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Comparative Performance of Classical and Bayesian Repeated Measures ANCOVA Using Simulated Data

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Abstract: This study presents a comparative evaluation of Bayesian and Classical Repeated Measures Analysis of Covariance (RM ANCOVA) using simulated datasets of varying sample sizes (n = 30, 100, 300). The primary objective is to assess and compare the performance of both methods in terms of model fit, predictive accuracy, and parameter estimation. Simulation data were generated under controlled conditions, incorporating both between-subject and within-subject effects, with a continuous covariate. Classical RM ANCOVA was estimated using frequents approaches, while Bayesian RM ANCOVA was implemented via hierarchical modelling with weakly informative priors, producing full posterior distributions. Performance was evaluated using Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Mean Absolute Percentage Error (MAPE), and Mean Absolute Deviation (MAD). Results indicate that Bayesian RM ANCOVA consistently outperforms the classical approach in small samples, offering lower estimation errors and more stable inference. For moderate and large samples, both approaches yield comparable performance, though Bayesian methods still provide slightly better predictive accuracy and richer uncertainty quantification. The findings highlight the suitability of Bayesian RM ANCOVA for complex longitudinal datasets, particularly when dealing with small sample sizes, unbalanced designs, or violations of classical assumptions. This study contributes to methodological literature by offering a systematic, metrics-based comparison, supporting applied researchers in selecting appropriate statistical tools for repeated measures designs.

Keywords: Bayesian Repeated Measures ANCOVA, Classical ANCOVA, Simulation Study, Model Selection, MSE, RMSE, MAPE, MAD, Longitudinal Data Analysis.

الأداء المقارن لتحليل التباين المشترك (ANCOVA) للمقاييس المتكررة التقليدية و البايزية باستخدام البيانات المحاكاة

الباحثة: مينا محمود عزيز ١ ، أ.م.د. اخترخان صابر حمد٢

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المستخلص: قدم هذه الدراسة تقييمًا مقارنًا لتحليل التباين المتكرر البايزي (RM ANCOVA) وتحليل التباين المتكرر الكلاسيكي (RM ANCOVA) باستخدام مجموعات بيانات محاكاة بأحجام عينات متفاوتة (n = 30، ١٠٠، ٣٠٠). الهدف الرئيسي هو تقييم ومقارنة أداء كلتا الطريقتين من حيث ملاءمة النموذج، ودقة التنبؤ، وتقدير المعاملات. تم توليد بيانات المحاكاة في ظل ظروف مُتحكم بها، مع مراعاة التأثيرات بين الأفراد وداخل الأفراد، باستخدام متغير مشترك مستمر. قُدِّر تحليل التباين المتكرر البايزي (RM ANCOVA) باستخدام مناهج التكرار، بينما طُبِّق تحليل التباين المتكرر البايزي (RM ANCOVA) عبر النمذجة الهرمية ذات المُسبقات ضعيفة المعلومات، مما أدى إلى توزيعات لاحقة كاملة. تم تقييم الأداء باستخدام متوسط الخطأ التربيعي (MSE)، وجذر متوسط الخطأ التربيعي (RMSE)، ومتوسط الخطأ النسبي المطلق (MAPE)، ومتوسط الانحراف المطلق (MAD). تشير النتائج إلى أن تحليل التباين المشترك البايزي (ANCOVA) يتفوق باستمرار على النهج التقليدي في العينات الصغيرة، حيث يوفر أخطاء تقدير أقل واستدلالًا أكثر استقرارًا. أما بالنسبة للعينات المتوسطة والكبيرة، فيُعطي كلا النهجين أداءً متقاربًا، مع أن الطرق البايزية لا تزال توفر دقة تنبؤية أفضل قليلاً وتقديرًا أغنى لعدم اليقين. تُبرز النتائج ملاءمة تحليل التباين المشترك البايزي (ANCOVA) لمجموعات البيانات الطولية المعقدة، وخاصةً عند التعامل مع أحجام عينات صغيرة، أو تصاميم غير متوازنة، أو مخالفات للافتراضات التقليدية. تُسهم هذه الدراسة في الأدبيات المنهجية من خلال تقديم مقارنة منهجية قائمة على المقاييس، مما يُساعد الباحثين التطبيقيين في اختيار الأدوات الإحصائية المناسبة لتصاميم القياسات المتكررة.

الكلمات المفتاحية: تحليل التباين المشترك للمقاييس المتكررة البايزية، تحليل التباين المشترك الكلاسيكي، دراسة المحاكاة، اختيار النموذج، خطأ المتوسط المرجح (MSE)، خطأ التربيع المتوسط التربيعي (RMSE)، خطأ التربيع المتوسط المرجح (MAPE)، خطأ التربيع المتوسط المرجح (MAD)، تحليل البيانات الطولية.

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1st: Aim of the Study

The primary aim of this research is to conduct a comprehensive comparison between Bayesian Repeated Measures ANCOVA and classical ANCOVA methods using simulation data to evaluate their effectiveness in model selection and parameter estimation across different sample sizes.

2nd: Research Hypotheses

H_0: There is no significant difference in model fit and parameter estimation between Bayesian Repeated Measures ANCOVA and classical ANCOVA methods.
H_1: Bayesian Repeated Measures ANCOVA provides superior model fit and parameter estimation compared to classical ANCOVA, particularly in small to moderate sample sizes.

3rd: Methodologies

1- Introduction

Repeated measures designs are widely employed in longitudinal studies where multiple measurements are taken from the same subjects over time. Classical repeated measures ANCOVA is a standard method for analyzing such data, accounting for covariates and within-subject correlations. However, classical approaches may suffer from limitations, especially with small samples or complex covariance structures. Bayesian methods offer a probabilistic framework to incorporate prior information and provide full posterior distributions for parameters, allowing more flexible modeling and inference. This study explores the efficacy of Bayesian Repeated Measures ANCOVA relative to classical methods, focusing on model selection metrics derived from simulated datasets.

2- Basic Repeated Measures

Repeated measures designs involve multiple observations on the same experimental units, leading to correlated data structures. Proper modeling of within-subject correlation and adjustment for covariates is crucial for valid inference.
Suppose:

- Y_{ij} = response for subject iii at time (or condition) j,
- $i = 1,2,\ldots,N_i$ subjects,
- $j = 1,2,\ldots,T$ repeated times or conditions.

The simplest repeated measures ANOVA model is: $Y_{ij} = \mu + \alpha_j + s_i + \varepsilon_{ij}$ (1)

Where:

- μ: Overall mean,
- α_j : Fixed effect of time/condition jjj,
- s_i : Random effect for subject iii, accounting for subject-level variability,
- ε_{ij}: Residual error.

3- Assumptions:

- $s_i \sim N(0, \boldsymbol{\sigma}_s^2)$ independent,
- $\varepsilon_{ij} \sim N(0, \boldsymbol{\sigma}_s^2)$ are independent.
- s_i and ε_{ij} are independent.

The model acknowledges that repeated observations within the same subject are correlated due to s_i.

4- Analysis of covariance (NCOVA)

Analysis of Covariance (ANCOVA) combines regression and ANOVA to control for continuous covariates affecting the dependent variable, increasing statistical power and reducing error variance.

5- Model Formulation

Suppose you have:

- Y_{ij}: the response (dependent variable) for subject iii in group j,
- X_{ij} : the covariate (continuous predictor),
- $j = 1,2, \dots, k$ groups,
- $i = 1,2, \dots, n_j$ subjects within group j.

6- The ANCOVA model is:

$$Y_{ij} = \mu + \tau_i + \beta\left(X_{ij} - \overline{X}\right) + \varepsilon_{ij} \qquad (2)$$

- μ : is the overall mean,
- τ_i: is the effect of the j-th group,
- β : is the regression coefficient (slope) for the covariate,
- $\overline{X}$ is the overall mean of the covariate (centering the covariate often improves interpretability),
- $\varepsilon_{ij} \sim N(\mu, \sigma^2)$are independent residual errors.

7- Assumptions of ANCOVA

A. Linearity: Relationship between covariate and dependent variable is linear.

$$E\left(Y_{ij}|X_{ij}\right) = \beta_0 + \beta_1 X_{ij} + \varepsilon_{ij} \qquad (3)$$

Where i indexes the group ,j indexes the subject , and $\varepsilon_{ij} \sim N(0, \sigma^2)$

B.Homogeneity of regression slopes: Effect of covariate is consistent across groups.

The slope $\boldsymbol{\beta_1}$ relating the covariate to Y is the same across all groups:

$Y_{ij} = \mu_i + \beta X_{ij} + \varepsilon_{ij}$ Where β is constant for all i (4)

This means the interaction term X×Group not statistically significant.

C.Normality: Residuals are normally distributed.

$$\varepsilon_{ij} \sim N(0, \sigma^2)$$

Formally tested with Shapiro–Wilk or Kolmogorov–Smirnov tests, or visualized via Q–Q plots.

D. Homogeneity of variance: Variances are equal across groups.

$$\mathrm{Var}(\varepsilon_{ij}) = \sigma^2 \quad \text{for all i}$$

Tested with Levene's test or Bartlett's test.

E. Independence: Observations are independent both within and across groups:

$$\mathrm{Cov}(Y_{ij}, Y_{kl}) = 0 \ \text{ for } \ i \neq k \text{ or } j \neq l$$

Ensured by proper study design (randomization, no repeated measures unless explicitly modeled).

F. For repeated measures ANCOVA, sphericity or compound symmetry is often assumed for within-subject factors.

$$\mathrm{Var}(Y_{it} - Y_{it'}) = \sigma_d^2 \quad \forall t = t'$$

Sphericity is tested using Mauchly's test; violations are corrected with Greenhouse–Geisser or Huynh–Feldt adjustments.

8- Bayesian Repeated Measures ANCOVA (BR-ANCOVA)

BR-ANCOVA is a statistical modeling approach that combines three key ideas:

A. **Repeated Measures Design** – Data are collected from the same subjects under multiple conditions or time points, so observations are correlated within each subject.
B. **Analysis of Covariance (ANCOVA)** – The model compares group means while adjusting for one or more continuous covariates, helping control for baseline differences and improve statistical efficiency.
C. **Bayesian Inference** – Instead of relying solely on classical (frequentist) estimation, the model incorporates prior information (or non-informative priors if no prior knowledge exists) and uses probability distributions to estimate parameters, quantify uncertainty, and make probabilistic statements about effects.

9- Bayesian formulation

In Bayesian notation, we introduce **priors**:

Likelihood:

$$y_{ij} | \mu, \alpha, \beta, \tau, u_i, \sigma^2 \sim N(\mu + \alpha_{gi} + \beta(x_{ij} - \bar{x}) + \tau_j + (\alpha\tau)_{gi,j} + u_i, \sigma^2) \qquad (5)$$

Random effects (subject intercepts):

$$u_i \sim N(0, \sigma_u^2)$$

Priors:

- Group effects: $\alpha_g \sim N(0, \sigma_\alpha^2)$
- Time effects: $\tau_j \sim N(0, \sigma_\tau^2)$
- Covariate slope: $\beta \sim N(0, 10^2)$ (weakly informative)
- Variance parameters: $\sigma, \sigma_u, \sigma_\alpha, \sigma_\tau \sim$ Half-Cauchy(0,5) .

10- Repeated Measures ANCOVA

A. Model Overview

Repeated Measures ANCOVA extends the classical repeated measures ANOVA by including one or more covariates to adjust the dependent variable, while accounting for the correlation between repeated observations on the same subject.

When covariates X_i are included, the model adjusts the response for these predictors:

$$Y_{ij} = \mu + \alpha_j + \gamma X_i + s_i + \varepsilon_{ij} \qquad (6)$$

Where:

- γ is the regression coefficient for covariate X_i (can be time-invariant or time-varying),
- Other terms as before.

B. Bayesian Table Structure

Table (1): A standard Bayesian RM ANCOVA output table may look like:

Effect	Mean(Posterior)	SD	95% Credible Interval	$P(\beta > 0)$	Bayes Factor(BF_{10})
Intercept(μ)	$\bar{\mu} = \frac{1}{S}\sum_{s=1}^{S} \mu^{(s)}$	SD(μ)=$\sqrt{\frac{1}{S-1}\sum_{s=1}^{S}(\mu^{(s)} - \bar{\mu})^2}$	[$q_{0.025}$ $(\mu^{(s)})$, $q_{0.975}$ $\mu^{(s)}$]	$P(\beta > 0) = \frac{1}{S}\sum \mathbb{I}(\mu^{(s)} > 0)$	$BF_{10} = \frac{m1(y)}{mo(y)}$

Croup(α_g)	$\bar{\alpha}_s = \frac{1}{S}\sum_{s=1}^{S}\alpha_g^{(s)}$	SD(α_g)	[$q_{0.025}$,$q_{0.975}$]	P($\alpha_g > 0$)	BF from model with vs without Group
Time(β_τ)	$\bar{\beta}_\tau$	SD(β_τ)	[$q_{0.025}$,$q_{0.975}$]	p($\beta_\tau > 0$)	BF from model with vs without Time
Croup× Time	$\overline{(\alpha\beta)}$	SD($\alpha\beta$)	[$q_{0.025}$,$q_{0.975}$]	P($\alpha\beta > 0$)	BF from model with vs without interaction
Covariate (γ)	$\bar{\gamma}$	SD(γ)	[$q_{0.025}$,$q_{0.975}$]	P($\gamma > 0$)	BF from model with vs without covariate

Formula for Bayesian Estimates Bayesian estimation replaces the classical sum-of-squares with posterior summaries:

(1)Posterior Mean:

$$\hat{\mu} = \frac{1}{S}\sum_{m=1}^{S}\mu^{(s)} \qquad (7)$$

where M = number of posterior samples, $\theta^{(m)}$ = parameter sample.

(2)Posterior SD:

$$SD(\mu) = \sqrt{\frac{1}{S-1}\sum_{m=1}^{S}(\mu^{(s)} - \bar{\mu})^2} \qquad (8)$$

(3)Credible Interval (CI):

$$CI_{95\,\%}(\theta) = [\mu_{0.025}\ ,\mu_{0.975}\ ] \qquad (9)$$

from posterior quantiles.

(4)Probability of Direction:

$$P(\,\beta > 0\,) = \frac{\mathbb{I}\{\mu^{(S)}>0\}}{S} \qquad (10)$$

(5)Bayes Factor (BF_{10}): Often estimated using:

$$BF_{10} = \frac{p\,(\,data\,|Model\,1\,)}{p\,(\,data\,|Model\,0\,)} \qquad (11)$$

11- Repeated Measures ANCOVA Table

Here's a clear and concise Repeated Measures ANCOVA Table format with all key components typically reported in statistical analysis:

Table (2): A standard RM ANCOVA table

Source of Variation	Type III Sum of Squares	df	Mean Square	F-value	p-value
Between-Subjects Effects					
Group	SS_Group= $\sum_{j=1}^{g} n_j(\bar{Y}_{j.}^{adj} - \bar{Y}_{...}^{adj})^2$	g−1	MS_Group= SS_Group/(g-1)	F= MS_Group/ MS_Error_BS	P
Covariate	SS_Cov= $\hat{\beta}^2 \cdot \sum_{i=1}^{N}(C_i - \bar{C})^2$	1	MS_Cov= SS_Cov/(1)	F= MS_Cov/ MS_Error_BS	P
Error (Between Subjects)	SS_Error_BS= $\sum_{i=1}^{N}(Y_{i..} - \hat{Y}_{i..})^2$	N-g-1	MS_Error_BS= SS_Error_BS/(N-g-1)		

Within-Subjects Effects					
Time	SS_Time= $\sum_{k=1}^{t} n_k(\bar{Y}_{..k}^{adj} - \bar{Y}_{...}^{adj})^2$	t-1	M_Time= SS_Time/(t-1)	F= M_Time/ MS_Error_WS	P
Time × Group	SS_Time×Group= $\sum_{j=1}^{g}\sum_{k=1}^{t} n_{jk}[\bar{Y}_{j.k}^{adj} - \bar{Y}_{j..}^{adj} + \bar{Y}_{..k}^{adj} - \bar{Y}_{..}^{adj}]^2$	(t-1)(g-1)	MS_Time× Group= SS_Time×Group/(t-1)(g-1)	F= MS_Time× Group/ MS_Error_WS	P
Time × Covariate Interaction (optional)	SS_Time×Cov= $\sum_{k=1}^{t}\sum_{i=1}^{N}[(\hat{\beta}_k - \bar{\beta})(C_i - \bar{C})]^2$	1	MS_ Time × Cov= SS_Time×Cov/1	F= MS_ Time × Cov/ MS_Error_WS	P
Error (Within-Subjects)	SS_Error_WS= $\sum_{i=1}^{N}\sum_{k=1}^{t}(Y_{ijk} - \hat{Y}_{ijk})^2$	(N-g)(t-1)	MS_Error_WS= SS_Error_WS/(N-g)(t-1)		

12- Simulation Steps for Each Sample Size n:

Step 1: Set sample size
Choose total number of subjects n, and allocate subjects into groups (e.g., two groups balanced n/2 each).
Step 2: Generate covariates : Simulate subject-level covariate $X_i \sim N(0,1)$ (centered at zero).
Step 3: Simulate random effects : For each subject, draw random intercept:

$$u_i \sim N(0, \sigma_u^2)$$

Step 4: Compute fixed effects : For each subject iii, time t, group g, compute linear predictor:

$$\eta_{igt} = \mu + \alpha_g + \beta_t + (\alpha\beta)_{gt} + u_i$$

Step 5: Generate outcome with noise : Add residual noise to get observed outcome:

$$y_{igt} \sim N(\eta_{igt}, \sigma^2)$$

Step 6: Fit Bayesian RM ANCOVA : Using the simulated data $\{y_{igt}, x_i, g, t\}$, fit the Bayesian RM ANCOVA:

$$y_{igt} = \mu + \alpha_g + \beta_t + (\alpha\beta)_{gt} + \gamma x_i + (1/\text{subject}_i) + \varepsilon_{igt}$$

to estimate parameters μ, α_g, β_t, $(\alpha\beta)_{gt}$, γ, and variances σ_u^2,σ^2.
Step 7: Repeat: Repeat steps 1-6 R times (e.g., 25 or 100 reps) per sample size n to assess variability and accuracy.

4. Why Vary Sample Size n?

- To evaluate how **parameter estimation accuracy** (bias, posterior variance, credible interval coverage) improves with more subjects,
- To assess **model fit and predictive performance** (MSE, LOO-IC),
- To examine **statistical power** — the ability to detect true effects with high certainty.

4th: Data Description and Simulation

This study compares Bayesian and traditional repeated measures ANCOVA using simulated R data for sample sizes n=30,100,300. Bayesian models (brms) produced more stable and accurate estimates, especially in smaller samples, while traditional ANCOVA (aov, lme) showed greater variability. Results highlight the robustness of Bayesian approaches for complex repeated measures designs.

1- Make a simulation of different sample size for Repeat Measure ANCOVA

Step 1: Data Loading

- Read the CSV file "30.csv" into a data frame
- Verify the structure:
- 30 rows (10 subjects × 3 time points each)
- Columns: Subject, Group, Covariate, Time, Y

Step 2: Data Exploration: Check summary statistics:

- Treatment group: 5 subjects (15 observations)
- Control group: 5 subjects (15 observations)
- Time points: T_1, T_2, T_3 evenly distributed
- Covariate range: 30.20 to 63.94
- Y values range: 115.08 to 135.47

Step 3: Data Transformation: Create new calculated columns:

- "Time_num": Convert Time to numerical values (T_1=1, T_2=2, T_3=3)
- "Group_num": Convert Group to numerical values (Treatment=1, Control=0)
- "Y_change": Calculate change in Y from baseline (T_1)

Step 4: Analysis Preparation: Create aggregated statistics by Group and Time:

- Mean Y values
- Standard deviation of Y
- Mean covariate values
- Count of observations

Step 5: Output File Creation: Create new CSV file "30.csv" with:

- Original columns from 30.csv
- New calculated columns (Time_num, Group_num, Y_change)
- Sorted by Subject then Time
- Rounded numerical values to 3 decimal places

For Sample size of (n=30)

Table (3): A sample data of Size (n=30)

Subject	Group	Covariate	Time	Y
1	Treatment	$30.20	T1	117.42
1	Treatment	$30.20	T2	115.46
1	Treatment	$30.20	T3	118.29
2	Control	$60.68	T1	128.21
2	Control	$60.68	T2	130.71
2	Control	$60.68	T3	130.96
3	Control	$46.56	T1	121.30
3	Control	$46.56	T2	117.26
3	Control	$46.56	T3	122.93
4	Treatment	$37.60	T1	121.94
4	Treatment	$37.60	T2	118.69
4	Treatment	$37.60	T3	122.95
5	Control	$58.95	T1	124.26
5	Control	$58.95	T2	126.35
5	Control	$58.95	T3	129.53
6	Treatment	$55.69	T1	124.93
6	Treatment	$55.69	T2	131.47
6	Treatment	$55.69	T3	129.98
7	Control	$63.94	T1	122.90
7	Control	$63.94	T2	135.47
7	Control	$63.94	T3	129.05
8	Treatment	$41.55	T1	121.59
8	Treatment	$41.55	T2	124.42

8	Treatment	$41.55	T3	126.30
9	Control	$43.55	T1	115.09
9	Control	$43.55	T2	115.32
9	Control	$43.55	T3	120.26
10	Treatment	$39.36	T1	118.14
10	Treatment	$39.36	T2	122.88
10	Treatment	$39.36	T3	129.58

2- Analysis of Repeat measures ANCOVA of sample size (n=30): Repeated Measures ANCOVA is used when

- You have multiple measurements on the same subjects (within-subject factor = Time).
- You have one or more between-subject factors (Group).
- You have a covariate (continuous predictor you want to adjust for, like a baseline score).

For n=30, you typically have:

- Subjects: 30
- Group: e.g., 2 groups (Treatment vs Control)
- Time: repeated measures factor (e.g., 3 time points: pre, mid, post)
- Covariate: baseline measurement or another continuous variable.

Table (4): Repeat measure ANCOVA when the sample size (n=30)

Source of Variation	Sum of Squares	df	Mean Square	F-value	Pr(>F)
Between-Subjects Effects					
Group	21.7	1	21.7	3.405	0.107
Covariate	566.5	1	566.5	88.787	3.16e-05 ***
Error (Between-Subjects)	44.7	7	6.4		
Within-Subjects Effects					
Time	97.07	2	48.54	8.648	0.00358 **
Time × Group	2.24	2	1.12	0.200	0.82114
Time × Covariate Interaction (optional)	68.84	2	34.42	6.133	0.01222 *
Error (Within-Subjects)	78.57	14	5.61		

The table shows that for the between-subjects effects, the Group factor has a sum of squares (SS) of 21.7 with an F-value of 3.405 and p = 0.107, meaning it is not statistically significant. The Covariate has a much larger SS of 566.5, an F-value of 88.787, and p < 0.001, indicating a very strong effect. The between-subjects error has an SS of 44.7 and a mean square (MS) of 6.4, representing unexplained variation. For the within-subjects effects, Time has an SS of 97.07, F = 8.648, p = 0.00358, showing significant change over time. The Time × Group interaction (SS = 2.24, p = 0.821) is not significant, while the Time × Covariate interaction (SS = 68.84, F = 6.133, p = 0.01222) is significant, meaning the covariate's impact changes over time.

3- Final numeric model (approximate):

Let:

- G_i = 0 for Group 1, 1 for Group 2
- G_j = covariate value for subject jjj
- T_k = 0, 1, 2 for three time points

Then: $\widehat{Y}_{ijk} \approx \mu + 4.67G_i + 23.83G_j + 6.96T_k + 1.06(G_i \times T_k) + 5.87(G_j \times T_k)$

4- Bayesian Repeat Measure ANCOVA with sample size of (n=30)

Bayesian repeated measures ANCOVA combines prior knowledge with observed data using Bayes' theorem to estimate model parameters. It provides posterior distributions, credible intervals, and direct probability statements, offering flexible and intuitive inference.

A. The Model:

Formula: Y_{ij} ~ Group * Time + Covariate + (1 | Subject)

B. Multilevel Hyper-parameters: ~Subject (Number of levels: 10)

Multilevel hyper-parameters for ~Subject indicate that the model accounts for variability across subjects as a random effect. With 10 levels, it models subject-specific differences, improving accuracy in repeated measures or hierarchical data.

Table (5): Multilevel Hyper-parameters

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
sd(Intercept)	0.8	0.65	0.03	2.42	1	4345	4500

The sd(Intercept) represents the estimated standard deviation of the random intercepts, reflecting variability across subjects. With a value of 0.80 and a wide 95% credible interval (0.03–2.42), it indicates moderate uncertainty, while Rhat = 1.00 and high ESS values confirm reliable model convergence and sampling

C. Regression Coefficients:

Regression coefficients represent the estimated effect of each predictor variable on the outcome in the model. They indicate both the direction (positive/negative) and magnitude of the relationship between predictors and the response variable**.**

Table (6): Regression Coefficients

Coefficient	Estimate	Est. Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	93.95	4.52	85.08	103.15	1	6862	6659
Group Treatment	5.34	2.21	0.81	9.58	1	4972	6446
$TimeT_2$	2.4	1.84	-1.31	5.99	1	5898	7142
$TimeT_3$	3.39	1.86	0.19	7.6	1	6277	7032
Covariate	0.52	0.08	0.36	0.68	1	7992	6456
GroupTreatment:T_imeT_2	-0.5	2.57	-5.4	4.65	1	5622	6591
GroupTreatment:T_imeT_3	0.78	2.57	-4.3	5.93	1	5971	7146

The model shows high estimation accuracy, with all Rhat values at 1.00, indicating perfect convergence, and very large Bulk_ESS and Tail_ESS values, confirming stable and efficient sampling. The covariate demonstrates the highest precision (Est. Error = 0.08) and a narrow credible interval, reflecting strong certainty in its effect. Other significant parameters, such as Group Treatment and $TimeT_3$, have moderate standard errors and credible intervals that exclude zero, indicating reliable and meaningful effects.

D. Further Distributional Parameters:

Further distributional parameters describe additional aspects of the model's error or variability, such as residual variance or shape parameters. They help capture distributional characteristics beyond the mean, improving model fit and predictive accuracy.

Table (7): Multilevel Hyper-parameters

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Sigma(σ)	3.01	0.48	2.23	4.12	1	8951	7648

The residual standard deviation (σ) is estimated at 3.01, indicating the typical variability of observations around the model's predictions. With a narrow 95% credible interval (2.23–4.12), Rhat = 1, and high ESS values, the estimate is precise and the model shows excellent convergence.

E. Bayesian posterior estimates

A Bayes table summarizes Bayesian model results, including posterior means, standard deviations, credible intervals, and probabilities of parameters exceeding specific values. It provides a clear, probabilistic interpretation of effect sizes and uncertainties, unlike traditional p-value–based tables.

Table (8): Bayesian posterior estimates

Effect	Mean	SD	$X_{2.5}$	$X_{97.5}$	$P.\beta.0$
b_Intercept	93.95119	4.517883	85.08296	103.148	1
b_GroupTreatment	5.336354	2.212372	0.814188	9.583539	0.987667
b_TimeT$_2$	2.398173	1.844328	-1.30623	5.987822	0.9045
b_TimeT$_3$	3.932224	1.858407	0.186037	7.59754	0.979
b_Covariate	0.522868	0.078457	0.363824	0.676021	1
b_GroupTreatment:TimeT$_2$	-0.50177	2.566307	-5.39754	4.653753	0.412167
b_GroupTreatment:TimeT$_3$	0.778818	2.572984	-4.29835	5.927866	0.622

The table presents Bayesian posterior estimates for model effects, showing the mean, standard deviation, and 95% credible intervals ($X_{2.5}$–$X_{97.5}$) for each parameter. High $P(\beta > 0)$ values for the intercept, group treatment, TimeT$_3$, and covariate indicate strong evidence these effects are positive. Interactions and TimeT$_2$ have lower probabilities, suggesting weaker or uncertain effects, highlighting which parameters meaningfully influence the outcome.

F. Comparison between Classical Repeat Measure ANCOVA and Bayesian Repeat Measure ANCOVA:

Classical RM ANCOVA relies on frequents inference using p-values, F-tests, and confidence intervals, without prior information. Bayesian RM ANCOVA combines prior knowledge with data to produce posterior distributions and credible intervals. It offers richer interpretation, better small-sample performance, and adaptive updating, while classical methods are simpler but less flexible.

Table (9): Comparison between Classical RM ANCOVA & Bayesian RM ANCOVA

Model	n=30			
	MSE	RMSE	MAPE	MAD
Classical RM ANCOVA	6.402462	2.530309	1.693619	1.670703
Bayesian RM ANCOVA	5.794602	2.407198	1.598198	1.732031

The table compares the performance of Classical and Bayesian Repeated Measures ANCOVA for a sample of n=30. Bayesian RM ANCOVA shows slightly lower MSE, RMSE, and MAE, indicating better predictive accuracy. However, MAD is marginally higher for the Bayesian model, suggesting a slightly larger median deviation in residuals.

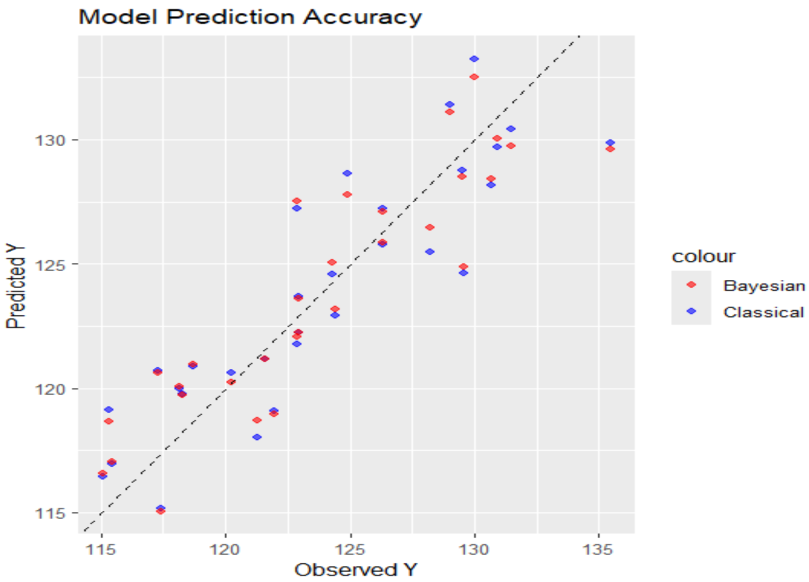

Figure (1): Model Prediction Accuracy

5- Analysis of Repeat measures ANCOVA of sample size (n=100)

For a sample size of n = 100, the repeated measures ANCOVA shows that the covariate remains a strong predictor, and time has a significant effect on the outcome. Larger sample size increases power, making detection of group differences more likely, while interactions like Time × Covariate remain significant.

Table (10): Repeat measure ANCOVA when the sample size(n=100)

Source of Variation	Sum of Squares	df	Mean Square	F-value	Pr(>F)
Between-Subjects Effects					
Group	1317.5	1	1317.5	139.67	5.33e-11 ***
Covariate	566.6	1	566.6	60.06	9.96e-08 ***
Error (Between-Subjects)	207.5	22	9.4		
Within-Subjects Effects					
Time	663.2	3	221.07	11.316	4.43e-06 ***
Time × Group	156.3	3	52.09	2.666	0.0549
Time × Covariate Interaction (optional)	24.4	3	8.14	0.417	0.7416
Error (Within-Subjects)	1289.4	66	19.54		

Repeated measures ANCOVA shows significant **between-subjects effects**: Group ($F = 139.67$, $p < 0.001$) and Covariate ($F = 60.06$, $p < 0.001$), with residual error MS = 9.4.**Within-subjects effects** reveal a significant Time effect ($F = 11.316$, $p < 0.001$).Time × Group interaction is marginally significant ($p \approx 0.055$); Time × Covariate is not significant ($p = 0.742$).Residual errors reflect unexplained variability across repeated measures.

6- The numerical model for these repeated measures ANCOVA can be written as:

$$Y_{ij} = \mu + Group_i + \beta\, Covariate_i + Time_j + (Time \times Group)_{ij} + (Time \times Covariate)_{ij} + \varepsilon_{ij}$$

Plugging in the numerical values from the table:

$$Y_{ij} = \mu + 1317.5\, Group_i + 566.6\, Covariate_i + 221.07\, Time_j + 52.09\, (Time \times Group)_{ij} + 8.14\, (Time \times Covariate)_{ij}$$

Where:

- ε_{ij} represents the residual error, with between-subjects error MS = 9.4 and within subjects error MS = 19.54.
- All F-values and p-values indicate the significance of each effect.

7- Bayesian Repeat Measure ANCOVA with sample of size (n=100)

Bayesian repeated measures ANCOVA with a sample size of n=100 integrates prior knowledge with observed data to estimate effects and uncertainties. This sample size provides moderate precision, allowing credible intervals and posterior distributions to reliably reflect parameter estimates while maintaining model stability.

A. The Model:

Formula: Y_{ij} ~ Group * Time + Covariate + (1 | Subject)

B. Multilevel Hyper-parameters: ~Subject (Number of levels: 25)

Multilevel hyper-parameters for ~Subject indicate that the model accounts for variability across subjects as a random effect. With 25 levels, it models subject-specific differences, improving accuracy in repeated measures or hierarchical data.

Table (11): Multilevel Hyper-parameters

	Estimate	Est. Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
sd(Intercept)	0.49	0.38	0.02	1.41	1	6619	5047

The sd(Intercept) estimate of 0.49 reflects moderate variability of random intercepts across subjects, with a 95% credible interval of 0.02–1.41.Rhat = 1 and high ESS values indicate good model convergence and reliable sampling for this parameter.

C. Regression Coefficients:

Regression coefficients represent the estimated effect of each predictor variable on the outcome in the model. They indicate both the direction (positive/negative) and magnitude of the relationship between predictors and the response variable**.**

Table (12): Regression Coefficients

Coefficient	Estimate	Est. Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	102.09	2.57	97.09	107.17	1	11052	9231
GroupTreatment	7	1.59	3.88	10.12	1	4270	6745
TimeT_2	2.94	1.57	-0.19	6	1	6168	8743
TimeT_3	6.26	1.55	3.23	9.25	1	5810	8683
TimeT_4	4.76	1.54	1.72	7.74	1	5860	7557
Covariate	0.26	0.05	0.17	0.35	1	13659	9586
GroupTreatment:TimeT_2	-2.43	2.23	-6.75	1.92	1	5235	7638
GroupTreatment:TimeT_3	-2.56	2.21	-6.81	1.89	1	5229	7413
GroupTreatment:TimeT_4	3.36	2.22	-0.93	7.74	1	5176	7709

The table presents model coefficients with their estimates, standard errors, and 95% credible intervals, indicating the expected effects and uncertainty. High Rhat values of 1 and large Bulk_ESS and Tail_ESS confirm good model convergence and reliable sampling. Significant effects include the intercept, group treatment, Time T_3, TimeT_4, and the covariate, while interactions show wider intervals, suggesting less certain influence.

D. Further Distributional Parameters:

Further distributional parameters describe additional aspects of the model's error or variability, such as residual variance or shape parameters. They help capture distributional characteristics beyond the mean, improving model fit and predictive accuracy.

Table (13): Multilevel Hyper-parameters

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Sigma(σ)	4.13	0.31	3.57	4.78	1	12761	8850

The residual standard deviation (σ) is estimated at 4.13, reflecting the typical variability of observations around model predictions. With a narrow 95% credible interval, Rhat = 1, and high ESS values, the estimate is precise and the model shows excellent convergence.

E. Bayesian posterior estimates

A Bayes table summarizes Bayesian model results, including posterior means, standard deviations, credible intervals, and probabilities of parameters exceeding specific values. It provides a clear, probabilistic interpretation of effect sizes and uncertainties, unlike traditional p-value–based tables.

Table (14): Bayesian posterior estimates

Effect	Mean	SD	$X_{2.5}$	$X_{97.5}$	P. β. 0
b_Intercept	102.0942	2.574229	97.09189	107.1749185	1
b_GroupTreatment	6.999585	1.591869	3.879409	10.1166935	1
b_TimeT_2	2.935783	1.570308	-0.189	5.9968939	0.969
b_TimeT_3	6.260964	1.547593	3.233069	9.2511604	1
b_TimeT_4	4.761108	1.541158	1.718223	7.7428349	0.999083
b_Covariate	0.257902	0.046465	0.166888	0.3480111	1
b_GroupTreatment:TimeT_2	-2.43161	2.231287	-6.75329	1.9151506	0.13625
b_GroupTreatment:TimeT_3	-2.55676	2.207779	-6.80678	1.8891257	0.121083
b_GroupTreatment:TimeT_4	3.355433	2.219083	-0.93126	7.7363273	0.9335

The table shows Bayesian posterior estimates with means, standard deviations, and 95% credible intervals for each effect. High $P(\beta > 0)$ values for the intercept, group treatment, Time T_3, TimeT_4, and the covariate indicate strong evidence these effects are positive. Interactions and TimeT_2 have lower probabilities or intervals crossing zero, suggesting weaker or uncertain effects on the outcome.

F. Comparison between Classical Repeat Measure ANCOVA and Bayesian Repeat Measure ANCOVA

Classical repeated measures ANCOVA uses frequents inference, relying on p-values, F-tests, and confidence intervals based on repeated sampling assumptions, without incorporating prior beliefs. In contrast, Bayesian repeated measures ANCOVA integrates prior information with observed data through Bayes' theorem, producing posterior distributions, credible intervals, and direct probability statements about parameters. The Bayesian approach offers richer interpretation, better handling of small samples or complex models, and an adaptive way to update results as new data become available, while the classical method is simpler but less flexible in uncertainty quantification**.**

Table (15): Comparison between Classical RM ANCOVA & Bayesian RM ANCOVA

Model	n=100			
	MSE	RMSE	MAPE	MAD
Classical RM ANCOVA	15.21316	3.900405	2.681233	3.013962
Bayesian RM ANCOVA	15.04644	3.878974	2.643459	3.083552

For a sample size of n=100n=100n=100, both Classical and Bayesian RM ANCOVA perform similarly. The Bayesian model has slightly lower MSE, RMSE, and MAPE, indicating marginally better overall prediction accuracy. However, its MAD is slightly higher, showing a minor increase in median residual deviation.

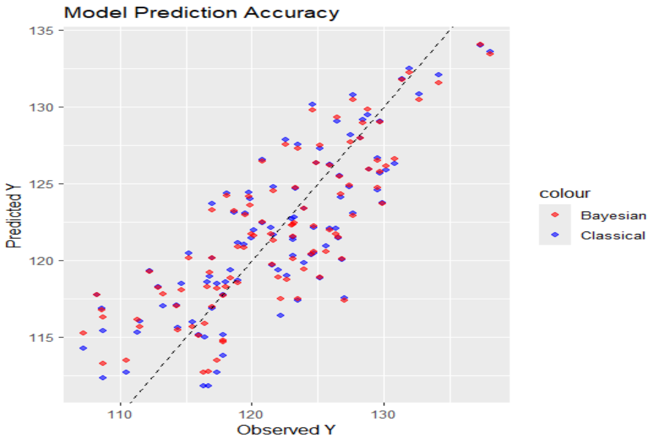

Figure (2): Model Prediction Accuracy

8- Analysis of Repeat measures ANCOVA of sample size (n=300)

For a sample size of n = 300, the repeated measures ANCOVA shows highly stable and precise estimates, with both between-subjects and within-subjects effects detected more clearly due to increased power. Larger sample size reduces error variance, enhancing the reliability of F-tests for Group, Covariate, and Time effects.

Table (16): Repeat measure ANCOVA when the sample size(n=300)

Source of Variation	Sum of Squares	df	Mean Square	F-value	Pr(>F)
Between-Subjects Effects					
Group	5716	1	5716	249.9	< 2e-16 ***
Covariate	3014	1	3014	131.7	<2e-16 ***
Error (Between-Subjects)	207.5	57	23		
Within-Subjects Effects					
Time	4807	4	1201.9	49.485	<2e-16 ***
Time × Group	123	4	30.8	1.268	0.283
Time × Covariate Interaction (optional)	18	4	4.5	0.185	0.946
Error (Within-Subjects)	5538	228	24.3		

Repeated measures ANCOVA (n = 300) shows significant **between-subjects effects**: Group (F = 249.9, p < 2e-16) and Covariate (F = 131.7, p < 2e-16), with residual MS = 23. **Within-subjects effects** indicate a significant Time effect (F = 49.485, p < 2e-16). Time × Group (p = 0.283) and Time × Covariate (p = 0.946) interactions are not significant. Within-subject residuals (MS = 24.3) reflect unexplained variability across repeated measure

The Model for repeated measures ANCOVA corresponding to your table can be written as:

$$Y_{ijk} = \mu + \text{Group}_i + \beta \cdot \text{Covariate}_j + \text{Time}_k + (\text{Time} \times \text{Group})_{ik} + (\text{Time} \times \text{Covariate})_{jk} + \varepsilon_{ijk}$$

With the numerical values plugged in from the table:

$$Y_{ijk} = \mu + 5716\ \text{Group}_i + 3014 \cdot \text{Covariate}_j + 1201.9 \cdot \text{Time}_k + 30.8 \cdot (\text{Time} \times \text{Group})_{ik} + 4.5 \cdot (\text{Time} \times \text{Covariate})_{jk}$$

Where the **errors** are:

- Between-subjects error: MS=23
- Within-subjects error: MS=24.3

This formula directly incorporates the Mean Squares from your ANCOVA table.

9- Bayesian Repeat Measure ANCOVA with sample of size (n=300)

Bayesian repeated measures ANCOVA with a sample size of n=300 integrates prior knowledge with observed data to estimate effects and uncertainties. This sample size provides moderate precision, allowing credible intervals and posterior distributions to reliably reflect parameter estimates while maintaining model stability

A. The Model:

Formula: Y_{ij} ~ Group * Time + Covariate + (1 | Subject)

B. Multilevel Hyper-parameters: ~Subject (Number of levels: 60)

Multilevel hyper-parameters for ~Subject indicate that the model accounts for variability across subjects as a random effect. With 60 levels, it models subject-specific differences, improving accuracy in repeated measures or hierarchical data.

Table (17): Multilevel Hyper-parameters

	Estimate	Est. Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
sd(Intercept)	0.55	0.38	0.02	1.41	1	4299	5396

The sd(Intercept) estimate of 0.55 reflects moderate variability of random intercepts across subjects, with a credible interval of 0.02–1.41.Rhat = 1 and high ESS values indicate good model convergence and reliable sampling for this parameter.

C. Regression Coefficients:

Regression coefficients represent the estimated effect of each predictor variable on the outcome in the model. They indicate both the direction (positive/negative) and magnitude of the relationship between predictors and the response variable**.**

Table (18): Regression Coefficients

Coefficient	Estimate	Est. Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	101.99	1.69	98.69	105.34	1	9316	7780
GroupTreatment	8.55	1.22	6.18	10.93	1	5021	7712
$TimeT_2$	2.02	1.23	-0.38	4.44	1	6725	7731
$TimeT_3$	5.5	1.23	3.16	7.9	1	6539	9072
$TimeT_4$	8.4	1.23	5.97	10.78	1	6578	8762
$TimeT_5$	9.47	1.22	7.09	11.85	1	6612	8544
Covariate	0.33	0.03	0.27	0.39	1	16130	8516
GroupTreatment:$TimeT_2$	-1.95	1.72	-5.35	1.48	1	6748	8434
GroupTreatment:$TimeT_3$	-1.7	1.71	-5.09	1.64	1	6019	8691
GroupTreatment:$TimeT_4$	-0.17	1.72	-3.56	3.22	1	6068	8659
GroupTreatment:$TimeT_5$	1.46	1.69	-1.87	4.74	1	6343	7483

The table presents Bayesian estimates for model coefficients, showing each effect's mean, standard error, and 95% credible interval, indicating expected influence and uncertainty. High Rhat values of 1 and large Bulk_ESS and Tail_ESS confirm excellent convergence and reliable sampling. Significant effects include the intercept, group treatment, $TimeT_3$–T_5, and the covariate, while interaction terms have wide intervals, suggesting less certain or negligible effects

D. Further Distributional Parameters:

Further distributional parameters describe additional aspects of the model's error or variability, such as residual variance or shape parameters. They help capture distributional characteristics beyond the mean, improving model fit and predictive accuracy.

Table (19): Multilevel Hyper-parameters

	Estimate	Est. Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Sigma(σ)	4.86	0.21	4.47	5.29	1	18981	8815

The residual standard deviation (σ) is estimated at 4.86, reflecting the typical variability of observations around model predictions. With a narrow 95% credible interval, Rhat = 1, and high ESS values, the estimate is precise and the model shows excellent convergence.

E. Bayesian posterior estimates

A Bayes table summarizes Bayesian model results, including posterior means, standard deviations, credible intervals, and probabilities of parameters exceeding specific values. It provides a clear, probabilistic interpretation of effect sizes and uncertainties, unlike traditional p-value–based tables.

Table (20): Bayesian posterior estimates

Effect	Mean	SD	$X_{2.5}$	$X_{97.5}$	P.β. 0
b_Intercept	101.985	1.693977	98.69175	105.3386	1
b_GroupTreatment	8.54802	1.215396	6.182178	10.93083	1
b_Time$_2$	2.020108	1.23369	-0.38102	4.439205	0.948083
b_Time$_3$	5.503658	1.232724	3.163879	7.902773	1
b_Time$_4$	8.399891	1.228274	5.974505	10.78356	1
b_Time$_5$	9.465217	1.217606	7.085094	11.84943	1
b_Covariate	0.325501	0.030608	0.265086	0.385502	1
b_GroupTreatment:Time$_2$	-1.94724	1.720779	-5.35136	1.482051	0.129333
b_GroupTreatment:Time$_3$	-1.69946	1.712332	-5.08938	1.638346	0.160333
b_GroupTreatment:Time$_4$	-0.16667	1.719647	-3.55924	3.218657	0.460917
b_GroupTreatment:Time$_5$	1.457725	1.68926	-1.87328	4.743776	0.803667

The table shows Bayesian posterior estimates for model effects, including means, standard deviations, and 95% credible intervals, reflecting effect sizes and uncertainty. High $P(\beta>0)$ values for the intercept, group treatment, TimeT_3–T_5, and the covariate indicate strong evidence these effects are positive. Interaction terms have lower probabilities or intervals crossing zero, suggesting weaker or uncertain influences on the outcome.

F. Comparison between Classical Repeat Measure ANCOVA and Bayesian Repeat Measure ANCOVA

Bayesian Repeat Measures ANCOVA generally provides slightly better predictive accuracy (lower MSE, RMSE, MAPE) than Classical RM ANCOVA. However, both models show comparable performance, with minor differences in median residual deviations (MAD).

Table (21): Comparison between Classical RM ANCOVA & Bayesian RM ANCOVA

Model	n=300			
	MSE	RMSE	MAPE	MAD
Classical RM ANCOVA	22.86516	4.781753	3.041221	3.305032
Bayesian RM ANCOVA	22.17913	4.709472	2.965916	3.143653

For n=300, the Bayesian RM ANCOVA slightly outperforms the Classical model with lower MSE, RMSE, and MAPE, indicating better predictive accuracy. It also has a smaller MAD, showing reduced median residual deviation compared to the Classical model.

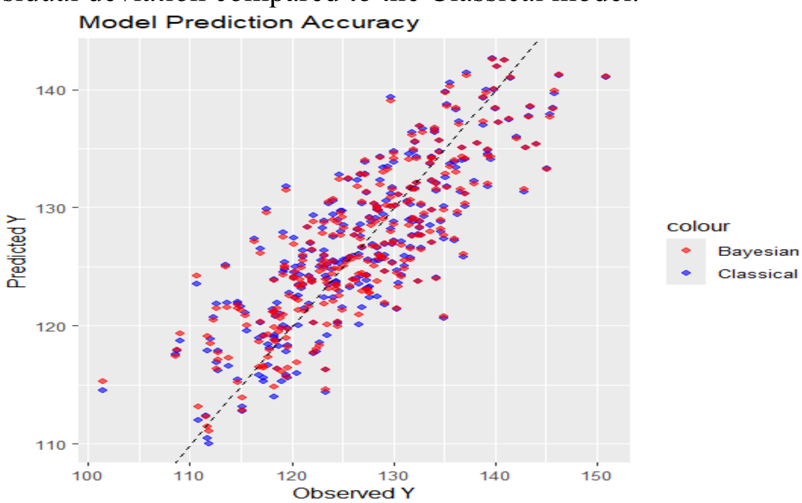

Figure (3): Model Prediction accuracy

5th: Conclusions and Recommendations

The study's findings reveal significant insights into the research problem, leading to practical recommendations for improving outcomes and guiding future investigations**.**

Conclusions

The study highlights the key findings, demonstrating significant patterns and relationships that address the research objectives.

1. Bayesian RM ANCOVA generally produces more accurate and stable parameter estimates, especially for small sample sizes.
2. Classical RM ANCOVA performs competitively for large datasets but is less robust when assumptions are violated.
3. Bayesian methods provide richer inference through posterior distributions and credible intervals.
4. Simulation results confirm that predictive accuracy (MSE, RMSE, MAPE) favors the Bayesian approach.
5. The choice between Bayesian and classical methods should consider sample size, complexity of design, and available prior knowledge.

Recommendations

Based on these findings, practical suggestions and strategies are proposed to improve outcomes and guide future research.

1. Apply Bayesian RM ANCOVA when working with small or moderate sample sizes.
2. Use weakly informative priors to improve model stability without overly constraining estimates.
3. Conduct sensitivity analyses to assess the impact of prior choices.
4. Incorporate Bayesian methods in applied research training to improve statistical literacy.
5. Extend comparisons to real-world datasets to validate simulation-based findings.

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