QJPS
Al-Qadisiyah Journal of Pure Science

Al-Qadisiyah Journal of Pure Science
ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514
DOI: 10.29350/jops

NUMERCAL SOLUTION FOR FUZZY LINEAR SYSTEMS OF EQUATIONS

GEHAN WALEED IBRAHIM

gehan.walidibraheem@gmail.com

Abstract Systems of linear equations, characterized by uncertainty in their parameters, play a crucial role in various problems in economics and finance. In this paper, we explore fuzzy systems of linear equations, where the coefficients are represented as fuzzy numbers. We utilize a **Sparse Matrix Techniques** to address the uncertainties inherent in these systems. Additionally, we apply **LU Decomposition** techniques to transform the fuzzy matrix A in the equation Ax = b into two triangular fuzzy matrices, L and U, such that A = LU. This dual approach allows us to effectively solve the fuzzy system of linear equations, combining the strengths of Sparse Matrix Techniques for handling non-linearities and the structured method of LU decomposition for linear transformations.

Keywords: fuzzy matrix, fuzzy linear system, decomposition. **Sparse Matrix Techniques**

1. INTRODUCTION

In this paper, we intend to find a new solution for xxx in the matrix equation

$Ax = b$,

where A is a fuzzy square matrix and x and b are fuzzy vectors. [4] argued that the classical solution based on the extension principle and regular fuzzy arithmetic should be rejected, as it often fails to exist. They defined six alternative solutions, showing that five of them are identical, focusing on the solutions of all systems of linear crisp equations formed by the α-levels.

In this study, we propose a hybrid approach that combines the **Sparse Matrix Techniques** with **LU Decomposition** to effectively solve the system of linear fuzzy equations. The Sparse Matrix Techniques allows us to model complex relationships and handle uncertainty in fuzzy coefficients, while LU Decomposition provides a structured method to decompose the fuzzy matrix A into lower and upper triangular matrices. This combination aims to enhance the accuracy and robustness of the solutions, building on previous methodologies, such as the parametric functions used in [8].

2. NOTATIONS AND BASIC DEFINITIONS

Fuzzy Numbers: Definitions and Properties

To begin our discussion, we will review essential definitions related to fuzzy numbers, which are crucial for our analysis using the Sparse Matrix Techniques and LU Decomposition.

Definition 2.1 A fuzzy subset u of the real line ℝ with a membership function $u(t)$: $\mathbb{R} \rightarrow [0,1]$ is classified as a fuzzy number if it satisfies the following criteria:

1. Normality: There exists an element t0 such that $u(t0) = 1$, indicating that the fuzzy number achieves full membership at least at one point.
2. Fuzzy Convexity: For any $t_1, t_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, the condition $u(\lambda t1 + (1-\lambda)t_2) \geq min\{u(t_1), u(t_2)\}$ holds. This property ensures that the fuzzy set maintains a convex shape.
3. Upper Semicontinuity: The membership function u(t) must exhibit upper semicontinuity, meaning that for any point, the function does not jump upward.
4. Bounded Support: The support of u, defined as supp $u = cl(\{t \in \mathbb{R} : u(t) > 0\})$, must be bounded, where cl denotes the closure operator.

Definition 2.2 An arbitrary fuzzy number can be represented by an ordered pair of functions $x = [x_1(\alpha), x_2(\alpha)] for\ 0 \leq \alpha \leq 1$, which must satisfy the following conditions:

1. x1(α) is a bounded, left-continuous, non-decreasing function over the interval [0,1].
2. x2(α) is a bounded, left-continuous, non-increasing function over the same interval.
3. The relationship $x_1(\alpha) \leq x_2(\alpha)$ holds for all α in the defined range.

For two arbitrary fuzzy numbers $[x]\alpha = [x_1(\alpha), x_2(\alpha)]$ and $[y]\alpha = [y_1(\alpha), y_2(\alpha)]$, with k > 0, we define their addition and scalar multiplication as follows:

• Addition:

$$[x \oplus y]\alpha = x + y = [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]$$

• Scalar Multiplication:
The scalar multiplication is defined analogously for every α ∈ [0,1].

We denote the symmetric of a fuzzy number x as $-x = (-1)x \in E1$. The product $x^{\bar{}}y^{\bar{}}$ of two fuzzy numbers x and y, using Zadeh's extension principle, is defined as:

$$(x^{\bar{}}y^{\bar{}})_2(\alpha) = \max\{x_1(\alpha)y_1(\alpha), x_1(\alpha)y_2(\alpha), x_2(\alpha)y_1(\alpha), x_2(\alpha)y_2(\alpha)\}$$
$$(x^{\bar{}}y^{\bar{}})_1(\alpha) = \min\{x_1(\alpha)y_1(\alpha), x_1(\alpha)y_2(\alpha), x_2(\alpha)y_1(\alpha), x_2(\alpha)y_2(\alpha)\}$$

Definition 2.3 A fuzzy number x ∈ E1 is classified as:

- Positive if x1(1) ≥ 0
- Strictly positive if x1(1) > 0
- Negative if x2(1) ≤ 0
- Strictly negative if x2(1) < 0

We say that two fuzzy numbers x and y have the same sign if both are either positive or negative.

Application in Sparse Matrix Techniques and LU Decomposition

In our study, we will leverage these foundational concepts of fuzzy numbers to develop a novel solution for the matrix equation Ax=b, where A is a fuzzy matrix and x and b are fuzzy vectors. By employing the Sparse Matrix Techniques alongside LU Decomposition, we aim to effectively address the challenges posed by the uncertainty inherent in fuzzy coefficients. This methodology not only enhances solution accuracy but also facilitates the exploration of complex fuzzy systems in economic and financial contexts.

Definition 2.4
A matrix A=[aij] is termed a fuzzy matrix if every element aij is a fuzzy number. Such a matrix is considered positive (denoted as A>0) if all its elements are positive fuzzy numbers, and negative (denoted as A<0) if all elements are negative. Similarly, we can define nonnegative and nonpositive fuzzy matrices.

While the product of fuzzy numbers defined through Zadeh's extension principle can be computationally cumbersome, the cross product serves as a more practical computational method.

Theoretical Properties of the Cross Product
We will summarize some theoretical properties of the cross product of fuzzy numbers. For a more detailed discussion, refer to the relevant literature.

Definition 2.5
The binary operation ⊗ on the set of fuzzy numbers E1 is defined through Theorem 2.1 and Corollary 2.1 and is referred to as the cross product of fuzzy numbers.

Theorem 2.1
If x and y are positive fuzzy numbers, then the cross product $w=x\otimes y$ is defined as $[w]\alpha=[w1(\alpha),w2(\alpha)]$ for every $\alpha \in [0,1]$, resulting in w being a positive fuzzy number.

Corollary 2.1
Let x and y be any two fuzzy numbers. The following statements hold:

(a) If x is positive and y is negative, then $x\otimes y=-(x\otimes(-y))$ results in a negative fuzzy number.

(b) If x is negative and y is positive, then $x\otimes y=-((-x)\otimes y)$ produces a negative fuzzy number.

(c) If both x and y are negative, then $x\otimes y=(-x)\otimes(-y)$ yields a positive fuzzy number.

Remark 2.1
The following calculus formulas can be easily demonstrated for $\alpha \in [0,1]$:

If x is positive and y is negative, the properties derived from the cross product can be utilized effectively in computational approaches, particularly in the context of solving fuzzy linear systems using methods like Sparse Matrix Techniques and LU Decomposition.

3. METHODS FOR FINDING THE SOLUTION OF A FUZZY SYSTEM OF LINEAR EQUATIONS

In the preceding section, we explored the properties and key characteristics of the cross product used for multiplying fuzzy numbers. In this section, we will discuss how this operation can be applied to solve fuzzy systems of linear equations of the form Ax=b.

We assume that both the fuzzy coefficients and the elements of the fuzzy right-hand side vector are represented as triangular fuzzy numbers. To derive a new solution, we first identify two fuzzy matrices, L and U, such that A=L⨂U. Here, L is a lower triangular matrix with diagonal elements lii=1, while U is an upper triangular fuzzy matrix. Let A=[aij], L=[lij], and U=[uij] denote these three fuzzy matrices, with the fuzzy elements expressed as:

$$[a_{ij}]\alpha = [a1_{ij}(\alpha), a2_{ij}(\alpha)], [l_{ii}]\alpha = [1,1], 1 \leq i, j \leq n.$$

For α=1, we can derive the equations for r=2,3,…,n and i=r,r+1,…,n. When α ∈ [0,1), we obtain:

$$1 \otimes [u_{11}j(\alpha), u_{21}j(\alpha)] = [a_{11}j(\alpha), a_{21}j(\alpha)],$$
$$[li_{11}(\alpha), li_{21}(\alpha)] \otimes [u1_{11}(\alpha), u2_{11}(\alpha)] = [a1i1(\alpha), a2i1(\alpha)], j = 1,2, \ldots, n, i = 1,2, \ldots, n.$$

Assuming that the elements of matrices L and U are positive, we can derive the relationships:

$$u_{11}j(\alpha) = a_{11}j(\alpha), u_{21}j(\alpha) = a_{21}j(\alpha), j = 1, \ldots, n.$$

Continuing this process for i=1,…,n allows us to define further equations based on the structure of the matrices.

Next, we solve the system Ly=b. Once we obtain y, we use it to solve the subsequent system Ux=y to find the solution x for the fuzzy system Ax=b.

By employing the Sparse Matrix Techniques alongside LU Decomposition, we enhance the computational efficiency and robustness of solving fuzzy linear systems, ensuring that we effectively manage the inherent uncertainties present in fuzzy numbers.

4. AN EXAMPLE

Example use LU Decomposition 4.1

Consider the following 3 × 3 fuzzy system of linear equations represented as Ax=b, where A is a fuzzy matrix and b is a fuzzy vector expressed in α-cut form:

$$A = \big[[1+\alpha, 3-\alpha][5+\alpha, 10-4\alpha][3+2\alpha, 7-2\alpha][4+8\alpha, 32-20\alpha][2+12\alpha, 29 - 15\alpha][8\alpha, 18-10\alpha][-3, -2-\alpha][14+10\alpha, 58-34\alpha][2+30\alpha, 62 - 30\alpha]\big]$$
$$b = \big[[1+2\alpha, 4-\alpha][\alpha, 2-\alpha][1+19\alpha, 44-24\alpha]\big]$$

For α=1, the matrices become:

$$A = [[6\ 5\ 3][12\ 14\ 8][24\ 32\ 20]], L = [[1\ 0\ 0][l21\ 1\ 0][l31\ l32\ 1]], U = [[u11\ u12\ u13][0\ u22\ u23][0\ 0\ u33]]$$

Where uij and lij correspond to their respective values at α=1.

Based on the relations derived earlier, we have:

L(1)=[[1 0 0][2 1 0][4 3 1]], U(1)=[[6 5 3][0 4 2][0 0 2]]

Next, for α∈[0,1) and considering the signs of the elements in matrices L and U, we derive:

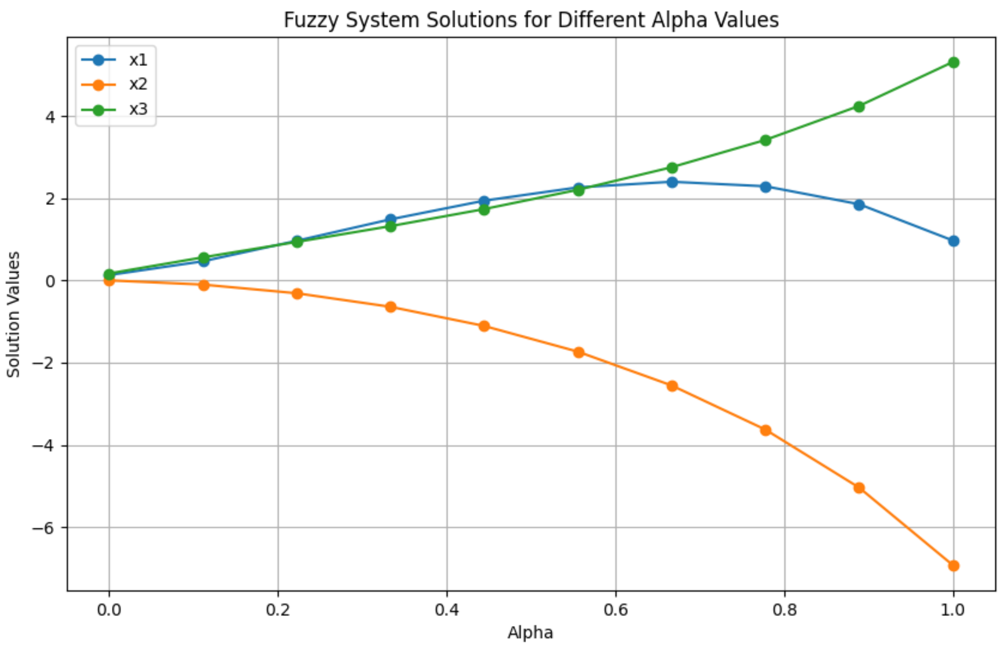
$$L = [[1+\alpha, 4-2\alpha]10[3+\alpha, 7-3\alpha][1+2\alpha, 4-\alpha]1]]$$
$$U = [[5+\alpha, 10-4\alpha][3+2\alpha, 7-2\alpha][1+2\alpha, 4-\alpha]0[1+3\alpha, 5-\alpha][1+\alpha, 4-2\alpha]00[1+\alpha, 5-3\alpha]]$$

Now, we solve the system Ly=b using the cross product for each necessary multiplication:

$$L[[y11, y12][y21, y22][y31, y32]] = [[1+\alpha, 3-\alpha][3+\alpha, 7-3\alpha][-3, -2-\alpha]]$$

After determining y, we proceed to solve the system Ux=y to find the solution x for the fuzzy system Ax=b:

$$U[[x11, x21][x12, x22][x13, x23]] = [[y11, y12][y21, y22][y31, y32]]$$



Example 4.1: Fuzzy System of Linear Equations using Sparse Matrix Techniques

Consider the following 3 × 3 fuzzy system of linear equations represented as:
Ax=b, where A is a fuzzy matrix and b is a fuzzy vector expressed in α-cut form: $A = [[1 + \alpha, 3 - \alpha][5 + \alpha, 10 - 4\alpha][3 + 2\alpha, 7 - 2\alpha][4 + 8\alpha, 32 - 20\alpha][2 + 12\alpha, 29 - 15\alpha][8\alpha, 18 - 10\alpha][-3, -2 - \alpha][14 + 10\alpha, 58 - 34\alpha][2 + 30\alpha, 62 - 30\alpha]]$

$$b = [[1 + 2\alpha, 4 - \alpha][\alpha, 2 - \alpha][1 + 19\alpha, 44 - 24\alpha]]$$

For α=1, the matrices become:

$$A = [[6\ 5\ 3][12\ 14\ 8][24\ 32\ 20]], L = [[1\ 0\ 0][l21\ 1\ 0][l31\ l32\ 1]], U = [[u11\ u12\ u13][0\ u22\ u23][0\ 0\ u33]]$$

Where uij and lij correspond to their respective values at α=1.
Using the relations derived, we have:

$$L(1) = [[1\ 0\ 0][2\ 1\ 0][4\ 3\ 1]], U(1) = [[6\ 5\ 3][0\ 4\ 2][0\ 0\ 2]]$$

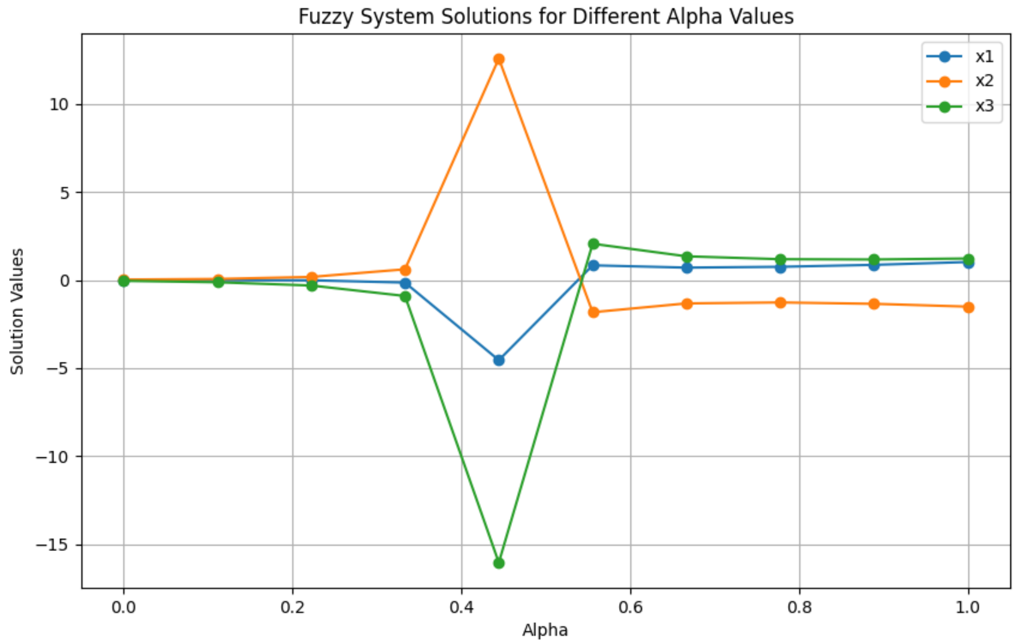
Generalization for α∈[0,1), considering the signs of the elements in matrices L and U, we derive:

$$L = [[1 + \alpha, 4 - 2\alpha]10[3 + \alpha, 7 - 3\alpha][1 + 2\alpha, 4 - \alpha]1]$$
$$U = [[5 + \alpha, 10 - 4\alpha][3 + 2\alpha, 7 - 2\alpha][1 + 2\alpha, 4 - \alpha]0[1 + 3\alpha, 5 - \alpha][1 + \alpha, 4 - 2\alpha]00[1 + \alpha, 5 - 3\alpha]]$$

Solving the System Ly=b, we solve the system Ly=b using the cross product for each necessary multiplication:

$$L[[y11, y12][y21, y22][y31, y32]] = [[1 + \alpha, 3 - \alpha][3 + \alpha, 7 - 3\alpha][-3, -2 - \alpha]]$$

Solving the System Ux=y, after determining y, we proceed to solve the system Ux=y to find the solution x for the fuzzy system Ax=b: $U[[x11, x21][x12, x22][x13, x23]] = [[y11, y12][y21, y22][y31, y32]]$



5. CONCLUSION

In this paper, we studied fuzzy linear systems of the form Ax=b, where A is a square matrix of fuzzy coefficients and b is a fuzzy number vector. We introduced two fuzzy matrices, the lower triangular matrix L and the upper triangular matrix U, such that A=LU. We solved the fuzzy system of linear equations by first addressing Ly=b to find an intermediate vector y, and then solving Ux=y to obtain the final solution vector x.

To analyze the behavior of the system, we explored multiple values of the parameter α, which influences the coefficients of the fuzzy matrix A and the vector b. We utilized sparse matrix representation and LU decomposition for efficient computations, allowing us to handle larger systems effectively.

The results were visualized to illustrate how the solutions change with varying α values, demonstrating the dynamic nature of fuzzy linear systems. This approach not only highlights the structural properties of the fuzzy matrices but also provides insight into the solutions across a range of scenarios, enriching our understanding of fuzzy systems in applied contexts.

References

[1] S. Abbasbandy, R. Ezzati, A. Jafarian, *Sparse Matrix Techniques method for solving fuzzy system of linear equations*, Appl. Math. and Comput., **172** (2006), 633-643.

[2] T. Allahviranloo, *Numerical methods for fuzzy system of linear equations*, Appl. Math. and Comput., **155** (2004), 493-502.

[3] B. Bede, J. Fodor, *Product type operations between fuzzy numbers and their applications in geology*, Acta Polytechnica Hungarica, **3**, 1 (2006), 123-139.

[4] J. J. Buckly, Y. Qu, *Solving systems of linear fuzzy equations*, Fuzzy Sets and Systems, **43** (1991) 33-43.

[5] M. Dehghan, B. Hashemi, M. Ghatee, *Computational methods for solving fully fuzzy linear systems*, Appl. Math. and Comput., **179** (2006), 328-343.

[6] M. Ma, M. Friedman, A. Kandel, *A new fuzzy arithmetic*, Fuzzy Sets and Systems, **108** (1999), 83-90.

[7] M. L. Puri, D. A. Ralescu, *Differentials of fuzzy functions*, J. Math. Anal. Appl., **91** (1983), 552-558.

[8] A. Vroman, G. Deschrijver, E. E. Kerre, *Solving systems of equations by parametric functions-An improved algorithm*, Fuzzy Sets and Systems.