], then the GFEM was introduced [4], a modified quantic trigonometric B- Spline collocation technique [5], and the polynomial cubic Spline method [6]. Later in 2022, other methods were introduced to solve the PPDEs, such as the explicit FDM [7], the mixed HPM with integral transform method [8] and a novel collocation technique [9]. Recently, this problem was solved by using Crank- Nicolson method [10] and many other methods which they used other methods [11-13]. All the above methods together with the method in [14] and the applications [15] and [16] which are used to solve couple PDEs motivated us to introduced the study of the solution for a new model of PDEs, described by CPS using a new technique "the proposed method IGFEM".

This paper discusses the description of the continuous CPS, then the weak formulation (WF) of the considered problem is formulated and then it is approximated from dsicretting of the continuous CPS, through utilizing the GFEM in the variable of space with the IFDM in the variable of time, therefore the abbreviation IGFEM is utilized to express on the method. In this method at each discrete value of time t_j, the CPS convert into a Galerkin system of algebraic equations (GLAS) and later it is resolved by utilizing the method of Cholesky Decomposition (ChDeM). These steps of finding for the solution is given by algorithm. Lastly, Examples of explanations are awarded to solve two different types problems utilizing the IGFEM with the succor of MATLAB software in mathematics and the illustration for the results are presented through tables and figures and they exhibit that the method which is proposed her is accurate and efficient.

2. Couple Parabolic System Description

Let $\Omega = \{\vec{x} = (x_1, x_2) \in R^2\colon\ 0 < x_1, x_2 < 1\} \subset R^2$, be the region with boundary $\partial\Omega$, and let $I = [0, T]$, $Q = \Omega \times I$, $0 < T < \infty$, then the CPS are given by:

$$U_{1t} - \Delta U_1 + U_1 - U_2 = w_1(\vec{x}, t),\ \text{in } Q \tag{1}$$

$$U_{2t} - \Delta U_2 + U_2 + U_1 = w_2(\vec{x}, t),\ \text{in } Q \tag{2}$$

$$U_1(\vec{x}, 0) = U_1^0(\vec{x}),\ \text{in } \Omega \tag{3}$$

$$U_2(\vec{x}, 0) = U_2^0(\vec{x}),\ \text{in } \Omega \tag{4}$$

$$U_1(\vec{x}, t) = 0,\ \text{on } \partial\Omega \times I \tag{5}$$

$$U_2(\vec{x}, t) = 0,\ \text{on } \partial\Omega \times I \tag{6}$$

where$\Delta = \sum_{r=1}^{2} \frac{\partial^2}{\partial x_r^2}$, $U_1 = U_1(\vec{x}, t),\ U_2 = U_2(\vec{x}, t) \in C^2(Q),\ w_1 = w_1(\vec{x}, t),\ w_2 = w_2(\vec{x}, t)$ are given functions in $C(Q)$ for all $\vec{x} \in \Omega$. Here $C^2(Q)$ is the space of all continuous functions in Q that has continuous partial derivatives up to the 2nd order in Q. The "classical solution" of system ((1)-(6)) is $\vec{U}(\vec{x}, t) = (U_1(\vec{x}, t), U_2(\vec{x}, t)) \in (C^2(Q))^2, s.t\ \vec{U} = \vec{0}\ on\ \partial\Omega$, for all $\vec{x}$ in Ω

2.1 The Weak Form of the Couple Parabolic System

Let $V = \{v\colon v = v(\vec{x}) \in C^1(\Omega), \forall \vec{x} \in \Omega, with\ v = 0\ on\ \partial\Omega\}$, multiply both sides of ((1)-(6)) by $v_r \in V$ (for $r = 1,2$) then using the Green's theorem for boating the 1st term in the L.H.S. then the WF of the CPS ((1)-(6)) is resulted as follows:

$$\langle U_{1t}, v_1\rangle + a_1(t, U_1, v_1) - (U_2, v_1) = (w_1, v_1), \forall\, v_1 \in V, \vec{U} \in (C_0^1(Q))^2 \quad (7)$$
$$(U_1(0), v_1) = (U_1^0, v_1), \text{ in } V \quad (8)$$
$$\langle U_{2t}, v_1\rangle + a_2(t, U_2, v_2) + (U_1, v_2) = (w_2, v_2), \forall\, v_2 \in V, \vec{U} \in (C_0^1(Q))^2 \quad (9)$$
$$(U_2(0), v_2) = (U_2^0, v_2), \text{ in } V \quad (10)$$

Where the following bilinear form are defined

$a_1(t, U_1, v_1) = \sum_{r,s=1}^{2}\left(\frac{\partial U_1}{\partial x_s}, \frac{\partial v_1}{\partial x_r}\right) + (U_1, v_1)$

$a_2(t, U_2, v_2) = \sum_{r,s=1}^{2}\left(\frac{\partial U_2}{\partial x_s}, \frac{\partial v_2}{\partial x_r}\right) + (U_2, v_2)$.

MAIN RESULTS

3. Discretization of the Couple Parabolic System

The discretization for the WF((7)-(10)), begin with assuming $M_1 > 0$ be an integer and let $O = O_i^n, i = 1,2,\dots,N, N = M^2$ where $M = M_1 - 1$ be an admissible regular triangulation of $\overline{\Omega} = \Omega \cup \partial\Omega$, then the region Ω is splitting to $\overline{\Omega} = \bigcup_{i-1}^{N} O_i$, let $h = 1/M_1$, $x_{i1} = x_{i2} = ih$, ($i = 1,2,\dots,M_1$)be points in $\overline{\Omega} = [0,1] \times [0,1]$, s.t.

$0 = x_{01} < x_{11} < \cdots < x_{i1} < \cdots < x_{(M_1)1} = 1$ and

$0 = x_{02} < x_{12} < \cdots < x_{i2} < \cdots < x_{(M_1)2} = 1$.

The interval of the variable of time is divided into the subintervals $I_j = I_j^n \coloneqq [t_j^n, I_{j+1}^n]$, with $t_j^n = j\Delta t, j = 0,1,\dots,NT-1$.

To use the proposed GFEM [17], consider $V_N \subset V$ represents the space of piecewise linear function, s.t.

$V_N = \{V_i, i = 1,2,\dots,N, with\ V_i(\vec{x}) = 0\ on\ \partial\Omega\}$, let $\vec{V}_N = V_N \times V_N$, then the discrete WF(DWF) of ((7)-(10)) is,

$$\langle U_{1t}^n, v_1\rangle + a_1(t, U_1^n, v_1) - (U_2^n, v_1) = (w_1, v_1), \forall\, v_1 \in V_N,\ \vec{U}^n \in \vec{V}_N \quad (11)$$
$$(U_1(0), v_1) = (U_1^0, v_1), \text{ in } V_N \quad (12)$$
$$\langle U_{2t}^n, v_2\rangle + a_2(t, U_2^n, v_2) + (U_1^n, v_2) = (w_2, v_2), \forall\, v_2 \in V_N,\ \vec{U}^n \in \vec{V}_N \quad (13)$$
$$(U_2(0), v_2) = (U_2^0, v_2), \text{ in } V_N \quad (14)$$

4. The Approximate Solution of the Couple Parabolic System

To find the approximate solution (APS) $\vec{U}^n = (U_1^n, U_2^n)$ of ((11)-(14)), using the GFEM, the following proceedings can be applied:

Step1: Utilizing the mixing of the both methods (GFEM and IFDM) in [18], then the basis $(v_1, v_2, \dots, v_N)$ of V_N is utilize to approximate the APS $U_i^n(\forall\, i = 1,2)$, i.e.

$U_1^n(\vec{x}, t_j) = \sum_{k=1}^{N} C_k(t_j)V_k(\vec{x})$, $U_2^n(\vec{x}, t_j) = \sum_{k=1}^{N} C_{k+N}(t_j)V_k(\vec{x})$

$U_1^n(\vec{x}, 0) = \sum_{k=1}^{N} C_k(0)V_k(\vec{x})$, $U_2^n(\vec{x}, 0) = \sum_{k=1}^{N} C_{k+N}(0)V_k(\vec{x})$

Step2: Substitute $v_1 = v_2 = v_m$ in $\vec{U}^n$ then using it in the above DWF((11)-(14)), the following couple GLAS with their ICs are obtained

$$(A + \Delta t B)C_k^{j+1} - \Delta t D C_{k+N}^{j+1} = A C_k^j + \Delta t \vec{b}_1(t_{j+1}) \quad (15)$$
$$A C_k(0) = \vec{b}_1^0 \quad (16)$$
$$(A + \Delta t E)C_{k+N}^{j+1} + \Delta t D C_k^{j+1} = A C_{k+N}^j + \Delta t \vec{b}_2(t_{j+1}) \quad (17)$$
$$A C_{k+N}(0) = \vec{b}_2^0 \quad (18)$$

where $A = (a_{mk})_{N\times N}$, $a_{mk} = (v_k, v_m)$, $B = (b_{mk})_{N\times N}$, $b_{mk} = a_1(v_k, v_m)$, $a_1(v_k, v_m) = \sum_{r,s=1}^{2}\left(\frac{\partial v_k}{\partial x_s}, \frac{\partial v_m}{\partial x_r}\right) + (v_k, v_m)$, $D = (d_{mk})_{N\times N}$, $d_{mk} = (v_k, v_m)$, $E = (e_{mk})_{N\times N}$, $e_{mk} = a_2(v_k, v_m)$, $a_2(v_k, v_m) = \sum_{r,s=1}^{2}\left(\frac{\partial v_k}{\partial x_s}, \frac{\partial v_m}{\partial x_r}\right) + (v_k, v_m)$,

$C_k(t_j) = \left(c_k(t_j)\right)_{N\times 1}, C_{k+N}(t_j) = \left(c_{k+N}(t_j)\right)_{N\times 1}, \vec{b}_1 = (b_{1i})_{N\times 1},\ \ b_{1i} = \left(w_1(\vec{x}, t_{j+1}), v_m\right)$,
$\vec{b}_2 = (b_{2i})_{N\times 1}, b_{2i} = \left(w_2(\vec{x}, t_{j+1}), v_m\right)$, $\vec{b}_1^0 = (b_{1i}^0)_{N\times 1}, b_{1i}^0 = (U_1^0, v_i), \vec{b}_2^0 = (b_{2i}^0)_{N\times 1}$,
$b_{2i}^0 = (U_2^0, v_i)$, $\forall\, m, k = 1,2, \dots, N$.
Step3: Use the ChDeM to solve the couple GLAS ((15)-(18)), to obtain the APS for the WF ((7)-(10)).

Remark: The uniqueness of the couple GLAS ((15)-(18)) is obtained from the positive definite property of the matrices A,B,D and E [19].

5. The Cholesky Decomposition Method

This method is utilized to obtain the solution of the couple GLAS with the necessary conditions that the coefficients matrix A must be positive definite, in this method, the matrix $A = L.L^T$ is written as a product of two unique matrices (the lower triangular matrix L and L^T is its transpose) [20]. The following steps deals with the ChDeM:

Step1: Set $L_{11} = \sqrt{a_{11}}$, and $L_{pp} = \left(a_{pp} - \sum_{z=1}^{p-1} L_{pz}^2\right)^{1/2}$ $for\ \ p = 2,3, \dots, N$

Step2: $L_{pq} = \frac{a_{pq} - \sum_{z=1}^{q-1} L_{qz}.L_{pz}}{L_{qq}}$ $for\ q = p+1, \dots, N$.

6. Algorithm for Solving the Galerkin Linear Algebraic System

Step1: Calculate C_k^0 and C_{k+N}^0 from solving (16) , (18) resp.
Step2: Put $j = 0$
Step3: Calculate C_k^1 and C_{k+N}^1 from solving the couple GLAS (15) ,(17) resp.
Step4: Put $j = j + 1$
Step5: Repeated Step3- Step4, till to $j = NT - 1$
Step6: Substitute the coefficients c_k , c_{k+N} $(k = 1,2, \dots, N)$ in the APS on $(U_1^n(\vec{x}, t), U_2^n(\vec{x}, t)$.

7. Numerical Examples:

In this part, and in order to examine the accuracy and the efficiency of the proposed IGFEM two numerical examples are introduced.

Example (1): Let $Q = \Omega \times I, \Omega = (0,1) \times (0,1)$, and $I = [0,1]$, then the CPS are given by

$U_{1t} - \frac{\partial^2 U_1}{\partial x_1^2} - \frac{\partial^2 U_1}{\partial x_2^2} + U_1 - U_2 = w_1(\vec{x}, t),\ in\ Q$

$U_{2t} - \frac{\partial^2 U_2}{\partial x_1^2} - \frac{\partial^2 U_2}{\partial x_2^2} + U_1 + U_2 = w_2(\vec{x}, t)\ \ in\ Q$

$U_1(\vec{x}, 0) = U_1^0(\vec{x}) = 0, in\ \Omega$

$U_2(\vec{x}, 0) = U_2^0(\vec{x}) = \sin(x_1 x_2) \tan\left(\sin(1 - x_1)(1 - x_2)\right),$ in Ω

$U_1(\vec{x}, t) = 0,\ on\ \partial\Omega \times I$

$U_2(\vec{x}, t) = 0,\ on\ \partial\Omega \times I$

Such that the right-hand term $w_1(\vec{x}, t, u_1)$ and $w_2(\vec{x}, t, u_2)$ are

$w_1(\vec{x}, t, u_1) = 2\tan\left(t/_7\right)\left(x_1(x_1 - 1) + x_2(x_2 - 1)\right)$
$- e^{-t} \sin(x_1 x_2) \tan\left(\sin(x_1 - 1)(x_2 - 1)\right) - x_1 x_2 (x_1 - 1)(x_2 -$
$1)(\tan\left(t/_7\right) + \tan^{2/7}\left(t/_7\right) + 7)$

$w_2(\vec{x}, t, u_2) = (x_1^2 + x_2^2)(e^{-t} \sin(x_1 x_2) \tan\left(\sin(x_1 - 1)(x_2 - 1)\right)) - x_1 x_2 (x_1 - 1)(x_2 -$
$1) tan\left(t/_7\right) + (((x_1 - 1)^2 + (x_2 - 1)^2)\, e^{-t} \sin(x_1 x_2)\, (\tan^2(\sin(x_1 - 1)(x_2 - 1) +$
$1))\, (\sin(x_1 - 1)(x_2 - 1) - 2cos^2\left((x_1 - 1)(x_2 - 1)\right) \tan\left(\sin(x_1 - 1)(x_2 - 1)\right)$

$-2e^{-t}\big(x_1(x_1-1)+x_2(x_2-1)\big)\cos(x_1-1)\,(x_2-1)\cos(x_1x_2)(\tan^2(\sin(x_1-1)\,(x_2-1)+1))$

The exact solution (ES) of the CPS is

$U_1(\vec{x},t)=x_1x_2(x_1-1)(x_2-1)\tan\left({}^{-t}/_{7}\right)$

$U_2(\vec{x},t)=\mathrm{e}^{-\mathrm{t}}\sin(x_1x_2)\tan\left(\sin(1-x_1)\,(1-x_2)\right)$

The IGFEM is used to solve this CPS for $M=9, NT=20$ and $T=1$, then the APS $\vec{U}^n$ and the ES $\vec{U}$ at x_1 and x_2 at the value of time $\hat{t}=0.5$ are given in Table (1) and are illustrated in Figure (1), with the maximum absolute error (ABER) is (0.0038).

Table 1: The ES, the APS and the ABER between them

x_1	x_2	**ES** U_1	**APS** $\mathrm{U}_1^{\mathrm{n}}$	ABER	x_1	x_2	**ES** U_2	**APS** $\mathrm{U}_2^{\mathrm{n}}$	ABER
0.1	0.1	-0.0006	-0.0010	0.0004	0.1	0.1	0.0054	0.0055	0.0001
0.3	0.1	-0.0014	-0.0024	0.0010	0.3	0.1	0.0122	0.0124	0.0002
0.5	0.1	-0.0016	-0.0028	0.0012	0.5	0.1	0.0141	0.0144	0.0003
0.7	0.1	-0.0014	-0.0023	0.0009	0.7	0.1	0.0116	0.0118	0.0002
0.9	0.1	-0.0006	-0.0010	0.0004	0.9	0.1	0.0049	0.0050	0.0001
0.1	0.3	-0.0014	-0.0024	0.0010	0.1	0.3	0.0122	0.0124	0.0002
0.3	0.3	-0.0032	-0.0057	0.0025	0.3	0.3	0.0277	0.0284	0.0007
0.5	0.3	-0.0038	-0.0069	0.0031	0.5	0.3	0.0324	0.0331	0.0007
0.7	0.3	-0.0032	-0.0057	0.0025	0.7	0.3	0.0267	0.0274	0.0007
0.9	0.3	-0.0014	-0.0023	0.0009	0.9	0.3	0.0113	0.0116	0.0003
0.1	0.5	-0.0016	-0.0028	0.0012	0.1	0.5	0.0141	0.0144	0.0003
0.3	0.5	-0.0038	-0.0069	0.0031	0.3	0.5	0.0324	0.0331	0.0007
0.5	0.5	-0.0045	-0.0083	0.0038	0.5	0.5	0.0379	0.0388	0.0009
0.7	0.5	-0.0038	-0.0068	0.0030	0.7	0.5	0.0313	0.0321	0.0008
0.9	0.5	-0.0016	-0.0028	0.0012	0.9	0.5	0.0132	0.0135	0.0003
0.1	0.7	-0.0014	-0.0023	0.0009	0.1	0.7	0.0116	0.0118	0.0002
0.3	0.7	-0.0032	-0.0057	0.0025	0.3	0.7	0.0267	0.0274	0.0007
0.5	0.7	-0.0038	-0.0068	0.0030	0.5	0.7	0.0313	0.0321	0.0008
0.7	0.7	-0.0032	-0.0056	0.0024	0.7	0.7	0.0257	0.0263	0.0006
0.9	0.7	-0.0014	-0.0023	0.0009	0.9	0.7	0.0107	0.0110	0.0003
0.1	0.9	-0.0006	-0.0010	0.0004	0.1	0.9	0.0049	0.0050	0.0001
0.3	0.9	-0.0014	-0.0023	0.0009	0.3	0.9	0.0113	0.0116	0.0003
0.5	0.9	-0.0016	-0.0028	0.0012	0.5	0.9	0.0132	0.0135	0.0003
0.7	0.9	-0.0014	-0.0023	0.0009	0.7	0.9	0.0107	0.0110	0.0003
0.9	0.9	-0.0006	-0.0009	0.0003	0.9	0.9	0.0044	0.0045	0.0001

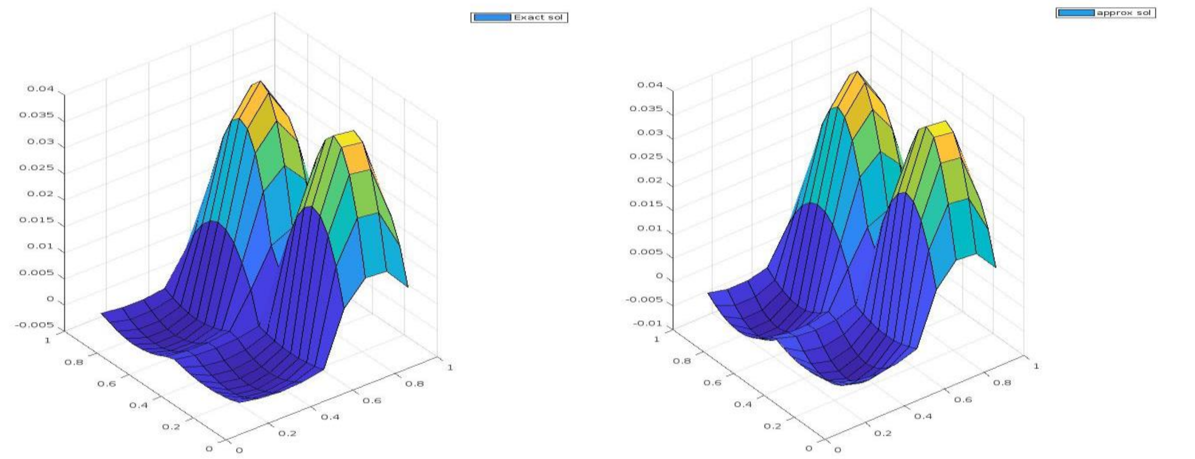

Figure 1: The ES and the APS

Example (2): Let $Q = \Omega \times I$, $\Omega = (0,1) \times (0,1)$, and $I = [0,1]$, then the CPS are given by

$$u_{1t} - \frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial^2 u_1}{\partial x_2^2} + u_1 - u_2 = w_1(\vec{x}, t), \ \ in\ Q$$

$$u_{2t} - \frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial^2 u_2}{\partial x_2^2} + u_1 + u_2 = w_2(\vec{x}, t) \ \ in\ Q$$

$$u_1(\vec{x}, 0) = u_1^0(\vec{x}) = 0.2(1 - x_2)\sin(\pi x_1)\ \tan(x_2)\, , in\ \Omega$$

$$u_2(\vec{x}, 0) = u_2^0(\vec{x}) = 0.1\sin(\pi x_1)\ (1 - e^{x_2(1-x_2)}) \quad , in\ \Omega$$

$$u_1(\vec{x}, t) = 0, \ \ on\ \partial\Omega \times I$$

$$u_2(\vec{x}, t) = 0, \ \ on\ \partial\Omega \times I$$

Such that the right hand term $w_1(\vec{x}, t, u_1)$ and $w_2(\vec{x}, t, u_2)$ are

$$w_1(\vec{x}, t, u_1) = 0.1\sqrt{1 + \mathrm{t}}\ \left(\sin(\pi x_1) e^{-x_2(x_2-1)} - 1\right) + 0.4\ \sin(\pi x_1)\cos(t)\,(\tan^2(x_2) + 1)$$
$$-0.2(x_2 - 1)\sin(\pi x_1)\tan(x_2)\,(\cos(t)\,(1 + \pi^2 + 2(\tan^2(x_2) + 1)) - \sin(t))$$

$$w_2(\vec{x}, t, u_2) = \ \sin(\pi x_1)(0.1\sqrt{1 + \mathrm{t}}\left((4x_2^2 - 4x_2 - 1)e^{-x_2(x_2-1)}\right.$$
$$\left. - (1 + \pi^2)\left(e^{-x_2(x_2-1)} - 1\right)\right) - \left(0.5 / \sqrt{1 + \mathrm{t}}\right)\left(e^{-x_2(x_2-1)} - 1\right)$$
$$- 0.2(x_2 - 1)\tan(x_2)\cos(t))$$

The ES of the above CLPSCC is

$$u_1(\vec{x}, t) = 0.2(1 - x_2)\sin(\pi x_1)\cos(t)\tan(x_2)$$

$$u_2(\vec{x}, t) = 0.1\sqrt{1 + \mathrm{t}}\sin(\pi x_1)\ (1 - e^{x_2(1-x_2)})$$

This problem is solved using the IGFEM for $M = 9, NT = 20\ \ and\ \ T = 1$, then the APS $\vec{U}^n$ and the ES $\vec{U}\ at\ x_1\ and\ x_2$ are given at the value of time $\hat{t} = 0.5$ are given in Table (2) and are illustrated in Figure (2), with maximum ABER (0.0035).

Table 2: The ES, the APS and the ABER between them

x_1	x_2	ES U_1	APSU_1^n	ABER	x_1	x_2	ES U_2	APS U_2^n	ABER
0.1	0.1	0.0049	0.0052	0.0003	0.1	0.1	-0.0036	-0.0035	0.0001
0.3	0.1	0.0128	0.0137	0.0009	0.3	0.1	-0.0093	-0.0092	0.0001
0.5	0.1	0.0158	0.0169	0.0011	0.5	0.1	-0.0115	-0.0114	0.0001
0.7	0.1	0.0128	0.0137	0.0009	0.7	0.1	-0.0093	-0.0092	0.0001
0.9	0.1	0.0049	0.0053	0.0004	0.9	0.1	-0.0036	-0.0035	0.0001
0.1	0.3	0.0117	0.0126	0.0009	0.1	0.3	-0.0088	-0.0087	0.0001
0.3	0.3	0.0307	0.0330	0.0023	0.3	0.3	-0.0232	-0.0228	0.0004
0.5	0.3	0.0380	0.0408	0.0028	0.5	0.3	-0.0286	-0.0283	0.0003
0.7	0.3	0.0307	0.0331	0.0024	0.7	0.3	-0.0232	-0.0229	0.0003
0.9	0.3	0.0117	0.0126	0.0009	0.9	0.3	-0.0088	-0.0088	0.0000
0.1	0.5	0.0148	0.0159	0.0011	0.1	0.5	-0.0107	-0.0106	0.0001
0.3	0.5	0.0388	0.0416	0.0028	0.3	0.5	-0.0281	-0.0278	0.0003
0.5	0.5	0.0479	0.0514	0.0035	0.5	0.5	-0.0348	-0.0344	0.0004
0.7	0.5	0.0388	0.0416	0.0028	0.7	0.5	-0.0281	-0.0278	0.0003
0.9	0.5	0.0148	0.0159	0.0011	0.9	0.5	-0.0107	-0.0106	0.0001
0.1	0.7	0.0137	0.0146	0.0009	0.1	0.7	-0.0088	-0.0088	0.0000
0.3	0.7	0.0359	0.0381	0.0022	0.3	0.7	-0.0232	-0.0229	0.0003
0.5	0.7	0.0444	0.0472	0.0028	0.5	0.7	-0.0286	-0.0283	0.0003
0.7	0.7	0.0359	0.0382	0.0023	0.7	0.7	-0.0232	-0.0228	0.0004
0.9	0.7	0.0137	0.0146	0.0009	0.9	0.7	-0.0088	-0.0087	0.0001
0.1	0.9	0.0068	0.0072	0.0004	0.1	0.9	-0.0036	-0.0035	0.0001
0.3	0.9	0.0179	0.0187	0.0008	0.3	0.9	-0.0093	-0.0092	0.0001
0.5	0.9	0.0221	0.0232	0.0011	0.5	0.9	-0.0115	-0.0114	0.0001
0.7	0.9	0.0179	0.0188	0.0009	0.7	0.9	-0.0093	-0.0092	0.0001
0.9	0.9	0.0068	0.0072	0.0004	0.9	0.9	-0.0036	-0.0035	0.0001

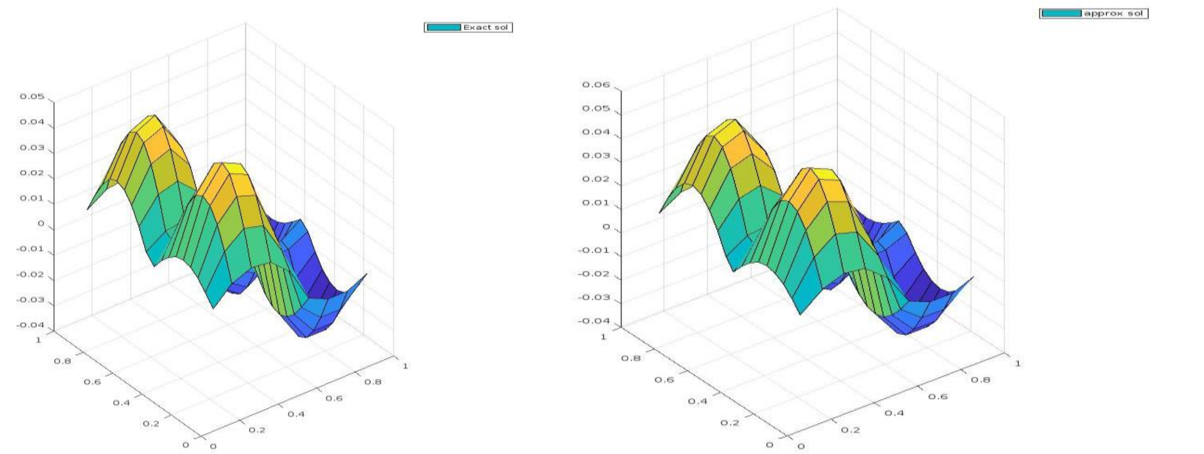

Figure 2: The ES and the APS

Conclusions:

In this research, the IGFEM is applied successfully to solve the CPS. Based on the result in Figures 1 and 2 and in the Tables 1 and 2, the maximum ABER (0.0038, and 0.0035) between the ES and the APS in the two given examples exhibit that the method is efficient, although the discretization for the variable of space is only for ten grid $(m = 9)$ with $NT = 20$. The ChDeM is utilized to solve the couple GLAS in step 3 in the algorithm and it is faster than some other methods like Gauss elimination and the Haar wavelets methods because the ChDeM saves a lot of number of calculations, and the elements in the GNAS are in analytic form comparing with other methods that the elements are in approximate or in a full discrete form. Finally, the APS, the ES and their ABER between them in the examples are calculated at each values of the discrete points for the space and the times variables but the tables indicated only to the half of these values of variable space and at $\hat{t} = 0.5$ to save the space. The method gives the investigators an indicator of ability to solve systems of two PDEs and more than of two PDEs.

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