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Using Improved Operationally Matrix for Volterra Integral Equation of the First Type

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Abstract

The purpose of this paper is to find a practical method with simple arithmetic operations to solve the first type of Volterra's integral equation. Using of the piecewise functions and the improved operationally matrix of their integration, the integrally equation of the first type can be decreased to a lower linear sparse relation that should be easily solved directly by replacing from forward. Numerical examples show that the approximate solution has an acceptable and appropriate value.

Keywords: Block pulse functions, Volterra's integral equation of the first type, improved operationally matrices.

Introduction

We know that one of the important ill-posed problems are first type integrally equations. [1, 2]. In addition,the Volterra's integral equation of the first type appears in the several problems in the scientific aspects. Therefore, studying such issues is very important in application. Many ways such as expansion way, regularization way [3], Galerkin method [4] and other methods have been found and suggested.

In recent papers [4, 5], delayed integral equations (DIEs) and delayed integro-differential equations (DIDEs) and stability of nonlinear neutral these are solved by different methods. Collocation methods for solving special kind of integral equations are obtained [6].

The methods of numerical solution of different types of ordinary and partial differential equations have also been studied in recent years [7-10].

Integral equations have many applications in other sciences. For example, in [11] Fredholm's integral equations are used in plasma physics calculations. Many studies have been done on the numerical solution of these equations and many types of numerical methods have been developed to quickly and accurately obtain the approximation of y(x). Literature reviews and references of many existing methods are available in [12]. Collocation methods [13], since methods [14-16], and general spectral methods [17, 18] are also considered.

In the past years and up to now, orthogonally functions or polynomials like orthogonal Block pulse functions, Hat functions, wavelets, Legendre polynomials, Genocchi polynomials and Lager polynomials estimation some systems such as integral equations are solved.

Previously, orthogonally functions or polynomials, like hyperbolic mass orthogonal functions, Hat functions, wavelets, Legendre polynomials, Genocchi polynomials and Laguerre polynomials, firstly introduced the multi constant simple orthogonally functions (block pulse functions), its attributes and it's operational and improved operationally matrices. Then by improved operationally matrix we convert the Volterra's integral equation of the first type to a less spare linear relationship.

Finally, in[19,20] we can explain the Spline-collocation method, it has evolved as valuable techniques for the solution of a broad class of problems covering ordinary and partial differential equations, functional equations,integral equations and integro-differental equations, also this method involves the determination of an approximate solution in a suitable set of functions, sometimes called trial functions,by requiring the approximate solution to satisfy the boundary conditions and the differential equation at certain points,called the collocation points.

Preliminaries

Here is the interpretation of the symbols and the feature of the (block-pulse functions), which are fully described in [9].

Definition 1. The *m* term of block pulse funs- can be written as follows: ,

$$\phi_i^{(m)}(t) = \begin{cases} 1 & (i-1)h \le t < ih \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

Where $t \in [0,T),\ i = 1,2,\ldots,m$ and $h = \frac{T}{m}$.

Its main primary attributes of these functions are disjoint from each other and orthogonality of them that can be expressed as follows:

$$\phi_i^{(m)}(t)\phi_j^{(m)}(t) = \delta_{ij}\phi_i^{(m)}(t),$$

$$\int_0^T \phi_i^{(m)}(t)\phi_j^{(m)}(t)dt = h\delta_{ij}, \qquad i,j = 1,2,\ldots,m \tag{2}$$

Function approximation

A functionally $f(t) \in L^2[0,T)$, that is real and bounded can be written and expanded in the form of(block pulse series) as,

$$f(t) \simeq \hat{f}_m(t) = \sum_{i=1}^{m} f_i \phi_i^{(m)}(t), \tag{3}$$

where $f_i = \frac{1}{h}\int_0^T f(t)\phi_i^{(m)}(t)\,dt$.

Also we can see,

$$f(t) \simeq \hat{f}_m(t) = F^T\Phi(t) = \Phi^T(t)F,$$

or

$$F = \left(f_1 \quad f_2 \quad \ldots \quad f_m\right)^T, \ \Phi(t) = \left(\phi_1(t) \quad \phi_2(t) \quad \ldots \quad \phi_m(t)\right)^T.$$

and

$$k(s,t) = \Psi^T(s)K\Phi(t) = \Phi^T(t)K^T\Psi(s),$$

Where

$$K = (k_{ij}),\ i = 1,2,\ldots,m_1,\ j = 1,2,\ldots,m_2 \quad , \quad k_{ij} = \frac{1}{h_1 h_2}\int_0^{T_1}\int_0^{T_2} k(s,t)\Psi_i^{(m_1)}(s)\Phi_j^{(m_2)}(t)dtds.$$

Integration operationally matrix

After some computing, $\int_0^t \phi_i^{(m)}(s)ds$ and simplifying it, we will have:

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{4}$$

where the operationally matrix of integral is given by

$$P = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m\times m}$$

As a result, for the arbitrary function *f(x)*, we can write:

$$\int_0^t f(x)dx \cong \int_0^t F^T\Phi(x)dx \cong F^T P\Phi(t)$$

Improved operationally matrix

If we set

$$g(t) = \int_0^t f(\tau)d\tau \cong F^T P\,\Phi(t) = (\tilde{g}_1 \quad \tilde{g}_2 \quad \ldots \quad \tilde{g}_m)\Phi(t)$$

where

$$\tilde{g}_i = h(f_1 + f_2 + \ldots. + f_{i-1}) + \frac{h}{2}f_i$$

using previous relations ,we can write $\tilde{g}_i$ as follows:

$$\begin{aligned}\tilde{g}_i &= h\left(\frac{1}{h}\int_0^h f(t)dt + \frac{1}{h}\int_h^{2h} f(t)dt + \cdots + \frac{1}{h}\int_{(i-2)h}^{(i-1)h} f(t)dt\right) + \frac{h}{2}\left(\frac{1}{h}\int_{(i-1)h}^{ih} f(t)dt\right) \\ &= \int_0^{(i-1)h} f(t)dt + \frac{1}{2}\int_{(i-1)h}^{ih} f(t)dt \\ &= g((i-1)h) + \frac{g(ih) - g((i-1)h)}{2} = \frac{g((i-1)h) + g(ih)}{2}\end{aligned}$$

With the aim of improving the scheme of block pulse coefficients resulting one of the rules integrated operation, we should firstly attain a little simple better approximation of *g(t)*, by use of the formula of Lagrange's interpolation, with 3 points $t_0 = (i-2)h$, $t_1 = (i-1)h$ and $t_2 = ih$, we can attain *g(t)* for values of *t* in the interval $[(i-1)h, ih)$ as:

$$\bar{g}(t) = g((i-2)h)\frac{(t-(i-1)h)(t-ih)}{2h^2}$$
$$-g((i-1)h)\frac{(t-(i-2)h)(t-ih)}{h^2} + g(ih)\frac{(t-(i-2)h)(t-(i-1)h)}{2h^2}$$

and then we attain the ith block pulse coefficient of *g(t)* from this approximation $\bar{g}(t)$ as a progress:

$$\bar{g}_i = \frac{1}{h}\int_{(i-1)h}^{ih} \bar{g}(t)\,dt = -\frac{1}{12}g((i-2)h) + \frac{8}{12}g((i-1)h) + \frac{5}{12}g(ih)$$

Finally, after some computations the operationally improved matrix $\bar{\mathrm{P}}$ will be:

$$\bar{\mathrm{P}} = \frac{h}{2}\begin{bmatrix} 1 & \frac{13}{6} & 2 & 2 & \cdots & 2 \\ 0 & \frac{5}{6} & \frac{13}{6} & 2 & \cdots & 2 \\ 0 & 0 & \frac{5}{6} & \frac{13}{6} & \ddots & \vdots \\ 0 & 0 & 0 & \frac{5}{6} & \ddots & 2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \frac{13}{6} \\ 0 & 0 & 0 & \cdots & 0 & \frac{5}{6} \end{bmatrix}_{m\times m}$$

Volterra's integral equation of the first type

It is considered following Volterra's integral equation of the first type,

$$f(x) = \lambda \int_0^x k(x,t)y(t)dt,\ 0 \le x < 1, \quad (5)$$

In (5), f and k are known and y is not known.

Also, we have, $(x,t) \in \mathcal{L}^2([0,1) \times [0,1)), f \in \mathcal{L}^2[0,1)$. Approximation the relations f, y, and k with respect to BPFs give

$$\begin{aligned} f(x) \simeq F^T\Phi(x) &\quad \simeq \Phi^T(x)F, \\ k(x,t) \simeq \Phi^T(x)K\Psi(t) &\quad \simeq \Psi^T(t)K^T\Phi(x), \\ y(t) \simeq Y^T\Psi(t) &\quad \simeq \Psi^T(t)Y \end{aligned}$$

by the previous relations, we know that $Fvectors, Y$, and Kmatrix are BPFs coefficients of $f(t), y(t)$, and $k(x,t)$, respectively. y is an unknown vector. Replace previous relation in above equation gives:

$$F^T\Phi(x) \simeq \lambda Y^T \int_0^x \Psi(t)\Psi^T(t)dtK^T\Phi(x),$$

Assume that in matrix K^T, we show the *ith* row of it by K_i and in modified integration operationally matrix $\bar{P}$, *jth* row be R_j, by the previous equations and relations and supposition $m_1 = m_2 = m$ we will have,

$$\int_0^x \Psi(t)\Psi^T(t)dtK^T\Phi(x) = \int_0^x \Phi(t)\Phi^T(t)dtK^T\Phi(x)$$
$$= \begin{pmatrix} R_1\Phi(x) & 0 & \dots & 0 \\ 0 & R_2\Phi(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_m\Phi(x) \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_m \end{pmatrix} \Phi(x)$$

$$= \begin{pmatrix} R_1\Phi(x)K_1\Phi(x) \\ R_2\Phi(x)K_2\Phi(x) \\ \vdots \\ R_m\Phi(x)K_m\Phi(x) \end{pmatrix} = \begin{pmatrix} R_1\Phi(x)\Phi^T(x)K_1^T \\ R_2\Phi(x)\Phi^T(x)K_2^T \\ \vdots \\ R_m\Phi(x)\Phi^T(x)K_m^T \end{pmatrix} = \begin{pmatrix} R_1 D_{k_1} \\ R_2 D_{k_2} \\ \vdots \\ R_m D_{k_m} \end{pmatrix} \Phi(x)$$

so, we have,

$$F^T\Phi(x) \simeq \lambda Y^T B\Phi(x),$$

where

$$B=\begin{pmatrix} R_1 D_{k_1} \\ R_2 D_{k_2} \\ \vdots \\ R_m D_{k_m} \end{pmatrix}=\begin{pmatrix} \frac{h}{2}k_{1,1} & \frac{13h}{12}k_{2,1} & hk_{3,1} & \cdots & hk_{m,1} \\ 0 & \frac{5h}{12}k_{2,2} & \frac{13h}{12}k_{3,2} & \ddots & \vdots \\ 0 & 0 & \frac{5h}{12}k_{3,3} & \ddots & hk_{m,m-2} \\ \vdots & \vdots & & \ddots & \frac{13h}{12}k_{m,m-1} \\ 0 & 0 & \cdots & 0 & \frac{5h}{12}k_{m,m} \end{pmatrix}$$

We can also write

$$F^T \simeq \lambda Y^T B,$$

Or, by substituting $\simeq$ by $=$ we will have it

$$F = \lambda B^T Y,$$

The matric B^T can be formulated as follows:

$$B^T = h\begin{pmatrix} \frac{1}{2}k_{1,1} & 0 & 0 & \cdots & 0 \\ \frac{13}{12}k_{2,1} & \frac{5}{12}k_{2,2} & 0 & & \vdots \\ k_{3,1} & \frac{13}{12}k_{3,2} & \frac{5}{12}k_{3,3} & & 0 \\ \vdots & \ddots & & \ddots & 0 \\ k_{m,1} & \cdots & k_{m,m-2} & \frac{13}{12}k_{m,m-1} & \frac{5}{12}k_{m,m} \end{pmatrix}$$

If desired equation has one solution after that obtained equation will have a good-condition inferior the linear trigonometric system of m algebraic equations for the unknowns $y_1, y_2, \cdots, y_m$, which can be easily solved by anterior replacement. As a result, the approximate solution could be calculated to desired equation without using any projection ways. So, form of B^T show that we do not need to evaluate k_{ij} for $j > i$.

Numerical examples

Some examples have been selected from various new references, therefore these may be the numerical results that we have here to compare with both of the exact solution with other numerical achieved results. The calculations of these examples have been done use Matlab 7.

Ex.1 Let the first following integral equation

$$\int_0^t \cos(x-t)y(t)dt = x\sin(x)$$

With the exact solution$y(t) = 2\sin(t)$, for $0 \le t < 1$. The numerically results achieved are presented in Table 1. That results have well accurate in compared to the other numericaly results obtained by using combination of Spline-collocation method and Lagrange interpolation.

Ex.2 Let the second following integral equation

$$\int_0^t \exp(x+t)y(t)dt = x\exp(x)$$

With exact solution $y(t) = \exp(-t)$ for $0 \le t < 1$, Table 2 We note the numerical achieved results.

Table 1: The results obtained for Ex.1

t	Exact solution	Approximate solution ($m = 32$)	Approximate solution ($m = 64$)
0	0	0.026819	0.016710
0.1	0.199666	0.200317	0.194916
0.2	0.397329	0.403089	0.399220
0.3	0.591042	0.585165	0.604607
0.4	0.778834	0.761575	0.779028
0.5	0.958848	0.986410	0.987823
0.6	1.129279	1.143686	1.116140
0.7	1.288428	1.298688	1.290559
0.8	1.434721	1.402992	1.413414
0.9	1.566649	1.693591	1.641487

Table 2: The results obtained for Ex. 2

t	Exact solution	Approximate solution (m = 16)	Approximate solution (m = 32)
0	1	0.975721	0.987874
0.1	0.904829	0.907888	0.895958
0.2	0.818727	0.802365	0.816203
0.3	0.740821	0.755727	0.743096
0.4	0.670318	0.668008	0.676617
0.5	0.606527	0.592751	0.597174
0.6	0.548809	0.544154	0.543308
0.7	0.496579	0.491203	0.496524
0.8	0.449331	0.466475	0.443770
0.9	0.406568	0.375927	0.404181

Conclusion

Using orthogonal piecewise functions as a base to answer the Volterra integral equation of the first kind is very easy with few arithmetic operations and effective in comparison with others with $O(m^2)$ operations. Its numerical application and efficiency have been checked in several examples, we find the approximate solution is abbreviation compared with exact solution only at 10 particular points and we see that the precision will increase as m increases.

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