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Cartan Soft Group Space

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Abstract
The main purpose of this work is to introduce a new definition namely Cartan soft group space in soft topological space and we investigate some properties of this concept.

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1. Introduction
The soft set theory introduced by Molodtsov in 1999 [2], this theory has been studied by mathematicians in different ways such as algebraic, topological and categorical [1, 3, 6, 8, 10, 12]. On the other hand, there are some concepts that are interdisciplinary, the action is one of the such concepts and it has a great important in mathematics and physics [5, 13]. It is an integral part of the theory of group and dynamical systems which represent a mathematical equation class that defines time-based systems with specific properties [4, 11]. Furthermore, the theory of dynamical systems, a mathematical theory such as soft set theory, is directly related to the developments in the understanding of complex and nonlinear systems of physics and mathematics [7, 9].

In this study, soft action is defined and studied as a new concept. Examples of this concept is given and some important properties are presented. Some concepts related to the action such as stabilizers, centralizers and normalizers are defined in the soft approach. Finally, the concept of cartan soft group space is introduced and some relations between it and the soft group space are investigated .

2. Notations and Basic Definitions
This section contained the basic definitions, propositions that are needed through this work.

Definition (2.1)[15]
Let $\mathbb{G}$ be a group, $\hat{E}$ be a set of parameters, a soft set $(H, \hat{E})$ is called ***a soft group*** over $\mathbb{G}$ if $H(\omega) < \mathbb{G}$ (i.e $H(\omega)$ is a subgroup of $\mathbb{G}$) for all $\omega \in \hat{E}$.

Definition (2.2)[14]

Let $(\mathbb{G}, \mu, T)$ be a topological group and $(F, \hat{E})$ be a non-null soft set over $\mathbb{G}$, then $(\mathbb{G}, T, F, \hat{E})$ is called a ***soft topological group*** (Stg) over $\mathbb{G}$ if:

i) $F(\omega) < \mathbb{G}$ for all $\omega \in \hat{E}$.

ii) The mapping $\mu_\omega(x, y) = xy$ of the topological space $F(\omega) \times F(\omega)$ onto $F(\omega)$ and the inversion $\psi_\omega : F(\omega) \to F(\omega)$, $\psi_\omega(x) = x^{-1}$ are continuous for all $\omega \in \hat{E}$.

Example (2.3)

Let $\mathbb{G} = S_3 = \{e_{\mathbb{G}}, (12), (13), (23), (123), (132)\}$ is the group of permutations on $S = \{1,2,3\}$ and T be a topology on $\mathbb{G}$ with bases $\beta = \{\{e_{\mathbb{G}}\}, \{(12)\}, \{(123)\}, \{(132)\}\}$, $\hat{E} = \{\omega_1, \omega_2, \omega_3\}$ and $(F, \hat{E})$ be a soft set, where $F(\omega_1) = \{e_{\mathbb{G}}\}$, $F(\omega_2) = \{e_{\mathbb{G}}, (12)\}$, $F(\omega_3) = \{e_{\mathbb{G}}, (123), (132)\}$, then $(\mathbb{G}, T, F, \hat{E})$ is Stg.

Definition (2.4)

A Stg $(\mathbb{G}, T, F, \hat{E})$ is called ***soft compact*** if $(F(\omega), T_{F(\omega)})$ are compact spaces for all $\omega \in \hat{E}$.

Example (2.5)

Let $\mathbb{G} = (Z_2, +_2)$ with discrete topology T , $\hat{E} = \{\omega_1, \omega_2\}$ and $(F, \hat{E})$ be a soft set where $F(\omega_1) = \{0\}, F(\omega_2) = Z_2$. Then $(\mathbb{G}, T, F, \hat{E})$ is a compact Stg.

Definition (2.6)[14]

Let $(\mathbb{G}, T, F, \hat{E})$ be a Stg and let $(X, \Gamma, \hat{E})$ be a soft topological space (briefly Sts). A ***soft action*** of $(\mathbb{G}, T, F, \hat{E})$ on $(X, \Gamma, \hat{E})$ is a continuous map $\varphi_\omega : F(\omega) \times X(\omega) \to X(\omega)$ for all $\omega \in \hat{E}$ such that:

(i) $\varphi_\omega(e_{\mathbb{G}}, x) = x$,for all $x \in X(\omega)$.

(ii) $\varphi_\omega(g, \varphi_\omega(g*, x)) = \varphi_\omega(gg*, x)$, for all $g, g* \in F(\omega)$ and $x \in X(\omega)$.

The Sts $(X, \Gamma, \hat{E})$ is called ***"Soft Group space"*** which is denoted by ($S\mathbb{G}$-space).

Example (2.7)

Let $(\mathbb{G}, T, F, \hat{E})$ be a Stg, then every Sts $(X, \Gamma, \hat{E})$ is $S\mathbb{G}$-space, where $\varphi_\omega : F(\omega) \times X(\omega) \to X(\omega)$ defined by $\varphi_\omega(g, g*) = gg*$, for all $\omega \in \hat{E}$.

Definition (2.8)[14]

Let $(X, \Gamma, \hat{E})$ be a $S\mathbb{G}$-space, then for all $x \in X(\omega)$ the set $Orb_\omega(x) = \{\varphi_\omega(g, x) : g \in F(\omega)\}$ for all $\omega \in \hat{E}$ is called ***the soft Orbit of*** $\boldsymbol{x}$.

Definition (2.9)[14]

Let $(X,\Gamma,\hat{E})$ be a $S\mathbb{G}$ -space, then the set $Stab_{\omega}(x)=\{g\in F(\omega):\varphi_{\omega}(g,x)=x\}$ for all $\omega\in\hat{E}$ is called ***the soft stabilizer of x***.

Definition (2.10)[14]

Let $(X,\Gamma,\hat{E})$ be a $S\mathbb{G}$ -space, then the soft kernel of the soft action $\boldsymbol{\varphi_{\omega}}$ is defined by $Ker\varphi_{\omega}=\{g\in F(\omega):\varphi_{\omega}(g,x)=x$ for all $x\in X(\omega)\}$ and $\omega\in\hat{E}$.

Proposition (2.11)

Let $(X,\Gamma,\hat{E})$ be a $S\mathbb{G}$-space, if X be a Hausdorff space, then for all $\omega\in\hat{E}$:

i) $Ker\varphi_{\omega}$ is a closed normal subgroup of $F(\omega)$.

ii) $Stab_{\omega}(x)$ is a closed subgroup of $F(\omega)$, for all $x\in X(\omega)$.

proof:

i) It is clear that $Ker\varphi_{\omega}$ is a normal subgroup of $F(\omega)$ for all $\omega\in\hat{E}$

Now let $g\in cl(Ker\varphi_{\omega})$, then there is a net $\{g_{\beta}\}_{\beta\in\Omega}$ in $Ker\varphi_{\omega}$ such that $g_{\beta}\to g$, then $\varphi_{\omega}(g_{\beta},x)\to\varphi_{\omega}(g,x)$ for all $x\in X(\omega)$, but $g_{\beta}\in Ker\varphi_{\omega}$, then $\varphi_{\omega}(g_{\beta},x)=x$, for all $x\in X(\omega)$.

Then $g\in Ker\varphi_{\omega}$.Hence $Ker_{\omega}\varphi$ is closed.

ii) It is clear that $Stab_{\omega}(x)<F(\omega)$, for all $x\in X(\omega)$.

Let $g\in cl(Stab_{\omega}(x))$, then there is a net $\{g_{\beta}\}_{\beta\in\Omega}$ in $Stab_{\omega}(x)$ such that $g_{\beta}\to g$.

But $\varphi_{\omega}(g_{\beta},x)=x$ & $\varphi_{\omega}(g_{\beta},x)\to\varphi_{\omega}(g,x)$

Since $X(\omega)$ is a Hausdorff Space, then $\varphi_{\omega}(g,x)=x$

So $g\in Stab_{\omega}(x)$.Thus $Stab_{\omega}(x)$ is closed set.

Definition (2.12)

Let $(X,\Gamma,\hat{E})$ be a $S\mathbb{G}$-space. A subset M of $X(\omega)$ is said to be **thin relative to a subset** N of $X(\omega)$ if the set $((M,N))=\{g\in F(\omega):\varphi_{\omega}(g,M)\cap N\neq\emptyset\}$ has a neighborhood whose closure is compact set in $F(\omega)$ for all $\omega\in\hat{E}$. If M is thin relative itself, then it is called ***thin***.

Remarks (2.13)

Let $(X,\Gamma,\hat{E})$ be a S$\mathbb{G}$-Space, then:

(i) Since $\varphi_{\omega}(g,M)\cap N=\varphi_{\omega}(g,M\cap\varphi_{\omega}(g^{-1},N))$, then if M relative to N, then N is relative to M.

(ii) Since $\varphi_{\omega}(g,\varphi_{\omega}(g_1,M))\cap\varphi_{\omega}(g_2,N)=\varphi_{\omega}(g_2,\varphi_{\omega}(g_2^{-1}gg_1,M)\cap N)$, it follows that if M and N relatively thin, then so any translates $\varphi_{\omega}(g_1,M)$ and $\varphi_{\omega}(g_2,N)$.

(iii) If M and N are relatively thin and $M'\subseteq M$, $N'\subseteq N$, then M' and N' are relatively thin. In particular a subset of a thin set is thin.

(iv) If M,N be compact subsets of $X(\omega)$, then $((M,N))$ is closed in $F(\omega)$

(v) If M, N are compact subsets of $X(\omega)$, then $((M, N))$ is compact in $F(\omega)$.

Theorem (2.14)

Let $(X, \Gamma, \hat{E})$ be a Hausdorff $S\mathbb{G}$-Space and M, N are compact subset of $X(\omega)$, then for all $\omega \in \hat{E}$:

(i) $((M, N))$ is closed subset of $F(\omega)$.

(ii) $((M, N))$ is compact when M and N are relatively thin.

Proof:

(i) Let $h \in cl((M, N))$, then there is a net $\{g_\beta\}_{\beta\in\Omega}$ in $((M, N))$ such that $g_\beta \to h$.

Since $g_\beta \in ((M, N))$, then there is a net $\{a_\beta\}_{\beta\in\Omega}$ in M which is compact such that $\varphi_\omega(g_\beta, a_\beta) \in N$.

Since $M$and N are compact and $X(\omega)$ is Hausdorff space, then there are $m \in M$, $n \in N$ such that $a_\beta \to m$ and $\varphi_\omega(g_\beta, a_\beta) \to n$.

Since $g_\beta \to h$, $\varphi_\omega(g_\beta, a_\beta) \to \varphi_\omega(h, m)$.

Since $X(\omega)$ is Hausdorff space, then $\varphi_\omega(h, m) = n$, thus $\varphi_\omega(h, M) \cap N \neq \emptyset$.

Then $h \in ((M, N))$.Thus $((M, N))$ is closed set.

(ii) Let $\{g_\beta\}_{\beta\in\Omega}$ be a net in $((M, N))$

Since $((M, N))$ is closed from(i), then there is $h \in cl((M, N))$ such that $g_\beta \to h$.

Then every net in $((M, N))$ has a subnet (say itself) convergent to h.Therefore $((M, N))$ is compact.

3. Cartan Soft Group Space

In this section we introduce a new concept namely cartan soft group space and introduce some results about it.

Definition (3.1)

A S$\mathbb{G}$-Space $(X, \Gamma, \hat{E})$ is called ***Cartan soft group space (CSG-space)*** if every point in $X(\omega)$ has a thin neighborhood, for all $\omega \in \hat{E}$.

Example (3.2)

Let $\mathbb{G} = (\mathcal{R}\backslash\{0\}, .)$ with usual topology, $\hat{E} = \{\omega_1, \omega_2\}$and let $(F, \hat{E})$ be a soft set over $\mathbb{G}$ such that $F(\omega_1) = F(\omega_2) = \mathcal{R}\backslash\{0\}$. Let $(X, \Gamma, \hat{E})$ be a sts such that $X(\omega_1) = X(\omega_2) = \mathcal{R}^2$. If we define $\varphi_{\omega_1} = \varphi_{\omega_2} = \varphi_\omega: \mathcal{R}\backslash\{0\} \times \mathcal{R}^2 \to \mathcal{R}^2$ by $\varphi_\omega(r, (x, y)) = (rx, y)$ for all $\omega \in \hat{E}$, then $(X, \Gamma, \hat{E})$ isn't $CS\mathbb{G}$ $-$space because $(0,0) \in \mathcal{R}^2$ has no thin neighborhood.

Theorem (3.3)

A $S\mathbb{G}$-space X is $CS\mathbb{G}$ $-$space if $(\mathbb{G}, \mathcal{T}, F, \hat{E})$ is soft compact.

Proof:

Let $x \in X(\omega)$ and M is a neighborhood of x, then $((M, M))$ has a neighborhood $F(\omega)$ whose closure is compact, hence $(X, \Gamma, \hat{E})$ is $CS\mathbb{G}$-space.

Theorem (3.4)

If $(X, \Gamma, \hat{E})$ be a $CS\mathbb{G}$-spac, then for all $\omega \in \hat{E}$:

(i) $Orb_\omega(x)$ is closed

(ii) $Stab_\omega(x)$ is compact

Proof:

(i) Let $y \in cl(Orb_\omega(x))$, then there is a net $\{y_\beta\}_{\beta\in\Omega}$ in $Orb_\omega(x)$ such that $y_\beta \to y$.

Since $(X, \Gamma, \hat{E})$ be a $CS\mathbb{G}$-space, then has a thin neighborhood M

Since $y_\beta \in Orb_\omega(x)$, then there is a net $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ such that $y_\beta = \varphi_\omega(g_\beta, x)$ for all $\beta \in \Omega$.

Then $\{\varphi_\omega(g_\beta, x)\}_{\beta\in\Omega}$ is a net in M where $\varphi_\omega(g_\beta, x) \to y$

There is $\beta_0 \in \Omega$ and $\varphi_\omega(g_\beta g_{\beta_0}^{-1}, \varphi_\omega(g_{\beta_0}, x)) = \varphi_\omega(g_\beta, x)$

Thus $\{g_\beta g_{\beta_0}^{-1}\} \in ((M, M))$, hence the net $\{g_\beta g_{\beta_0}^{-1}\}$ has a convergent subnet (say itself).

Then there is $g \in F(\omega)$ such that $g_\beta g_{\beta_0}^{-1} \to g$, hence $g_\beta \to g g_{\beta_0}$.

Then $\varphi_\omega(g_\beta, x) \to \varphi_\omega(g g_{\beta_0}, x)$ and $\varphi_\omega(g g_{\beta_0}, x) = y$, so $y \in Orb_\omega(x)$. Thus $Orb_\omega(x)$ is closed set.

(ii) Let $x \in X(\omega)$, then there is a compact thin neighborhood N of x

Since $Stab_\omega(x)$ is closed in $F(\omega)$ by (2.11), and since $Stab_\omega(x) \subseteq ((N, N))$ which is compact by theorem (2.14), hence $Stab_\omega$ is compact.

Theorem (3.5)

Let $(X, \Gamma, \hat{E})$ be a $CS\mathbb{G}$-space, let $(H, \hat{E})$ be a soft closed subgroup of $(F, \hat{E})$ over $\mathbb{G}$, and let Y be an open (closed) subspace of $X(\omega)$ which is an invariant under $H(\omega)$. Then $Y$is cartan H-space.

The proof is trivial.

Corollary (3.6)

Let $(X, \Gamma, \hat{E})$ be a $CS\mathbb{G}$-Space, and let Y be an open (closed) subspace of $X(\omega)$ which is an invariant under $F(\omega)$, then Y is cartan $\mathbb{G}$-space.

Corollary (3.7)

Let $(X, \Gamma, \hat{E})$ be a $CS\mathbb{G}$-Space, and let $(H, \hat{E})$ be a soft closed subgroup of $(F, \hat{E})$ over $\mathbb{G}$, then $X(\omega)$ is cartan soft H-space.

Definition (3.8)

Let $(X, \Gamma, \hat{E})$ be a $CS\mathbb{G}$-space, for any point $x \in X(\omega)$, then for all $\omega \in \hat{E}$:

(i) $J_\omega(x) = \{y \in X(\omega)$: there is a net $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ and there is a net $\{\chi_\beta\}_{\beta\in\Omega}$ in $X(\omega)$ with $g_\beta \to \infty$ and $\chi_\beta \to x$ such that $\varphi_\omega(g_\beta, \chi_\beta) \to y\}$ is called ***first soft prolongation limit set of $X(\omega)$***.

(ii) $\Lambda_\omega(x) = \{y \in X(\omega)$: there is a net $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ with $g_\beta \to \infty$ such that $\varphi_\omega(g_\beta, x) \to y\}$ is called ***soft limit set of $X(\omega)$***.

It is clear that the set $\Lambda_\omega(x)$ is a subset of $J_\omega(x)$ for all $\omega \in \hat{E}$.

Example (3.9)

Let $\mathbb{G} = (\mathcal{R}, .)$ with discrete space, $\hat{E} = \{\omega_1, \omega_2\}$ and let $(F, \hat{E})$ be a soft set over $\mathbb{G}$, such that $F(\omega_1) = F(\omega_2) = X(\omega_1) = X(\omega_2) = \mathcal{R}$, if we defined $\varphi_{\omega_1} = \varphi_{\omega_2} = \varphi_\omega : F(\omega) \times F(\omega) \to F(\omega)$ by $\varphi_\omega(g, h) = g.h$ for all $\omega \in \hat{E}$, then $(X, \Gamma, \hat{E})$ is a $S\mathbb{G}$-space and $\Lambda_{\omega_1}(x) = \Lambda_{\omega_2}(x) = \{0\}, J_{\omega_1}(x) = J_{\omega_2}(x) = \{0\}$.

Example (3.10)

Let $\mathbb{G} = (Z, +)$with discrete topology T, $\hat{E} = \{\omega_1, \omega_2\}$. Let $(F, \hat{E})$ be a soft set over $\mathbb{G}$, such that $F(\omega_1) = F(\omega_2) = Z$, let $(X, \Gamma, \hat{E})$ be a soft topological space over $\mathcal{R}$such that $X(\omega_1) = X(\omega_2) = \mathcal{R}\backslash\{0\}$. Define $\varphi_{\omega_1} : Z \times \mathcal{R} \to \mathcal{R}$by $\varphi_{\omega_1}(g, x) = x$, for all$x \in X(\omega_1)$. And $\varphi_{\omega_2} : Z \times \mathcal{R} \to \mathcal{R}$ by $\varphi_{\omega_2}(g, x) = g + x$, for all $x \in X(\omega_2)$. Then $(X, \Gamma, \hat{E})$ is $S\mathbb{G}$-space such that $\Lambda_{\omega_1}(x) = J_{\omega_1}(x) = \{x\}$, and $\Lambda_{\omega_2} = J_{\omega_2}(x) = \emptyset$.

Theorem (3.11)

Let $(X, \Gamma, \hat{E})$ be a $S\mathbb{G}$-space, then for all $x \in X(\omega)$ and for all $\omega \in \hat{E}$:

(i) $\Lambda_\omega(x)$ is an invariant set under $F(\omega)$.

(ii) $Orb_\omega(x)$ is closed iff $\Lambda_\omega(x) \subseteq Orb_\omega(x)$.

(iii) If $x \notin \Lambda_\omega(x)$, then $Stab_\omega(x)$ is compact.

(iv) $\Lambda_\omega(x)$ is closed set.

(v) $cl(Orb_\omega(x)) = Orb_\omega(x) \cup \Lambda_\omega(x)$.

(vi) $\varphi_\omega(g, \Lambda_\omega(x)) = \Lambda_\omega(\varphi_\omega(g, x)) = \Lambda_\omega(x)$.

Proof:

(i) Let $y \in \Lambda_\omega(x)$ and $g \in F(\omega)$,then there is a net $\{g_\beta\}$ in $F(\omega)$ with $g_\beta \to \infty$ such that $\varphi_\omega(g_\beta, x) \to y$.

It is clear that $\{gg_\beta\}_{\beta\in\Omega}$ is a net in $F(\omega)$ with $gg_\beta \to \infty$

By continuity of φ_ω, we have $\varphi_\omega(g, \varphi_\omega(g_\beta, x)) \to \varphi_\omega(g, y)$.

Thus $\varphi_\omega(gg_\beta, x) \to \varphi_\omega(g, y)$. Hence $\varphi_\omega(g, y) \in \Lambda_\omega(x)$, but $\varphi_\omega(g, y) \in F(\Lambda_\omega(x))$

Then $F(\Lambda_\omega(x)) = \Lambda_\omega(x)$. Therefore $\Lambda_\omega(x)$ is an invariant set under $F(\omega)$.

(ii) Suppose $Orb_\omega(x)$ is closed and let $y \in \Lambda_\omega(x)$

Then there is a net $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ with $g_\beta \to \infty$ such that $\varphi_\omega(g_\beta, x) \to y$

But $\varphi_\omega(g_\beta, x) \in Orb_\omega(x)$, then $y \in Cl(Orb_\omega(x)) = Orb_\omega(x)$. Therefore $\Lambda_\omega(x) \subseteq Orb_\omega(x)$.

Conversely

Suppose that $\Lambda_\omega(x) \subseteq Orb_\omega(x)$ and let $y \in Cl(Orb_\omega(x))$, then there is a net $\{y_\beta\}_{\beta\in\Omega}$ in $Orb_\omega(x)$ such that $y_\beta \to y$.

Then $y_\beta = \varphi_\omega(g_\beta, x) \to y$ for some net $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$.Hence either $g_\beta \to \infty$ or $g_\beta \to g$

If $g_\beta \to \infty$, then $y \in \Lambda_\omega(x)$. Hence $Orb_\omega(x)$ is closed

If $g_\beta \to g$, then $\varphi_\omega(g_\beta, x) \to \varphi_\omega(g, x) = y$. Therefore $Orb_\omega(x)$ is closed.

(iii) Let $x \notin \Lambda_\omega(x)$, suppose that $Stab_\omega(x)$ isn't compact

Then there is a net $\{g_\beta\}$ in $Stab_\omega(x)$ such that $g_\beta \to \infty$

Since $\varphi_\omega(g_\beta, x) = x$ for all $\beta \in \Omega$. Then $\varphi_\omega(g_\beta, x) \to x$ which is contradiction

Thus $Stab_\omega(x)$ is compact.

(iv) Let $y \in cl(\Lambda_\omega(x))$, then there is a net $\{y_\beta\}_{\beta\in\Omega}$ in $\Lambda_\omega(x)$ such that $y_\beta \to y$Then for all $\beta \in \Omega$, there is a net $\{g_\delta^\beta\}_{\delta\in M_\beta}$ in $F(\omega)$ with $g_\delta^\beta \to \infty$ and $\varphi_\omega(g_\delta^\beta, x) \to y_\beta$

Then there is a subnet $\{g_{\delta_i}^{\beta_i}\}_{\delta_{i\in M_\beta}}$ such that $\varphi_\omega(g_{\delta_i}^{\beta_i}, x) \to y$

Now either $g_{\delta_i}^{\beta_i} \to \infty$ or $g_{\delta_i}^{\beta_i} \to g$

If $g_{\delta_i}^{\beta_i} \to \infty$, then $y \in \Lambda_\omega(x)$.

If $g_{\delta_i}^{\beta_i} \to g$, then $\varphi_\omega(g_{\delta_i}^{\beta_i}, x) \to \varphi_\omega(g, x) = y$, hence $y \in \Lambda_\omega(x)$

Therefore $\Lambda_\omega(x)$ is closed for all $\omega \in \hat{E}$.

(v) If $Orb_\omega(x)$ is closed, then $cl(Orb_\omega(x)) = Orb_\omega(x)$

Hence $cl(Orb_\omega(x)) = \Lambda_\omega(x) \cup Orb_\omega(x)$ from(ii).

If $Orb_\omega(x)$ isn't closed, let $y \notin Orb_\omega(x)$ and $y \in \Lambda_\omega(x)$, then there is a net $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ with $g_\beta \to \infty$ such that $\varphi_\omega(g_\beta, x) \to y$

Since $\{gg_\beta\}_{\beta\in\Omega}$ is a net in $F(\omega)$ with $gg_\beta \to \infty$, so $\varphi_\omega(gg_\beta, x) \to \varphi_\omega(g, y)$.

That is $\varphi_\omega(g, y) \in \Lambda_\omega(x)$, but $\varphi_\omega(g, y) \in Orb_\omega(x)$

Then $\Lambda_\omega(x) \subseteq Orb_\omega(x)$, therefore by (ii) $Orb_\omega(x)$ is closed. Hence $cl(Orb_\omega(x)) = Orb_\omega(x) \cup \Lambda_\omega(x)$.

(vi) clear.

Theorem (3.12)

Let $(X, \Gamma, \hat{E})$ be a $S\mathbb{G}$-space, $x \in X(\omega)$, then for all $\omega \in \hat{E}$:

(i) $J_\omega(x)$ is an invariant under $F(\omega)$.

(ii) $J_\omega(x)$ is closed set.

(iii) $y \in J_\omega(x)$ iff $x \in J_\omega(y)$.

(iv) $\varphi_\omega(g, J_\omega(x)) = J_\omega(\varphi_\omega(g, x)) = J_\omega(x)$.

Proof:

(i) Let $y \in J_\omega(x)$ and $g \in F(\omega)$, then there are nets $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ with $g_\beta \to \infty$and $\{\chi_\beta\}_{\beta\in\Omega}$ in $X(\omega)$ with $\{\chi_\beta\} \to x$ such that $\varphi_\omega(g_\beta, \chi_\beta) \to y$.

It is clear that $\{gg_\beta\}_{\beta\in\Omega}$ is a net in $F(\omega)$ with $gg_\beta \to \infty$

Thus $\varphi_\omega(gg_\beta, \chi_\beta) \to \varphi_\omega(g, y)$, then $\varphi_\omega(g, y) \in J_\omega(x)$

But $\varphi_\omega(g, y) \in F(J_\omega(x))$, then $F(J_\omega(x)) = J_\omega(x)$

(ii) Let $y \in cl(J_\omega(x))$, then there is a net $\{y_\beta\}_{\beta\in\Omega}$ in $J_\omega(x)$ such that $y_\beta \to y$.

Then there is a net $\{g_\delta^\beta\}_{\delta\in M_\beta}$in $F(\omega)$ with $g_\delta^\beta \to \infty$ and $\chi_\beta \to x$ such that $\varphi_\omega(g_\delta^\beta, \chi_\beta) \to y_\beta$.

Then there is a diagonal subnet $\{g_{\delta_i}^{\beta_i}\}_{\delta_i\in M_\beta}$ such that $\varphi_\omega(g_{\delta_i}^{\beta_i}, \chi_{\beta_i}) \to y$.

Now either $g_{\delta_i}^{\beta_i} \to \infty$ or $g_{\delta_i}^{\beta_i} \to g$

If $g_{\delta_i}^{\beta_i} \to \infty$, then $y \in J_\omega(x)$, thus $J_\omega(x)$ is closed.

If $g_{\delta_i}^{\beta_i} \to g$, then $\varphi_\omega(g_{\delta_i}^{\beta_i}, \chi_{\beta_i}) \to \varphi_\omega(g, \chi_{\beta_i}) = y$. Hence $y \in J_\omega(x)$, therefore $J_\omega(x)$ is closed.

(iii) Let $y \in J_\omega(x)$, then there are nets $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ and $\{\chi_\beta\}_{\beta\in\Omega}$ in $X(\omega)$ with $g_\beta \to \infty$ and $\{\chi_\beta\} \to x$ such that $\varphi_\omega(g_\beta, \chi_\beta) \to y$

If we put $\varphi_\omega(g_\beta, \chi_\beta) = y_\beta \to y$

Since $g_\beta^{-1} \to \infty$ and $\varphi_\omega(g_\beta^{-1}, y_\beta) \to x$, then $x \in J_\omega(y)$.

The converse is similar

(iv) Is clear.

Theorem (3.13)

Let $(X, \Gamma, \hat{E})$ be a $S\mathbb{G}$-space, $x \in X(\omega)$, then for all $\omega \in \hat{E}$:

(i) If $(X, \Gamma, \hat{E})$is discrete, then $\Lambda_\omega(x) = J_\omega(x)$, for all $x \in X(\omega)$.

(ii) If $x \in J_\omega(x)$, then $y \in J_\omega(y)$ for all $y \in Orb_\omega(x)$.

(iii) If $y \in \Lambda_\omega(x)$, then $y \in J_\omega(y)$ for some $y \in X(\omega)$.

(iv) If $x \notin J_\omega(x)$, then $\Lambda_\omega(x) = \emptyset$ for all $x \in X(\omega)$.

Proof:

(i) Clear

(ii) Let $x \in J_\omega(x)$ and $y \in Orb_\omega(x)$

Since $J_\omega(x)$ is an invariant, then $y \in J_\omega(x)$ for all $y \in Orb_\omega(x)$

Then $x \in J_\omega(y)$, but $J_\omega(y)$ is an invariant

Then $y \in J_\omega(y)$.

(iii) let $y \in \Lambda_\omega(x)$, then there is a net $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ with $g_\beta \to \infty$ such that $\varphi_\omega(g_\beta, x) \to y$.

Put $y_\beta = \varphi_\omega(g_\beta, x) \to y$

Since $g_\beta^{-1} \to \infty$ and since $\varphi_\omega(g_\beta^{-1}, y_\beta) = \varphi_\omega(g_\beta^{-1}, \varphi_\omega(g_\beta, x)) \to x$

Then $x \in J_\omega(y)$

Since $J_\omega(y)$ is an invariant, then $\varphi_\omega(g_\beta, x) \in J_\omega(y)$.

Since $\varphi_\omega(g_\beta, x) \to y$ and $J_\omega(y)$ is closed, then $y \in J_\omega(y)$.

(iv) Let $x \notin J_\omega(x)$ for all $x \in X(\omega)$

If $y \in \Lambda_\omega(x)$, then $y \in J_\omega(y)$ from (iii) and it is contradiction

Thus $\Lambda_\omega(x) = \emptyset$ for all $x \in X(\omega)$.

Theorem (3.14)

Let $(X, \Gamma, \hat{E})$ be a $S\mathbb{G}$-space, if $\Lambda_\omega(x) = \emptyset$ for all $x \in X(\omega)$, then $Orb_\omega(x)$ isn't compact for all $\omega \in \hat{E}$.

Proof:

Suppose that $Orb_\omega(x)$ is compact

Let $\{g_\beta\}_{\beta\in\Omega}$ be a net in $F(\omega)$ with $g_\beta \to \infty$

Since $\{\varphi_\omega(g_\beta, x)\}_{\beta\in\Omega}$ is a net in $Orb_\omega(x)$ which is compact, then $\{\varphi_\omega(g_\beta, x)\}_{\beta\in\Omega}$ has a subnet (say itself) which is convergent to y for some $y \in X(\omega)$.Thus $y \in \Lambda_\omega(x)$, this is contradiction.

Then $Orb_\omega(x)$ isn't compact.

Theorem (3.15)

Let $(X, \Gamma, \hat{E})$ be a $CS\mathbb{G}$-space, then $\Lambda_\omega(x) = \emptyset$ for all $x \in X(\omega)$.

Proof:

Suppose $\Lambda_\omega(x) \neq \emptyset$, then there is a point $y \in X(\omega)$ such that $y \in \Lambda_\omega(x)$.

Then there is a net $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ with $g_\beta \to \infty$ such that $\varphi_\omega(g_\beta, x) \to y$

Let U be a compact thin neighborhood of y, then there is $\beta_0 \in \Omega$ such that $\varphi_\omega(g_\beta, x) \in U$ for all $\beta \geq \beta_0$, we get $\varphi_\omega(g_\beta g_{\beta_0}^{-1} g_{\beta_0}, x) = \varphi_\omega(g_\beta, x) \in U$.

Thus $g_\beta g_{\beta_0}^{-1} \in ((U, U))$ which is compact

Hence the net $\{(g_\beta g_{\beta_0}^{-1})\}_{\beta\in\Omega}$ has a convergent subnet (say itself), that is there is $g \in F(\omega)$ such that $g_\beta g_{\beta_0}^{-1} \to g$, this contradiction. Then $y \notin \Lambda_\omega(x)$. Hence $\Lambda_\omega(x) = \emptyset$.

Theorem (3.16)

Let $(X, \Gamma, \hat{E})$ be a $CS\mathbb{G}$-space, and $x \in X(\omega)$. If $x \in J_\omega(x)$, then x has no thin neighborhood.

Proof:

Let $x \in J_\omega(x)$ and suppose that x has a thin neighborhood

Then there is a compact neighborhood U of x such that $((U, U))$ has a compact closure.

Since $x \in J_\omega(x)$, then there are nets $\{g_\beta\}_{\beta\in\Omega}$ in $F(\omega)$ with $g_\beta \to \infty$and $\{x_\beta\}_{\beta\in\Omega}$ in $X(\omega)$ with $\chi_\beta \to x$ such that $\varphi_\omega(g_\beta, \chi_\beta) \to x$

Since U is a neighborhood of x, then there is $\beta_0 \in \Omega$ such that $x_\beta, \varphi_\omega(g_\beta, x_\beta) \in U$ for all $\beta \geq \beta_0$.

So $g_\beta \in ((U, U))$ for all $\beta \geq \beta_0$ which is compact closure, therefore the net $\{g_\beta\}_{\beta\in\Omega}$ has convergent subnet. This is contradiction.

Theorem (3.17)

Let (I_G, f_ω) be a morphism from $S\mathbb{G}$-space $(X, \Gamma, \hat{E})$ into $S\mathbb{G}$-space $(X', \hat{E}, \grave{\Gamma})$ such that $f_\omega: X(\omega) \to X'(\omega)$ be homeomorphism for all $\omega \in \hat{E}$. If $(X, \Gamma, \hat{E})$ is $CS\mathbb{G}$-space, then so is $(X', \hat{E}, \grave{\Gamma})$.

Proof:

Let $y \in X'(\omega)$, since f_ω is homeomorphism for all $\omega \in \hat{E}$, then there is $x \in X(\omega)$ such that $f_\omega(x) = y$.

Since $(X, \Gamma, \hat{E})$ is $CS\mathbb{G}$-space, then x has a thin neighborhood M.

Since f_ω is homeomorphism, then $f_\omega(M)$ is neighborhood of y in $X'(\omega)$.

Since f_ω is an injective and equivariant function for all $\omega \in \hat{E}$, then:

$$\begin{aligned} g \in ((M, M)) &\leftrightarrow \varphi_\omega(g, M) \cap M \neq \emptyset \\ &\leftrightarrow \varphi_\omega(g, f_\omega(M)) \cap f_\omega(M) \neq \emptyset \\ &\leftrightarrow g \in ((f_\omega(M), f_\omega(M))) \end{aligned}$$

Then $((M, M)) = ((f_\omega(M), f_\omega(M)))$

Since $((M, M))$ has compact closure, then so is $((f_\omega(M), f_\omega(M)))$.

Then $(X', \hat{E}, \grave{\Gamma})$ is $CS\mathbb{G}$-space.

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