# Comparing Estimators of Reliability Function of Modified Pareto Distribution

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**المستخلص:**

في هذا البحث قمنا بتطوير توزيع باريتو ذي المعلمتين $(c, \theta)$ إلى ثلاث معلمات $(c, \theta, a)$ هذا التعديل ضروري جداً للحصول على افضل توفيق للتوزيع الاحتمالي للبيانات وكذلك تعديل مقدرات الامكان الاعظم عندما تكون المقدرات متحيزة إلى مقدرات غير متحيزة وذات أقل تباين ممكن ويعمل هذا التعديل على الحصول على بيانات ذات اقل تذبذب وأكثر ملائمة للتوزيع الاحتمالي وأيضاً عندما تكون المشاهدات صغيرة يمكن تعديلها من خلال التحوير من خلال ادخال معلمة معينة وقد عملنا في هذا البحث على تحوير توزيع باريتو من ذي المعلمتين $(c, \theta)$ إلى توزيع ذي ثلاث معلمات $(c, \theta, a)$ ثم تم تقدير المعلمات بالإمكان الاعظم ومقدرات العزوم بعد ان تم اشتقاق العينة العامة للعزم الرائي ثم تم تقدير دالة المعولية $R_T(t)$ ومقارنة المقدرات بواسطة المقياس الاحصائي (متوسط مربعات الخطأ) ولعينات مختلفة الحجوم.

**Abstract:**

This paper deals with modifying a two parameters pareto $(c, \theta)$ distribution into three parameters pareto $(c, \theta, a)$, this modification is necessary to find the best fitting model for observation, and also to modify the maximum likelihood estimator when its biased to another which is minimum variance unbiased estimator, the modification gives stable and less sensitive data to a random fluctuations in the observed data, also when the observations are small we can modify them through powering to a certain parameters [6][7]. In this research we modify the two parameters $(c, \theta)$ into three parameters $(c, \theta, a)$ and also we estimate these parameters by maximum likelihood methods and Moments estimators which obtained after deriving for formula of rth moments about origin. Then the reliability function $R_T(t)$ is estimated and compared using different sample size (n=25,50,75) and applying simulation procedure, all the results of simulation are explained in table and the results are compared using statistical measures mean square error (MSE).

**Keyword:** modified Pareto distribution (MPD), reliability function $R_T(t)$, maximum likelihood estimator (MLE), moment estimator method (MOM).

### (1-1) Introduction

The two parameters Pareto distribution (which belong to exponential family distribution) is defined by equation (1):

$$F\ (x,\theta.c) = \theta\ c^{\theta}\ x^{-\theta-١} \dots\dots\dots\dots\dots (١)$$

When

$X > C$ X: Random Variable

$\theta > O$ $\theta$: *Shape Parameter*

$C > O$ C: Scale Parameter

and the cumulative distribution function :

$$F\ (x) = ١ - \left(\frac{c}{x}\right)^{\theta}, x > ٢ \dots\dots\dots\dots\dots (٢)$$

$$E\ (x) = \left(\frac{\theta}{\theta - ١} c\right), \theta > ١$$

$$Var\ (x) = \frac{\theta}{(\theta - ١)^{٢}(\theta - ٢)} C^{٢}, \theta > ٢ \dots\dots\dots\dots\dots. (٣)$$

Here we work on finding another modified distribution using new reparmetrazati on formula [5][3][9].

$$F\ (x,a) = \frac{f\ (x)}{f\ (x) +\ a\ \left(١ - f(x)\right)} \dots\dots\dots\dots\dots (٤)$$

According to eq (4), the new modified Pareto distribution is Which we obtain [1][4] p.d.f:

$$F\ (x,a) = \frac{af\ (x)}{[f\ (x) +\ a\ (١ - f\ (x))]^{٢}} \dots\dots\dots\dots\dots (٥)$$

$$F\ (x,a) = \frac{af\ (x)}{[a - (a - ١)\ f\ (x)]^{٢}} \dots\dots\dots\dots\dots (٦)$$

Applying equation (1) and (2) in equation (4) and (6), we obtain the new modified three parameters Pareto (a, $\theta$, c):

$$F\ (x, a) = \frac{١ - \left(\frac{c}{x}\right)^{\theta}}{١ - \left(\frac{c}{x}\right)^{\theta} + a\left(\frac{c}{x}\right)^{\theta}}$$

This modified three parameters CDF:

$$F\ (x, a) = \frac{١ - \left(\frac{c}{x}\right)^{\theta}}{١ - (١ - a)\left(\frac{c}{x}\right)^{\theta}} \dots\dots\dots\dots\dots (٧)$$

And new generated p.d.f (8): With three parameters (a, $\theta$, c) are given by:

$$F\ (x, a) = \frac{af\ (x)}{[a - (a - ١)\ f(x)]^{٢}}$$

$$F\ (x, a) = \frac{a\ \theta\ c^{\theta} x^{-\theta-١}}{\left[a - (a - ١)\ (١ - \left(\frac{c}{x}\right)^{\theta}\right]^{٢}}$$

Then we have:

$$F\ (x, c, \theta, a) = \frac{a\ \theta\ c^{\theta} x^{-\theta-١}}{\left[١ + (a - ١)\ \left(\frac{c}{x}\right)^{\theta}\right]^{٢}} \dots\dots\dots\dots\dots (٨)$$

X > C X: Random Variable

$\theta > O$ $\theta$: *Shape Parameter*

C > O C: Scale Parameter

a: Modified on parameter

equation (7) and (8) represent a new CDF and a new p.d.f. of modified three parameters Pareto distribution. Now, we derive equation No [2]:

**(2-1) Derivation formula for rth moments about origin:**

$$\mu^{|}r = E(x^{r}) = \int_{c}^{\infty} f(x, c, \theta, a)\ dx$$

$$\mu^{|}r = a\ \theta\ C^{\theta} \int_{c}^{\infty} x^{r-\theta-١}\ [\ ١ + (a - ١)\ \left(\frac{C}{x})^{\theta}\right)]^{-٢}\ dx$$

Thus meain: $[\,١ - (a - ١)\left(\frac{c}{x})^{\theta}\right)]^{-٢}$ { since x $> c \;\rightarrow \frac{c}{x} < ١$}

So that:

$$\frac{١}{[\,١ + (a - ١)\left(\frac{c}{x})^{\theta}\right)]^{٢}} = \frac{١}{\sum_{j=٠}^{٢} C_j^{٢}\left((a - ١)\left(\frac{c}{x})^{\theta}\right)^{j}}$$

$$\mu^{|}r = E(x^{r}) = \int_{c}^{\infty} f(x, c, \theta, a)\; dx$$

$$\mu^{|}r = a\,\theta\,C^{\theta}\int_{c}^{\infty} x^{r-\theta-١}\,[\,١ + (a - ١)\left(\frac{c}{x})^{\theta}\right)]^{-٢}\; dx$$

$$\mu^{|}r = a\,\theta\,C^{\theta}\int_{c}^{\infty} x^{r-\theta-١}\sum_{j=٠}^{\infty}(j + ١)\left((١ - a)\left(\frac{c}{x})^{\theta}\right)^{j}\; dx$$

$$\mu^{|}r = a\,\theta\,C^{\theta}\sum_{j=٠}^{\infty}(j + ١)\,(١ - a)^{j}\,C^{\theta j}\int_{c}^{\infty} x^{r-\theta-١}x^{-\theta j}\; dx$$

$$\mu^{|}r = a\,\theta\,C^{\theta}\sum_{j=٠}^{\infty} C^{\theta j}(j + ١)\,(١ - a)^{j}\int_{c}^{\infty} x^{r-(\theta+١)j-١}\; dx\,\frac{x^{r-(\theta+١)j}}{r - (\theta + ١)j}$$

$$\mu^{|}r = a\,\theta\,C^{\theta}\sum_{j=٠}^{\infty} C^{\theta j}(j + ١)\,(١ - a)^{j}\left[\frac{C^{r-(\theta+١)j}}{(\theta+١)j-r}\right]$$

$$\mu^{|}r = \mathrm{a}\,(\mathrm{a} - ١)\;\theta\,C^{\theta}C_{k}^{٢+k-١}(-١)^{k+١}(C^{r-٢\square})$$

$$\mu^{|}r = \mathrm{a}\,(\mathrm{a} - ١)\,\theta\,C^{r-\theta}C_{k}^{٢+k-١}(-١)^{k+١}, \mathrm{k} = ٠, ١, ٢$$

$$\mu^{|}r = \mathrm{a}\,(\mathrm{a} - ١)\;\theta\,C^{r-\theta}\left[(-١)^{١} + ٢ + ٣(-١)^{٢}\right]$$

$$\mu^{|}r = \mathrm{a}\,(\mathrm{a} - ١)\theta\,C^{r-\theta}(-٢)$$

$$\mu^{|}r = ٢\mathrm{a}\,(١ - \mathrm{a})\;\theta\,C^{r-\theta}$$

Then equating:

$$\mu^{|}r = \frac{\sum x_i^{r}}{n}$$

We obtain the r'th monument from: which is defined in equ (9)

$$\left.\begin{aligned}
&r = ١ \rightarrow \bar{x} = ٢a\,(١ - a)\ \theta\, C^{١-\theta} \\
&r = ٢ \rightarrow \frac{\sum x_i^{٢}}{n} = ٢a\,(١ - a)\ \theta\, C^{٢-\theta} \\
&r = ٣ \rightarrow \frac{\sum x_i^{٣}}{n} = ٢a\,(١ - a)\ \theta\, C^{٣-\theta} \\
&\frac{\bar{x}}{\sum x_i^{٢}} = \frac{C^{١-\theta}}{C^{٢-\theta}} \rightarrow \frac{١}{c} = \frac{\bar{x}}{\sum x_i^{٢}} \\
&\hat{C}_{mom} = \frac{\sum x_i^{٢}}{\bar{x}}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(9)$$

Then completing solving set (9), we obtain $\hat{C}_{mom}, \hat{a}_{mom}, \hat{\theta}_{mom}$

**(2-1-1) Moment's Estimators**

$$\mu^{|}r = \frac{\sum x_i}{n} \text{at} r = ١,٢,٣$$

$$\frac{\sum x_i^{r}}{n} = ٢a\,(١ - a)\ \theta\, C^{r-\theta}$$

$$\text{For } r = ١\ \frac{\sum x_i^{٢}}{n} = ٢a\,(١ - a)\ \theta\, C^{١-\theta}$$

$$\text{For } r = ٢\ \frac{\sum x_i^{٢}}{n}\, ٢a\,(١ - a)\ \theta\, C^{٢-\theta}$$

$$\text{For } r = ٣\ \frac{\sum x_i^{٣}}{n}\, ٢a\,(١ - a)\ \theta\, C^{٣-\theta}$$

**(2-1-2) Maximum Likelihood Estimators**

The estimation by this method depends on finding [6]:

$$L = \prod_{i=١}^{n} f(x_i, c, \theta, a) \text{ then}$$

$$L = a^n \theta^n C^{n\theta} \prod_{i=١}^{n} x_i^{-(\theta+١)} \prod_{i=١}^{n} [١ + (a - ١)\left(\frac{c}{x_i}\right)^{\theta}]^{-٢} \ldots\ldots\ldots\ldots(١٠)$$

$$\log L = n\log a + n\log\theta + n\,\theta\log c - (\theta + ١)\sum_{i=١}^{n}\log x_i$$
$$- ٢\sum_{i=١}^{n}\log\left[ ١ + (a - ١)\left(\frac{c}{x_i}\right)^{\theta}\right] \ldots\ldots\ldots\ldots\ldots (١١)$$

Since ($\hat{c} = Min\, x_i$), so we find MLE for $\theta$ and for a:

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} + n\log c - \sum_{i=١}^{n}\log x_i$$

$$-٢\sum_{i=١}^{n}\frac{١}{\left[١ + (a - ١)\left(\frac{c}{x_i}\right)^{\theta}\right]}(a - ١)\left(\frac{c}{x_i}\right)^{\theta}(١)\log\left(\frac{c}{x_i}\right)$$

$$\rightarrow \frac{n}{\hat{\theta}} = \sum_{i=١}^{n}\log x_i + ٢(a - ١)\sum_{i=١}^{n}\frac{\left(\frac{c}{x_i}\right)^{\theta}\log\left(\frac{c}{x_i}\right)}{\left[١ + (a - ١)\left(\frac{c}{x_i}\right)^{\theta}\right]} \ldots\ldots\ldots\ldots\ldots\ldots (١٢)$$

Solving this non-linear equation gives $\hat{\theta}_{MLE}$

$$\text{Now,}\ \frac{\partial \log L}{\partial a} = \frac{n}{a} - ٢\sum_{i=١}^{n}\frac{\left(\frac{c}{x_i}\right)^{\theta}}{\left[١ + (a - ١)\left(\frac{c}{x_i}\right)^{\theta}\right]}$$

$$\hat{a}_{MLE} = \frac{n}{٢\sum_{i=١}^{n}\frac{\left(\frac{c}{x_i}\right)^{\theta}}{\left[١+(a-١)\left(\frac{c}{x_i}\right)^{\theta}\right]}}$$

**(1-2) Simulation Procedure:**

This section deals with comparing two different estimators of parameters in order to estimate reliability through simulation depending on using modified (CDF) for Pareto distribution.

$$F\ (x,a) = \frac{١ - (\frac{C}{X})^{\theta}}{\left(١ - \left(\frac{C}{X}\right)^{\theta}\right) + a(\frac{C}{X})^{\theta}}$$

Let $Z = ١ - \left(\frac{C}{X}\right)^{\theta}$

$$\left(\frac{c}{x}\right)^{\theta} = ١ - Z$$

$$\left(\frac{C}{x}\right) = (١ - Z)^{\frac{١}{\theta}}$$

$$X = C(١ - Z)^{-\frac{١}{\theta}}$$

There for $Xi = C(١ - Zi)^{-\frac{١}{\theta}}$ generation equation of observation (Xi), Where $(Zi, C, \theta)$ are known.

This gives $F\ (x,a) = \frac{Z}{Z+a(١-Z)}$

Z is continues R.V [0,1]

Z= RND (1)

For $x \geq c$

We generate $z \geq 0$

According to z and using initial values for $(a, C, \theta)$

| Experimental | $\theta$ | c | a |
|---|---|---|---|
| 1 | 1 | 1.5 | 0.3 |
| 2 | 2.5 | 1.8 | 0.5 |
| 3 | 3.2 | 2 | 0.8 |

n = 25, 50, 70

L= 1000

The values of R.V.X are generated from

$X = C(١ - Z)^{\frac{-١}{\theta}}$

Z= RND (1)

for X ≥C

Z ≥ 0

According to different estimators of $(C, \theta and a)$

The reliability function of three modified Pareto distribution is estimated$\hat{R}(t)$ [8], and the results are comparing using:

$$MSE\left(\hat{R}(t)\right) = \frac{١}{L}\sum_{i=١}^{L}[\hat{R}_i(t) - R(t)]^{٢} \qquad \text{(L=1000)}$$

**Table (1):**Estimators of$\hat{R}(t)$ by different methods when $[\theta = ١, C = ١.٥, a = ٠.٣]$

| n | $t_i$ | Real R | $\hat{R}(ti)$MLE | $\hat{R}(ti)MOM$ |
|---|---|---|---|---|
| 25 | 1.6 | 0.7822 | 0.9458 | 0.8407 |
| | 2.06 | 0.6433 | 0.8336 | 0.7321 |
| | 2.89 | 0.35210 | 0.5142 | 0.6512 |
| | 3.85 | 0.33784 | 0.5201 | 0.3714 |
| | 4.62 | 0.2704 | 0.4372 | 0.2611 |
| | 5.61 | 0.2115 | 0.1922 | 0.1708 |
| | 7.86 | 0.1876 | 0.1604 | 0.1407 |
| | 8.93 | 0.1544 | 0.09107 | 0.0731 |
| | 9.22 | 0.1358 | 0.0983 | 0.0521 |
| | 9.48 | 0.10985 | 0.0867 | 0.0431 |
| 50 | 1.6 | 0.7822 | 0.9863 | 0.8503 |
| | 2.06 | 0.6433 | 0.8975 | 0.7645 |
| | 2.89 | 0.35210 | 0.7534 | 0.5915 |
| | 3.85 | 0.33784 | 0.4082 | 0.4557 |
| | 4.62 | 0.2704 | 0.3865 | 0.3362 |
| | 5.61 | 0.2115 | 0.3987 | 0.3042 |
| | 7.86 | 0.1876 | 0.2346 | 0.26702 |
| | 8.93 | 0.1544 | 0.2847 | 0.23781 |
| | 9.22 | 0.1358 | 0.1356 | 0.21407 |
| | 9.48 | 0.10985 | 0.1092 | 0.19208 |
| 75 | 1.6 | 0.7822 | 0.9448 | 0.8186 |
| | 2.06 | 0.6433 | 0.8306 | 0.7431 |
| | 2.89 | 0.35210 | 0.6142 | 0.7201 |
| | 3.85 | 0.33784 | 0.5502 | 0.6642 |
| | 4.62 | 0.2704 | 0.4306 | 0.5031 |
| | 5.61 | 0.2115 | 0.3937 | 0.4403 |
| | 7.86 | 0.1876 | 0.2392 | 0.3807 |
| | 8.93 | 0.1544 | 0.15546 | 0.2201 |
| | 9.22 | 0.1358 | 0.11278 | 0.1162 |
| | 9.48 | 0.10985 | 0.0847 | 0.04824 |

**Table (2):** Estimators of $\hat{R}(t)$ for modified pare to when ($\theta =$ ٢.٥$, C =$ ١.٨$, a =$ ٠.٥)

| n | $t_i$ | Real $R_i$ | $\hat{R}_{(MLE)}$ | $\hat{R}_{(MOM)}$ |
|---|---|---|---|---|
| 25 | 1.6 | 0.88245 | 0.9629 | 0.8832 |
| | 2.66 | 0.70631 | 0.8361 | 0.7362 |
| | 2.80 | 0.6041 | 0.6901 | 0.5331 |
| | 3.84 | 0.5016 | 0.5942 | 0.3955 |
| | 4.82 | 0.4386 | 0.4938 | 0.4721 |
| | 5.87 | 0.4201 | 0.4223 | 0.4601 |
| | 7.86 | 0.3504 | 0.3705 | 0.2413 |
| | 8.13 | 0.2918 | 0.2967 | 0.1902 |
| | 9.32 | 0.2618 | 0.2691 | 0.1706 |
| | 9.58 | 0.2503 | 0.2475 | 0.1501 |
| 50 | 1.6 | 0.88245 | 0.8714 | 0.8603 |
| | 2.66 | 0.70631 | 0.7087 | 0.6456 |
| | 2.80 | 0.6041 | 0.6525 | 0.5114 |
| | 3.84 | 0.5016 | 0.5206 | 0.5001 |
| | 4.82 | 0.4386 | 0.43048 | 0.4802 |
| | 5.87 | 0.4201 | 0.3162 | 0.3362 |
| | 7.86 | 0.3504 | 0.2784 | 0.2378 |
| | 8.13 | 0.2918 | 0.2224 | 0.21327 |
| | 9.32 | 0.2618 | 0.20178 | 0.1928 |
| | 9.58 | 0.2503 | 0.2006 | 0.1804 |
| 75 | 1.6 | 0.88245 | 0.8873 | 0.8607 |
| | 2.66 | 0.70631 | 0.8205 | 0.6993 |
| | 2.80 | 0.6041 | 0.7140 | 0.5943 |
| | 3.84 | 0.5016 | 0.6067 | 0.54102 |
| | 4.82 | 0.4386 | 0.5306 | 0.4558 |
| | 5.87 | 0.4201 | 0.4376 | 0.4103 |
| | 7.86 | 0.3504 | 0.3937 | 0.4012 |
| | 8.13 | 0.2918 | 0.3641 | 0.3721 |
| | 9.32 | 0.2618 | 0.2973 | 0.2106 |
| | 9.58 | 0.2503 | 0.2778 | 0.2001 |

**Table (3):** Estimators of $\hat{R}_{(t)}$ for modified pare to when ($\theta =$ ٣.٦$, C =$ ٢$, a =$ ٠.٨)

| n | $t_i$ | Real $R_i$ | $\hat{R}_{(MLE)}$ | $\hat{R}_{(MOM)}$ |
|---|---|---|---|---|
| 25 | 1.6 | 0.8789 | 0.9618 | 0.8724 |
| | 2.06 | 0.6706 | 0.8710 | 0.6615 |
| | 2.89 | 0.5973 | 0.7317 | 0.6013 |
| | 3.85 | 0.5384 | 0.7211 | 0.5022 |
| | 4.62 | 0.4266 | 0.6504 | 0.5011 |
| | 5.61 | 0.4102 | 0.5321 | 0.12411 |
| | 7.86 | 0.3519 | 0.1275 | 0.2513 |
| | 8.13 | 0.2617 | 0.1106 | 0.2401 |
| | 9.22 | 0.2601 | 0.0827 | 0.2311 |
| | 9.48 | 0.2504 | 0.0110 | 0.0847 |

| | | | | |
|---|---|---|---|---|
| 50 | 1.6 | 0.8789 | 0.9030 | 0.87138 |
| | 2.06 | 0.6706 | 0.7211 | 0.6704 |
| | 2.89 | 0.5973 | 0.7201 | 0.5703 |
| | 3.85 | 0.5384 | 0.5967 | 0.40367 |
| | 4.62 | 0.4266 | 0.4371 | 0.30292 |
| | 5.61 | 0.4102 | 0.4021 | 0.3011 |
| | 7.86 | 0.3519 | 0.24612 | 0.2351 |
| | 8.13 | 0.2617 | 0.19631 | 0.1841 |
| | 9.22 | 0.2601 | 0.1311 | 0.1211 |
| | 9.48 | 0.2504 | 0.0978 | 0.0854 |
| 75 | 1.6 | 0.8789 | 0.88943 | 0.86526 |
| | 2.06 | 0.6706 | 0.69834 | 0.65570 |
| | 2.89 | 0.5973 | 0.53137 | 0.648221 |
| | 3.85 | 0.5384 | 0.43137 | 0.5532 |
| | 4.62 | 0.4266 | 0.3354 | 0.3011 |
| | 5.61 | 0.4102 | 0.25481 | 0.2431 |
| | 7.86 | 0.3519 | 0.2246 | 0.14036 |
| | 8.13 | 0.2617 | 0.16826 | 0.1195 |
| | 9.22 | 0.2601 | 0.11056 | 0.1127 |
| | 9.48 | 0.2504 | 0.10273 | 0.0996 |

**Table (4):** Values of mean square error for estimating reliability function by three models

| Model | n | MLE | MOM | Best |
|---|---|---|---|---|
| I | 25 MOM | 0.010976 | 0.010964 | MOM |
| | 50 | 0.002115 | 0.005316 | MLE |
| | 75 | 0.00097 | 0.002987 | MLE |
| II | 25 MOM | 0.01664 | 0.01464 | MOM |
| | 50 | 0.00403 | 0.00758 | MLE |
| | 75 | 0.00254 | 0.0055 | MLE |
| III | 25 MOM | 0.012014 | 0.009148 | MOM |
| | 50 | 0.00342 | 0.00643 | MLE |
| | 75 | 0.001859 | 0.001992 | MLE |

**(1-3) Conclusion**

١. At Sample Size ($n = ٢٥$) the best estimator of Reliability Function is ($\hat{R}_{mom}$), due to the Value of Smallest Mean Square error.

٢. At Sample ($n = ٥٠$) and also ($n = ٥٧$) Then the best estimator for R is ($\hat{R}_{MLE)}$, according to the Property of estimators by M.L.E.

٣. The Statistical Measures for Comparison was Mean Square Error Which is defined by $\mathrm{MSE}\hat{R}(ti) = \frac{\sum_{i=١}^{n}(\hat{R}_i(ti) - R(ti)^{٢})}{L}$

$\mathrm{MSE}(\hat{R}_{mom}) < \mathrm{MSE}(\hat{R}_{mlm})$.

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