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Complex approximation by real approximation

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ABSTRACT

It is known that approximation of the complex functions does not, in general, depend on approximation of the real functions. However, if the complex function is viewed as a Cartesian product of two real functions, a relationship appears between the two types of approximation specified above. This means that if F is a complex function where $F : (x, y) \to (u(x, y), (v(x, y)), x, y \in R$, and f_1, f_2 are real functions with $F = f_1 \times f_2$, then there is a relation between an approximation of F and an approximations of f_1, f_2. This paper studies the relation between approximation of complex functions and approximation of real functions, in terms of existence and degree, in certain special cases. The first case is when u is a function only of x (i.e. $u(x)$) and v is a function only of y (i.e. $v(y)$); this means that the complex function F becomes $F : (x, y) \to (u(x), (v(y)),$ that is, $F = f_1 \times f_2$ with $f_1 : x \to u(x)$ and $f_2 : y \to v(y)$. The second case is when there exists a relation between x and y in the X-Y plane (i.e. $y = f(x)$), for example $y = x^2$ or $y = \sqrt{x}$; this case leads to the first one. In addition, the relationship between the polynomial of best approximation and any polynomial of the complex function $F(x, y)$ and its analogous real functions arising from it (i.e. $f_1(x), f_2(y)$) is considered in the sense of Kolmogorov's theorem. Furthermore, the relation between the modulus of continuity and the difference of the function $F(x, y)$ and the real functions f_1, f_2 is examined, together with other related results.

Keywords: *Approximation of complex functions, Approximation of real functions, Cartesian product of functions, Kolmogorov's theorem*

1 INTRODUCTION

An approximation of complex functions, measured by the modulus of continuity of the complex kind $w(F, h)$, has been studied in several books and research articles such as [1–3], but its relationship or influence on the approximation of real functions has not been investigated in general. If the complex function $F : (x, y) \to (u(x, y), (v(x, y))$ is viewed as a Cartesian product of two real functions (as shown later), a relationship appears between the two types of approximation mentioned above. This means that if $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are two real functions, then the Cartesian product of f_1, f_2 is written as $F = f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$, where $X_1 \times X_2 = (x_1, x_2)\, ; x_1 \in X_1$ and $x_2 \in X_2$ (and similarly for $Y_1 \times Y_2$). This representation makes clear the relationship between the two kinds of approximation for the two types of functions. Now, if $X_1 = X_2 = Y_1 = Y_2 = R$, then $F = f_1 \times f_2 : R^2 \to R^2$. In this paper the extent to which the approximation of the complex function F is affected by the approximation of the real functions $f_1(x), f_2(y)$ is studied, in terms of both existence and degree, in certain special cases. The first case is when $F : R^2 \to R^2$ is defined as $F((x, y)) = (u(x), v(y))$, which means that u is a function only of x and v is a function only of y. For example, let $F(z) = F(x, y) = F(x + iy) = 5x^2 + iy^3 = u(x) + iv(y) = (u(x), v(y)) = (5x^2, y^3)$ where $z = x+iy = (x, y) \in C = R^2$ = set of complex numbers. Another case occurs when there exists a relation between x and y in the domain of the function F, for example if $y = 2x$ in the domain of F and $F(x, y) = (xy^2, 2x + y)$, which implies $F(x, y) = (4x^3, 2y)$; as seen, this case leads to the first one. Note that the following symbols are used in this paper: $F(z) = F(x, y)$ denotes the complex function F, and $f_1(x), f_2(y)$ denote the real functions arising from F (i.e. if $F(x, y) = f_1(x) \times f_2(y)$). Also, $P_n(z) = P_n(x, y)$ denotes the complex polynomial of degree n (in x and y), and $P_n(x), P_n(y)$ denote the associated real polynomials of degree n that arise from $P_n(z)$. Furthermore, $\|(F(x, y))\| = \sup_{((x,y)\in A)} (|F(x, y)|), A \subset C : A = A_1 \times A_2$ and $\|f_1(x)\| = \sup_{(x\in A_1)} (|f_1(x)|), A_1 \subset R$ (and similarly for $f_2(y)$). From basic results in approximation theory, as in [4, 5], a real function (for example f) has a polynomial of best approximation $P_n \in P_n$ if f is continuous in that space, and the degree of this approximation satisfies $\|f - P_n\|_p \leq c, w(f, 1/n)$. Thus, approximation of complex functions (viewed as Cartesian products of two real functions) is studied together with approximation of the corresponding real functions, and conversely. For open sets in the real numbers R (with the usual topology), which appear as open intervals, recall that the Cartesian product of two open sets in R is an open set in R^2 (with the usual topology in R^2). This fact is used in the proof of Theorem 2.2. Through this observation, the effect of continuity of real functions on their domains, and its influence on the continuity of the corresponding complex functions, can be seen, and the converse as well. Throughout this work the trigonometric inequality in R^2 is also used, which states that the sum of two sides of any triangle (in this case, a right-angled triangle) is always longer than the third side. In the present setting, if the complex number $z = x + iy = (x, y)$ is considered, then $z = (x, y)$ forms a right-angled triangle with the points (0,0) and (x,0). By the trigonometric inequality $|z| = \sqrt{x^2 + y^2} \leq |x| + |y|$. Of course, the same result can be obtained in another way, since $|z| = |x + iy| \leq |x| + |iy| = |x| + |i||y| = |x| + |y|$, but the first approach is used here as it agrees with the common definition of absolute value. Kolmogorov in 1948 [6, 7] established and proved a theorem on approximation of complex functions that describes the relation between the polynomial of best approximation $P_n^*(z)$ of a complex function $F(z)$ and any complex polynomial $P_n(z)$ on a closed and bounded domain in the complex plane. The theorem states that the maximum value of the real part of the difference between a complex function and its best approximation polynomial, multiplied by the conjugate of an arbitrary complex polynomial, is always nonnegative, that is, $\max_{(x\in A_\circ)} Re[f(z) - P_n^*(z)]\overline{P(z)} \geq 0$. Therefore, in this paper the extent of the relationship between Kolmogorov's theorem for the complex function $F(z)$ and the corresponding theorem for the real functions $f_1(x), f_2(y)$ is examined, considering the case in which the imaginary parts are equal to zero, and other related results are also discussed.

2 MATERIALS AND METHODS

2.1 Auxiliary results

In this section, the relationships between the norm and the difference for the complex function $F(z)$ and the corresponding real functions $f_1(x)$ and $f_2(y)$ are studied, together with several related results.

In the first part, and as mentioned previously for the Cartesian product of two open sets in the real numbers R (with the usual topology), the following theorem shows that the Cartesian product of two continuous functions is continuous, and that the converse is also true. This theorem is proved here, with particular emphasis on the second part.

2.2 Theorem

Suppose $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two real continuous functions then necessary and sufficient condition that $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is continuous function in R^2.

Proof: Necessary let A be an open set in $Y_1 \times Y_2$ (with usual topology in R^2). Since $Y_1 \times Y_2$ has a basis $B_i \times C_i, i \in I$ such that B_i is open in Y_1 and C_i is open in Y_2

Then $A = \cup_{i=1}^{\infty} (B_i \times C_i) \in Y_1 \times Y_2$ be open in $Y_1 \times Y_2$.

So, $(f_1 \times f_2)^{-1} (A) = (f_1 \times f_2)^{-1} \left(\cup_{i=1}^{\infty} (B_i \times C_i)\right). =$

$\bigcup_{i=1}^{\infty} \left((f_1 \times f_2)^{-1} (B_i \times C_i)\right) =$

$\bigcup_{i=1}^{\infty} \left(f_1^{-1} (B_i) \times f_2^{-1} (C_i)\right)$

Since f_1 and f_2 are continuous then $f_1^{-1}(B_i)$ and $f_2^{-1}(C_i)$ are open in X_1 and X_2 (resp.) And then $f_1^{-1}(B_i) \times f_2^{-1}(C_i)$ is an open set in $X_1 \times X_2$

So, $\cup_{i=1}^{\infty} \left(f_1^{-1} (B_i) \times f_2^{-1} (C_i)\right) = A$ is open in $X_1 \times X_2$.

And then $f_1 \times f_2$ is continuous.

Sufficiency: Let B_1, B_2 be an open sets in Y_1, Y_2 (resp.) then $B_1 \times B_2$ be an open set in $Y_1 \times Y_2$

Since $f_1 \times f_2$ be continuous function, then $(f_1 \times f_2)^{-1}(B_1 \times B_2)$ be an open in $X_1 \times X_2$

But $(f_1 \times f_2)^{-1} (B_1 \times B_2) = f_1^{-1}(B_i) \times f_2^{-1}(B_2)$ be an open in $X_1 \times X_2$

So, $f_1^{-1} (B_i)$ is an open set in X_1 and $f_2^{-1} (B_2)$ is an open set in X_2. Hence, f_1 and f_2 are continuous functions. In the following part, the relation between the modulus of continuity of the complex function (modulus of continuity in the complex form) and the modulus of continuity of its associated real functions is discussed, since this relation links the approximation of the two types of functions mentioned above. Before that, the following relations are proved:

$(f_1(x) \times f_2(y)) - (P_1(x) \times P_2(y))$
$= \{(y_1, y_2) - (m_1, m_2)$
$\in (f_1(x) \times f_2(y) - (P_1(x) \times P_2(y))\}$
$= \{(y_1 - m_1, y_2 - m_2) \in (f_1(x) \times f_2(y)) -$
$= \{(y_1 - m_1, y_2 - m_2) \in (f_1(x)-$
$P_1(x)) \times (f_2(y) - P_2(y))\}\}$
$= (f_1(x) - P_1(x)) \times (f_2(y) - P_2(y))\}$ .... (2.2.1)

Now, suppose that $F : R^2 \to R^2$ is a continuous complex function, this means $F(x.y) \in \mathbb{C} = R^2$ and let $t = (t_1, t_2) \in \mathbb{C}$ where $t_1, t_2 \in R$

Now, the difference $\Delta_t^1(F(x, y)) = F(x + t_1, y + t_2) - F(x, y)$ And so, $\Delta_t^1((f_1 \times f_2)(x, y)) = f_1(x + t_1) \times f_1(y + t_2) - f_1(x) \times f_2(y)\ f_1(x)) \times (f_2(y + t_2) - f_2(y))$
$= \Delta_{t1}^1(f_1(x)) \times$
$\Delta_{t2}^1(f_2(y)) = \left(\Delta_{t_1}^1(f_1(x)), \Delta_{t_2}^1(f_2(y))\right)$ ... (2.2.2)

Since $|z| = |x + iy| = \sqrt{x^2 + y^2}, z \in \mathbb{C}$, let $t = (t_1 \cdot t_2) \in \mathbb{C}$ and $A \subset \mathbb{C}, A = A_1 \times A_2$

So, by (2.2.2) above and by the definition of the absolute value of complex numbers, we obtain:

$$\omega(F, h) = \max_{|t|\le h} \left\|\Delta_t^1(F(x, y))\right\| = \max_{|t|\le h}\Big(\sup_{(x,y)\in A} | F(x+ t_1, y + t_2) - F(x, y) |\Big) =$$
$$\max_{|t|\le h}\Big(\sup_{(x,y)\in A} \left|\left(\Delta_{t_1}^1(f_1(x)), \Delta_{t_2}^1(f_2(y))\right)\right|\Big)$$
$$= \max_{|t|\le h}\Big(\sup_{(x,y)\in A} \left(\sqrt{\left(\Delta_{t_1}^1(f_1(x))^2 + \Delta_{t_2}^1(f_2(y))^2\right)}\right)\Big)$$

And then by trigonometric inequality and properties of max and sup we get:

$$\omega(F.h) =$$
$$\max_{|t|\le h}\Big(\sup_{(x,y)\in A} \left(\sqrt{\left(\Delta_{t_1}^1(f_1(x))^2 + \Delta_{t_2}^1(f_2(y))^2\right)}\right)\Big)$$
$$\le \max_{|t|\le h}\Big(\sup_{(x,y)\in A} \left(\left|\Delta_{t_1}^1 f_1(x)\right| + \left|\Delta_{t_2}^1 f_2(y)\right|\right)\Big)$$
$$= \max_{\sqrt{t_1^2+t_2^2}\le h}\Big(\sup_{x\in A_1} \left|\left(\Delta_{t_1}^1 f_1(x)\right)\right|\Big)+$$
$$\max_{\sqrt{t_1^2+t_2^2}\le h}\Big(\sup_{y\in A_2} \left|\left(\Delta_{t_2}^1 f_1(y)\right)\right|\Big)$$

Again, by tringle inequality we get $\sqrt{t_1^2 + t_2^2} \le |t_1| + |t_2|$,So:

$$\omega(F, h) \le \max_{|t_1|+|t_2|\le h}\Big(\sup_{x\in A_1} \left|\left(\Delta_{t_1}^1 f_1(x)\right)\right|\Big)$$
$$+ \max_{|t_1|+|t_2|\le h}\Big(\sup_{y\in A_2} \left(\left|\left(\Delta_{t_2}^1 f_1(y)\right)\right|\right)\Big)$$
$$= \omega(f_1, h) + \omega(f_2, h) \ldots (2.2.3)$$

This result can be obtained for any order of the difference (that is, for $\Delta_t^2, \Delta_t^3, \ldots, \Delta_t^n, n \in \mathbb{N}$).

In addition, the following inequality will be proved: $|f \times g| \le |f| + |g|$.

let $(y_1, y_2) = y_1 + iy_2 \in f(x) \times g(y)$ then $|\ y_1 + |y_2| = \sqrt{y_1^2 + y_2^2}$

And by trigonometric inequality, properties of absolute value and max we get:

$|(y_1, y_2)| = |y_1 + iy_2| = \sqrt{y_1^2 + y_2^2} \le |y_1| + |y_2|$ where $y_1 = f(x), y_2 = g(y)$

By above and (2.2.1) we get: $\|(f \times g)(x, y)\| = \sup_{(x,y)\in C} |(f(x), g(y))|$

$$= \sup_{(x\cdot y)\in C} \left(\sqrt{y_1^2 + y_2^2}\right) \le \sup_{x\in R} |y_1| + \sup_{y\in R} |y_2| = \|f(x)\| + \|g(y)\| \ldots\ldots (2.2.4)$$

In addition to the preceding considerations, and with regard to the relationship between the modulus of continuity of the complex function $F(x, y)$ and the moduli of continuity of the real functions $f_1(x)$ and $f_2(y)$ and their norms, the following theorem examines the connection between the modulus of continuity of $F(x, y)$ and the norms of the real functions $f_1(x)$ and $f_2(y)$.

2.3 Theorem

If $F = f_1 \times f_2 : R^2 \to R^2$ and ω is the modulus of continuity in complex form then:

$\omega\left(F, \frac{1}{n}\right) \le \|f_1(x) - P_n(y)\| + \|f_2(y) - P_n(y)\|, (x, y) \in A \subset \mathbb{C}$ and $t = (t_1, t_2) \in \mathbb{C}, P_n(x), P_n(y)$ real polynomials.

Proof: By definition of modulus of continuity and (2.2.1) we get

$$\omega\left(F, \frac{1}{n}\right) = \max_{|t|\le h}(\|\Delta(F, h)\|) = \max_{|t|\le h}(\|F(x+ t_2.y + t_2) - F(x.y)\|) = \max_{|t|\le h}(\|f_1(x + t_2) \times f_2(y + t_2) - f_1(x) \times f_2(y)\|)$$
$$= \max_{|t|\le h}\Big(\sup_{(x,y)\in A} (|\ f_1(x + t_2) - f_1(x) \times (f_2(y\ + t_2) - f_2(y)\ |)$$
$$= \max_{|t|\le h}\Big(\sup_{(x,y)\in A} (|\ (f_1(x + t_2) - f_1(x) \times (f_2(y\ + t_2) - f_2(y))\ |)\Big) =$$

$$\max_{|t|\le h}\left(\sup_{(x,y)\in A}\sqrt{(f_1(x+t_2)-f_1(x))^2+(f_2(y+t_2)-f_2(y))^2}\right)$$

Notice that the last expression represents the length of the hypotenuse in a right-angled triangle whose sides are $|f_1(x+t_2)-f_1(x)|$ and $|f_2(y+t_2)-f_2(y)|$, and since the hypotenuse is the largest side of a right-angled triangle but is smaller than the sum of the other two sides, the last result must be:

$$\le \max_{|t|\le h}\Big(\sup_{|(x,y)\in A}\big(|\,(f_1\,(x+t_2)-f_1(x)\,|\big)+$$
$$\big(|\,f_2\,(y+t_2)-\ f_2(y)\,|\big)\Big)\Big)$$
$$=\max_{|t|\le h}\Big(\sup_{|(x,y)\in A}\big(|\,(f_1(x)-f_1\,(x+t_1)\,|\big)+$$
$$\big(|\,f_2(y)-f_2(y\ \ +t_2)\,|\big)\Big)\Big)$$
$$=\max_{|t|\le h}\Big(\sup_{|(x,y)\in A}\big(|\,(f_1(x)-f_1\,(x+t_1)+$$
$$P_n(x)-P_n(x)\,|\big)+\big(|\,f_2(y)-f_2\,(y+t_2)+P_n(y)-$$
$$P_n(y)\,|\big)\Big)$$
$$\max_{|t|\le h}\Big(\sup_{|(x,y)\in A}\big(|\,(f_1(x)-P_1(x))-$$
$$(f_1\,(x+t_1)-P_1(x))\,|+|f_2(y)-P_2(y))-$$
$$(f_2\,(y+t_2)-P_2(y)\,|\big)\Big)$$

As far as the two quantities $(f_1(x)-P_1(x))$ and

$(f_1\,(x+t_1)-P_1(x)$ (also for $(f_2(y)-P_2(y))$ and $(f_2\,(y+t_2)-P_2(y))$ which one is greater?

There are four possibilities for the function f_1 (also for f_2) and the polynomial $P_1(x)$ (also for $P_2(y)$), which is that the f_1 (also for f_2) is increasing or decreasing, and that the polynomial $P_1(x)$ (also for $P_2(y)$) is either above or under f_1 (also for f_2) this mean greater than it, or less than it, So:

First and second if f_1 (also for f_2) is decreasing(i.e. $f_1\,(x+t_1)\le f_1(x)$) and $P_1(x)$ (also for $P_2(y)$) is above of f_1 (also for f_2) this mean greater than it $(f_1(x)\le P_1(x))$ also for other or If f_1 (also for f_2) is increasing $f_1(x)\le f_1\,(x+t_1)$) and $P_1(x)$ (also for $P_2(y)$) is below of f_1 (also for f_2) this mean smaller than it $(P_1(x)\le f_1(x))$ also for other ,So the last end be:

$$\le \max_{|t|\le h}\Big(\sup_{|(x,y)\in A}\big(|(f_1(x)-P_1(x))|\big)+$$
$$\big(|f_2(y)-P_2(y)|\big)\Big)\Big)$$
$$=\|f_1(x)-P_1(x)\|+\|f_2(y)-P_2(y\|$$

Third and fourth if f_1 (also for f_2) is increasing and $P_1(x)$ (also for $P_2(y)$) is below of f_1 (also for f_2) or If f_1 (also for f_2) is decreasing and $P_1(x)$ (also for $P_2(y)$) is above of f_1 (also for f_2) the last end is:

$$\le \max_{|t|\le h}\Big(\sup_{|(x,y)\in A}\big(|(f_1\,(x+t_1)-P_1(x))|\big)+\big(|\,f_2(y+$$
$$t_2)-P_2(y)\,|\big)\Big)=\|f_1\,(x+t_1)-P_1(x)\|+$$
$$\|f_2\,(y+t_2)-P_2(y)\|$$

But, $f_1\,(x+t_1)$ is another value of values of $f_1(x)$ (also for other) so the last term is equal: $\|f_1(x)-P_1(x)\|+\|f_2(y)-P_2(y\|$ and the proof is completem

3 RESULTS AND DISCUSSION

3.1 Main results

In this section, the relationship between the approximation of the complex function $F(z)$ and the approximation of the real functions generated from it (i.e. $f_1(x), f_2(y)$) is studied. The relationship between the polynomial of best approximation and any other polynomial of the complex function is also examined, together with its effect on the corresponding relation between the polynomial of best approximation and any other polynomial of the associated real functions. In the first part, it is known that for continuous functions the difference in norm between the function and a sequence of polynomials cannot exceed the modulus of continuity, as stated in Theorem (3.2). In the following theorem, it is shown that if the difference between the complex function $F(x,y)$ and any complex polynomial $P_n(x,y)$ is less than the modulus of continuity (in complex form), then this property is inherited (transferred) to the real functions $f_1(x)$ and $f_2(y)$. Thus, by using Theorem 3.2 and relying on Theorem 2.2, the next theorem can be proved.

3.2 Theorem [4]

There exists a constant M such that for each function $f\in C[-1.1]$ there is a sequence of polynomials $P_n(x)$ for which $|f(x)-P_n(x)|\le M\omega\,(f,\Delta_n(x))\quad -1\le x\le 1\quad n=1,2,\cdots$.

Note that one can extend the interval $[-1.1]$ into general interval $[a.b]$.

3.3 Theorem

If $F=f_1\times f_2: X^2\to Y^2$ continuous in $\mathbb{C}$ where $X^2,Y^2\subset R^2=\mathbb{C}$ and ω is the modulus of continuity in complex form then:

$\|F(z)-P_n(z)\|\ \le\ c\omega\left(F,\frac{1}{n}\right)$ if and only if $\|f_1(x)-P_1(x)\|\le c\omega\left(f_1,\frac{1}{n}\right)$ and $\|f_2(y)-P_2(y)\|\le c\omega\left(f_2,\frac{1}{n}\right)$ where $(x,y)\in A\subset\mathbb{C}, A=A_1\times A_2$ and $t=(t_1,t_2)\in\mathbb{C}$.

Proof:(Necessary and sufficiently):_Let $\|F(x,y)-P_n(x,y)\| \le c\omega\left(F,\frac{1}{n}\right), h=\frac{1}{n}$
Then by theorem 2.2, 2.2.1 and trigonometric property we get:

$$\|f_1(x)\times f_2(y)-P_1(x)\times P_2(y)\| \le c\max_{|t|\le h}\|\Delta_t F(x,y)\|$$

$$\|\,(f_1(x)-P_1(x))\times(f_2(y)-P_2(y)\| \le$$
$$c\max_{|t|\le h}(\sup_{(x,y)\in A}(|\Delta F(x,y)|)))$$
$$\sup_{(x,y)\in A}|\,(f_1(x)-P_1(x))\times(f_2(y)-P_2(y)\,|$$
$$\le c\max_{|t|\le h}(\sup_{(x,y)\in A}(|(\Delta f_1(x),\Delta f_2(y))|))$$
$$\le c\max_{|t|\le h}(\sup_{(x,y)\in A}\left(\sqrt{(f_1(x)-P_1(x))^2+(f_2(y)-P_2(y))^2}\right)$$

As mentioned before, the sum of the two sides of a right-angled triangle is longer than the third. Therefore, if the hypotenuse in one right-angled triangle is longer than the hypotenuse in another right-angled triangle, then the sum of the two sides in the first triangle is greater than the sum of the corresponding two sides in the second (this conclusion can also be obtained from (2.2.4)). So:

$$\sup_{(x,y)\in A}(|\,f_1(x)-P_1(x)\,|+|f_2(y)-P_2(y)|)$$
$$\le c\max_{|t|\le h}(\sup_{(x,y)\in A}(|\Delta f_1(y)|$$
$$+|\Delta f_2(y)|))$$

By properties of supremum (and maximum) we get:

$$\sup_{x\in A_1}(|\,f_1(x)-\,P_1(x)\,|)+\sup_{y\in A_2}(|f_2(y)-P_2(y)|)$$
$$\le c\max_{|t_1|\le h}(\sup_{x\in A_1}(|\Delta f_1(y)|$$
$$+c\max_{|t_2|\le h}(\sup_{y\in A_2}(|\Delta f_2(y)|))$$

So, $\|f_1(x)-P_1(x)\|+\|f_2(y)-P_2(y)\| \le c\omega\left(f_1,\frac{1}{n}\right)+c\omega\left(f_2,\frac{1}{n}\right)$

By mean of theorem 3.2 we get
$\|f_1(x)-P_1(x)\| \le c\omega\left(f_1,\frac{1}{n}\right)$ and
$\|f_2(y)-P_2(y)\| \le c\omega\left(f_2,\frac{1}{n}\right)$ and the proof is end
For example: of this theorem:

If we choose $f_1(x)=|x|$ on $[-1.1]$ then the $P_2(x)$ of best approximation of $f_1(x)$ is
$P_2(x)=x^2+\frac{1}{8}$ [5]
So, $\sup_{-1\le x\le 1}|f_1(x)-P_1(x)| = \sup_{-1\le x\le 1}\left||x|-x^2-\frac{1}{8}\right| \le \frac{1}{8}$
And $\omega_{\boldsymbol{n}}\left(f_1\cdot\frac{1}{n}\right) = \max_{0\le t\le\frac{1}{n}}\|f_1(x)\| =$
$\max_{0\le t\le\frac{1}{n}}(\sup_{-1<x<1}|(|x|)| = \max_{0\le t\le\frac{1}{n}}(\sup_{-1<x<1}|x|) =$
$\max_{0\le t\le\frac{1}{n}}(\sup_{-1<x<1}|x+t-x|)$
$=\max_{0\le t\le\frac{1}{n}} t=\frac{1}{n}=\frac{1}{2}$ where $n\in N$ and $n=2$.
And then $\|f_1(x)-P_n(x)\| \le \omega_n\left(f_1\cdot\frac{1}{n}\right)$
If we choose $f_2(y)=e^y$ on $[-1.1]$ then the $P_2(y)$ of best approximation by Legendre of $f_2(y)$ is $P_2(y)=0\cdot 5367y^2+1\cdot 1036y+0\cdot 9963$ [5]
So, $\sup_{-1\le y\le 1}\{|f_2(y)-P_2(y)| = \sup_{-1\le y\le 1}\{|\,e^y-0\cdot 5367y^2-1\cdot 1036y-0\cdot 9963\,|\} \simeq 1.72$
And $\omega_{\boldsymbol{n}}\left(f_2\cdot\frac{1}{n}\right) = \max_{0\le t\le\frac{1}{n}}(\sup_{-1\le y\le 1}(|f_2(y)|)) =$
$\max_{0\le t\le\frac{1}{n}}(\sup_{-1\le y\le 1}(|e^y|))$

$$=\max_{0\le t\le\frac{1}{n}}(\sup_{-1\le y\le 1}(e^y)) = \max_{0\le t\le\frac{1}{n}}(\sup_{-1\le y\le 1}\left(e^{y+t}-e^y\right))$$
$$=\max_{0\le t\le\frac{1}{n}}\left\{e^{1+t}-e\right\}=e^{1+\frac{1}{n}}-e=e\left(e^{\frac{1}{2}}-\right.$$

1. where $2=n\in N$

So, $\omega_{\boldsymbol{n}}\left(f_2,\frac{1}{n}\right) > 1,72$ And then $\|f_2(y)-P_2(y)\| \le \omega_n\left(f_2,\frac{1}{n}\right)$.
Now, $\sup_{-1\le x,y\le 1}\{|F(x,y)-P(x,y)|\} =$
$\sup_{-1\le x.y\le 1}\left\{\left|(|x|,e^y)-\left(x^2+\frac{1}{8},0\cdot 5367y^2+1\cdot 1036y+0\cdot 9963\right)\,\right|\right\}$
$=\sup_{-1\le x.y\le 1}\left\{\left|\left(|x|-x^2-\frac{1}{8},e^y-5367y^2-1\cdot 1036y-0\cdot 9963\right)\right.\right. = \sup_{-1\le x.y\le 1}\left\{\left[\left(|x|-x^2-\frac{1}{8}\right)^2+(e^y-5367y^2-1\cdot 1036y-0\cdot 9963)^2\right]^{\frac{1}{2}}\right\}$
$=\left[(0\cdot 125)^2+(1.72)^2\right]^{\frac{1}{2}} \simeq 1.72453617$
$\omega_n\left(F,\frac{1}{n}\right) = \max_{|t|\le\frac{1}{n}}(\sup_{-1\le x,y\le 1}(|\Delta F(x,y)|)) =$
$\max_{|t|\le\frac{1}{n}}(\sup_{-1\le x,y\le 1}(|F(x+t_1,y+t_2)-F(x,y)|))$

$= \max_{|t|\le\frac{1}{n}} (\sup_{-1\le x,y\le 1} (||x+t_1|-|x|, e^{y+t_1}-e^y|)) =$

$\max_{|t|\le\frac{1}{n}} (\sup_{-1\le x,y\le 1} \left\{\left[(|x+t_1|-|x|)^2+(e^{y+t_1}-e^y)^2\right]^{\frac{1}{2}}\right)$

$= \left[\frac{1}{4}+e^2\left(e^{\frac{1}{2}}-1\right)^2\right]^{\frac{1}{2}}$

It seem that $\|F(x,y)-\boldsymbol{P}_2(x,y)\| \le \omega_{\boldsymbol{n}}\left(F.\frac{1}{n}\right)$

In the second part and as mentioned earlier for Kolmogorov theorem which explains the relation between best approximation polynomial and any polynomial for the complex functions $F(x.y)$ and because we considered $F(x,y)$ as a Cartesian product for f_1, f_2 and $P_n^*(x.y)$ is also Cartesian product for $P_1^*(x), P_2^*(y)$ (also for $P_n(x,y)$ as a $P_1(x)\times P_2(y)$)in the following theorem, we will prove that there exist a relationship between best approximation real polynomials $\left(P_1^*(x), P_2^*(y)\right)$ and any other real polynomials ($P_1(x), P_2(y)$)for the functions f_1, f_2 and vice versa for $F(x,y)$.

3.4 Theorem: Kolmogorov [4]

P is a polynomial of best approximation for a continuous F If and only if for each polynomial $Q \max_{x\in A_\circ} \mathrm{Re}\left\{[F(x)-P(x)]\overline{Q(x)} \ge 0\right.$ where $A_\circ$ denoted The set (which depends on f and P) of all $z\in A$ for which $|f(z)-P(z)| = \|f-P\|$ for the "real case" takes the form $\max_{x\in A_0} \mathrm{Re}\{[f(z)-P(z)]Q(z) \ge 0$.

Kolmogorov theorem can write with another form [7] as: The polynomial P_n is polynomial of best approximation of the function f in sense that

$\|F-P_n^*\| = \inf_{P_n^*\in P_n} \|F-P_n\|$ it is necessary and sufficient that the inequality:

$$\min_{z\in A_\circ} \mathrm{Re}\left\{\left(\overline{F(z)-P_n^*(z)}\right)Q(z)\right\} \le 0\}.$$

3.5 Theorem

Let $F(z) = f_1(x)\times f_2(y), P_n(z) = P_1(x)\times P_2(y)$ and $Q(z) = Q_1(x)\times Q_2(y)$ Then $P_n(z)$ is a polynomial of best approximation of best approximation for a continuous complex function $F(z)$ (that is for each polynomial $Q(z)$ $\max_{z\in A^\circ} \mathrm{Re}\left\{[F(z)-P_n(z)]\,\overline{Q(z)} \ge 0\right.$ (where $A_\circ = A_{\circ 1}\times A_{\circ 2}$)) if only if $\max_{x\in A_\circ} \mathrm{Re}\,\{[f_1(x) - P_1(x)\,|]\,Q_1(x)\} \ge 0$ and $\max_{y\in A_\circ} \mathrm{Re}\,\{[f_2(y) - P_2(y)]\,Q_2(y) \ge 0$ that is $P_1(x)$ best approximation of the continuous function $f_1(x)$ and $P_2(y)$ best approximation of the continuous function $f_2(y)$.

PROOF: (necessary and sufficiently)

Suppose that $\max_{x\in A^\circ} \mathrm{Re}\{[F(z)-P(z)]\overline{Q(z)}\} \ge 0$ And by theorem 2.2 we get

$\max_{(x.y)\in A_{\circ 1}\times A_{\circ 2}} \mathrm{Re}\,\{[(f_1\times f_2)\,(x,y) - (P_1\times P_2)\,(x,y)]\,\overline{(Q_1\times Q_2)}(x,y)\right\} \ge 0$

By (2.1.1) and properties of complex number conjugates the last resulting is equal:

$= \max_{(x.y)\in A_1^\circ\times A_2^\circ} \mathrm{Re}\,\{[(f_1-P_1)\,(x)\times(f_2 - P_2)\,(y)]\,(Q_1(x)\times -Q_2(y)\} \ge 0$

And since $[(f_1-P_1)\,(x)\times(f_2-P_2)\,(y)]$. $(Q_1(x)\times -Q_2(y)\}$ is multiply of two complex numbers then the last term is equal:

$= \max_{x\in A_{\circ^\circ}\times A_\circ} \mathrm{Re}\,\{[(f_1-P_1)\,Q_1(x)+(f_2-P_2)\,Q_2(y)]\times [-(f_1-P_1)\,(x)Q_2(y))+(f_2-P_2)\,(y)Q_1(x)]\} \ge 0$

$= \max_{x\in A_1^\circ\times A_2^\circ} \{[(f_1-P_1)\,Q_1(x)+(f_2-P_2)\,Q_2(y)] \ge 0$

$= \max_{x\in A_\circ}\{(f_1-P_1)\,Q_1(x) + \max_{y\in A_\circ}(f_2-P_2)\,Q_2(y)\} \ge 0,$

So there are three cases of this result:

First, either $\max_{x\in A_{\circ_1}} \{(f_1-P_1)\,Q_1(x) \ge 0$ and $\max_{y\in A_{\circ_2}} (f_2 - P_2)\,Q_2(y)\} \ge 0$

And then by Kolmogorov theorem $P_1(x)$ best approximation of the function $f_1(x)$ and $P_2(x)$ best approximation of the function $f_2(x)$.

Second, or $\max_{x\in A_1^\circ} \{(f_1-P_1)\,Q_1(x)\} \ge 0$ and $\max_{y\in A_1^\circ} \{(f_2 - P_2)\,Q_2(y) \le 0$

and $|\max_{x\in A_1^\circ}\{(f_1-P_1)\,Q_1(x)| \ge |\max_{y\in A_1^\circ 1} \{(f_2-P_2)\,Q_2(y)\,|$

Since $\max_{y\in A_1^\circ}\{(f_2-P_2)\,Q_2(y) \le 0$ then $\min_{y\in A_1^\circ} \{(f_2 - P_2)\,Q_2(y) \le 0$

And since all functions $f_2\cdot P_2\cdot Q_2$ are real functions

Then $\min_{x\in A_1^\circ}\left\{\overline{(f_2-P_2})\,Q_2(x)\right\} \le 0$ And since

$\max_{x\in A_1^\circ}\left\{(f_1-P_1)\,\overline{Q_1(x)}\right\} \ge 0$

This mean by Kolmogorov theorem $P_1(x)$ best approximation of the function $f_1(x)$ and $P_2(y)$ best approximation of the function $f_2(y)$.

Third, the other case $\max_{x\in A_1^\circ} \{(f_1-P_1)\,Q_1(x)\} \le 0$ and $\max_{y\in A_2^\circ} \{(f_2-P_2)\,Q_2(y) \ge 0$ and

$|\max_{x\in A_1^\circ}\{(f_1-P_1)\,Q_1(x)| \le |\max_{x\in A_1^\circ} \{(f_2-P_2)\,Q_2(x)\,|$ is treated with the same way of second and the proof is complete.

Now let us take the following examples:

Example: This example is mentioned in [7] and we use it to apply the theorem 3.4:

A polynomial $P^*_{n-1}(z_0) \equiv 0$ gives the best approximation of the function $f(z) = z^n$ defined on the unit disk $|z| \leq 1$ among all algebraic polynomials of degree $n-1$.

Now, as far as it comes to our subject let $n = 2$ then $f(z) = z^2 = (x^2 - y^2, 2xy)$ And $P^*_1(z_0) \equiv 0 = (0,0)$ let $z_0 = (x_0, y_0)$ note that the role of the set E is in this case, played by the unit circle

$E : |z| = 1$ (i.e. $x^2 + y^2 = 1$), So $x = \mp\sqrt{1-y^2}$ Then by theorem 3.4 we get: $P^*_1(x_0) \equiv 0$ and $P^*_1(y_0) \equiv 0$ gives the best approximation of the function $f_1(x) = x^2 - y^2 = x^2 - 1 + x^2 = 2x^2 - 1$ and $f_2(y) = 2xy = 2\sqrt{1-y^2}y$ (resp.) To access this result not that Let $P^*_1(x) = a_1x + b_1$ and $P^*_2(y) = a_2y + b_2$ where $a_1 \cdot b_1 \cdot a_2 \cdot b_2$ be a constants now, the product $P_{n-1}(x)\overline{\left(f(x) - P^*_{n-1}(x)\right.} = P_1(x)\left(2x^2 - 1 - 0\right) = P_1(x)\left(2x^2 - 1\right) = (a_1x + b_1)\left(2x^2 - 1\right)$. It clear that the last previous product is a continuous function. So, by continuity we get: it is non positive in at least one point $(x_0.0) \in E$. For example: if we take $x_0 = -1$ and choose $a_1 = b_1$ we get non positive result (or zero). Another example if we choose $x_0 = 0$ and b_1 is positive we get non positive result...etc.

So by Kolmogorov theorem $P^*_{n-1}(x)$ is best approximation of degree 1 of $f_1(x) = 2x^2 - 1$. With same way for the function $f_2(y) = -2y\sqrt{1-y^2}$

Another example for this for this purpose we go again of example in [5]:

Let $f_1(x) = e^x$ and $f_2(y) = |y|$, then It is clear that the polynomial of best approximation from degree 2 by Chebyshev's of $f_1(x)$ is $P_2(x) = 0\cdot 9890 + 1 \cdot 1302x + 0 \cdot 5540x^2$.

Also, polynomial of best approximation from degree 2 by of $f_2(y)$ is $P_2(y) = \frac{1}{8} + y^2$. Now, let $F(x,y) = e^x + i|y| = (e^x, |y|)$ and $\boldsymbol{P}^\circ_2(x,y) = P^\circ_2(x) + iP^\circ_2(y) = (0 \cdot 9890 + 1 \cdot 1302x + 0 \cdot 5540x^2) + i\left(\frac{1}{8} + y^2\right) = \left(0 \cdot 9890 + 1 \cdot 1302x + 0 \cdot 5540x^2, \frac{1}{8} + y^2\right)$. Now, we try to applying conditions of theorem 3.4. This mean: Is $\max\limits_{x\in A^\circ} \operatorname{Re}\left\{\left[F(z) - P^\circ_2(z)\right]\overline{P_2(z)}\right\} \geq 0$? for any polynomial $P_2(z)$ and $A_\circ = [1,-1] \times [1,-1]$, where $P_2(z) = a_0z^2 + a_1z + a_2 = a_0(x+iy)^2 + a_1(x+iy) + a_2$

$= a_0\left(x^2 - y^2 + 2ixy\right) + a_1(x+iy) + a_2$

$= a_0x^2 - a_2y^2 + 2a_0ixy + a_1x + ia_1y + a_2$

so, $\overline{P_2(z)} = a_0x^2 - a_2y^2 - 2a_0ixy + a_1x - ia_1y + a_2$

$\max\limits_{z\in A^\circ} \operatorname{Re}\left\{\left[\left(F - P^\circ_n\right)(z)\right]\overline{P_n(z)}\right\}$

$= \max\limits_{z\in A^\circ} \operatorname{Re}\left\{\left[F(z) - P^\circ_2(z)\right]\overline{P_2(z)}\right\}$

$= \max\limits_{z\in A_\circ} \operatorname{Re}\{[(e^x + i|y|)$

$-((0.9890 + 1.1302x +$

$0 \cdot 5540x^2) + i\left(\frac{1}{8} + y^2\right)\right)\Big]\overline{P_2(z)}\Big\}$

$= \max\limits_{z\in A^\circ} \operatorname{Re}\{[(e^x - (0 \cdot 9890 + 1 \cdot 1302x+$

$0.5540x^2)) + i\left(|y| - \left(\frac{1}{8} + y^2\right)\right)\Big]\overline{P_2(z)}\}$

$= \max\limits_{(z.y)\in A_\circ} \operatorname{Re}\{[(e^x - (9890 + 1.1302x+$

$0 \cdot 5540x^2)\ )$

$+i\left(|y| - \left(\frac{1}{8} + y^2\right)\right)\Big]\left(a_0x^2 - a_2y^2\right.$

$\left.-2a_0ixy + a_1x - ia_1y + a_2\right)\}$

$= \max\limits_{z\in A_\circ} \{[((e^x - (0.9890 + 1.1302x+$

$$0.5540x^2\Big)\left(a_0x^2 - a_2y^2 + a_1x + a_2\right) + \left(|y| - \left(\frac{1}{8} + y^2\right)\right)(2a_0xy + a_1y)\Big\}\Big)$$

Now, for example for the value of last amount in $z_0 = (0.0)$ and $a_2 \leq 0$ is negative

So, by theorem 3.3$P^\circ_2(z)$ be a best approximation of $F(x.y)$.

4 DISCUSSION

During the course of this research, the extent of the connection between the approximation of complex functions (in the special case described in the introduction) and the approximation of real functions has become clear. This connection appears in several aspects, including the polynomial of best approximation of the complex function under consideration and the polynomials of best approximation of the real functions that arise from it, as well as the degree of these approximations for both types of functions. It also involves the modulus that measures the smoothness of the complex function and its associated real functions, in addition to other related results.

5 CONCLUSION

Since a complex function can be written as the Cartesian product of two real functions, the main conclusion of this research is that the approximation of the complex function depends on the approximations of the two real functions that constitute it, according to the rule of the Cartesian product of functions. This dependence appears in the existence of the approximation, in its degree, and in the corresponding polynomial of best approximation.

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