P-ISSN 1991-8941
E-ISSN 2706-6703
Vol. 19, No. 2 (2025)

Journal of University of Anbar for Pure Science (JUAPS)

journal homepage: juaps.uoanbar.edu.iq

Open Access

Original Paper

Exploring strong Gelfand pairs in wreath products of specific finite groups

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ARTICLE INFO

Received: 16/02/2025
Accepted: 27/03/2025
Available online: 20/11/2025
December Issue
10.37652/juaps.2025.157548.1358

CITE @ JUAPS

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ABSTRACT

A strong Gelfand pair (G, H) consists of a group G and its subgroup H, characterized by the property that the induced representation Ind d_H^G of the permutation representation associated with the action of G on the set of cosets GH is multiplicity-free. In this study, the conditions under which G forms a strong Gelfand pair with certain subgroups are investigated, with emphasis on two main cases: i) the wreath product $G \wr S_n$ with its subgroup $G \wr S_{n-1}$, ii) the group $W_n = G \wr S_n = G^n \rtimes S_n$ with its subgroup $C_n = E_n \times S_n$, where E_n is the diagonal subgroup of the direct product of n copies of G. While such pairs are always Gelfand for abelian groups, in general they do not satisfy the strong Gelfand property. Specifically, there exists a positive integer $2 \le SN(G) < |G|$ such that $(G \wr S_{n-1}, G \wr S_n)$ and (W_n, C_n) are strong Gelfand pairs for $n < SN(G)$, but not for $n \ge SN(G)$. This result holds for both abelian and non-abelian finite groups. A method is developed to compute this critical number $SN(G)$, and explicit examples are provided for specific groups.

Keywords: *Computer algebra system GAP, Multiplicity-free, Strong Gelfand pair, Wreath product*

1 INTRODUCTION

The concept of a Gelfand pair was introduced by Gelfand and his collaborators in 1950. It provides a useful simplification in the study of representations, since the irreducible components then appear without repetition. The notion of a strong Gelfand pair was later introduced in the literature as an extension of this idea. In the context of character theory, a pair (G, H), consisting of a group G and a subgroup H of G, is called a Gelfand pair if the induced representation Ind $H^G 1H$ is multiplicity-free, where 1_H is the trivial representation of H. The pair is called a strong Gelfand pair if the same property holds for every irreducible character ϑ of H (see [1–3] for more details). This topic has rapidly spread into several areas, including representation theory, harmonic analysis, and mathematical physics, among others. Many attempts have been made to determine Gelfand and strong Gelfand pairs; see [4–6]. In [7], strong Gelfand pairs arising from groups constructed via the wreath product are studied. Using the computer algebra system MAGMA, Joseph E. Marrow determined all strong Gelfand pairs for certain finite groups, including dihedral, dicyclic, Frobenius, symplectic, and some sporadic groups [8]. An interesting family of groups and subgroup pairs is examined in [9], which shows that the pair $(G \wr S_{n-1}, G \wr S_n)$ is a Gelfand pair if and only if G is abelian, where S_n is the symmetric group and G is a finite group. In work related to parking functions, the authors conjecture the existence of a specific threshold beyond which pairs of groups and their subgroups fail to be Gelfand, and they provide such a bound for several groups, namely $S_3, A_4, GL(2,3)$ and $SL(3,2)$ [10]. This conjecture was confirmed in [11], where the pair (W_n, C_n) is studied, with $W_n = G \wr S_n = G^n \rtimes S_n$ and $C_n = E_n \times S_n$, where E_n is the diagonal subgroup of n copies of the Cartesian product of G with the symmetric group. In that work, a positive integer $N(G)$ is obtained, with $2 \le N(G) < |G|$, such that the pair (W_n, C_n) is a Gelfand pair for $n < N(G)$ and fails to be one for $n \ge N(G)$. The present study extends this result by asking whether an analogous statement holds when strong Gelfand pairs are considered.

Our approach to this problem is based on designing a

December 2025 issue – published 1 December 2025.

procedure that can be applied to any finite group together with its subgroups, in particular to the pair (W_n, C_n), in analogy with [11]. It is shown that even for abelian groups there exists a positive integer $2 \le SN(G) < N(G) < |G|$ such that (W_n, C_n) is a strong Gelfand pair for $n < SN(G)$, but fails to be strong Gelfand for $n \ge SN(G)$. This indicates the complexity of the irreducible representations of wreath products and direct products involving symmetric groups. In addition, it is shown that, for an arbitrary finite group G, the pair $(G \wr S_{n-1}, G \wr S_n)$ either is a strong Gelfand pair or fails to be one, independently of whether G is abelian or non-abelian.

The procedure derived in this work has been implemented using the computer algebra system GAP [12], although it can also be carried out in any other computer algebra system.

2 PRELIMINARIES

Let G be a finite group and H be a subgroup of G. Given a representation U of H, the induced representation of U from H to G is defined by:

$$\mathrm{Ind}_H^G U = \bigoplus_{g \in G} g \otimes U$$

The group G acts on this representation as,

$$g \cdot (s \otimes u) = (gs) \otimes u, \text{ where } g \in G, s \in G/H$$

The associated induced character is given by

$$\varphi_{Inde_H^G U}(g) = \sum_{\substack{s \in G/H \\ s^{-1}gs \in H}} \varphi_U \left(s^{-1}gs\right)$$

where φ_U is the character of the representation U of H. A pair (G, H) is called a Gelfand pair if the inner product of the induced character of the trivial character of H with all irreducible characters of G satisfies

$$\left\langle Ind_H^G id_H, \chi \right\rangle \le 1, \quad \forall \chi \in \mathrm{Irr}(G)$$

Analogously, (G, H) is a strong Gelfand pair if

$$\left\langle \mathrm{Ind}_H^G \vartheta, \chi \right\rangle \le 1, \quad \forall \chi \in \mathrm{Irr}(G), \forall \vartheta \in \mathrm{Irr}(H)$$

The symmetric group S_n acts on the Cartesian product G^n, permuting its coordinates and forming the wreath product

$$W_n = G \wr S_n = G^n \rtimes S_n$$

The action of S_n centralizes the diagonal subgroup

$$E_n := \{(g, \ldots, g) : g \in G\}$$

The subgroup $C_n = E_n \times S_n$ is a natural subgroup of $W_{n|}$. Moreover, the subgroup S_{n-1} is the stabilizes one element in the action of S_n on G in the wreath product. This enables to construct the pair ($G : S_{n-1}, G : S_n$) where $G \succ S_{n-1}$ is a subgroup of $G \succ S_n$.

In this work, a method is developed to test whether a pair (G, H), where G is a finite group and H is its subgroup, satisfies the strong Gelfand pair criteria. The analysis then focuses on the pairs $(G \succ S_{n-1}, G \succ S_n)$ and (W_n, C_n), proving the existence of a critical integer

$$2 \le SN(G) < N(G) < |G|$$

such that (W_n, C_n) is a strong Gelfand pair for $n < SN(G)$ but not for $n \ge SN(G)$. Obtaining such bounds simplifies the study of representations of wreath products of this type.

3 RESULTS AND DISCUSSION

3.1 Strong gelfand pair of a finite group

The following algorithm is designed to locate strong Gelfand pairs for any finite group.

Algorithm 1: Testing if a pair (G, H) form a Strong Gelfand Pair

Input: Two finite groups G and H, where H is a subgroup of G.
Output: True if (G, H) is strong Gelfand pair; False Otherwise.

1. irG = Irreducible characters of G
   # Compute irreducible characters of G
2. irH =Irreducible characters of H
   # Compute irreducible characters of H
3. for $x \in$ ir$G$ do
4. for $y \in$ irH do
5. $y^G(g) = \frac{1}{|H|} \sum_{r \in G} y\left(r^{-1}gr\right)$, where y extends to G
   by letting $y(g) = 0$ if $g \notin H$
   # Compute induced character of y from H to G
6. s_ p = $\left\langle x, y^G \right\rangle = \frac{1}{|G|} \sum_{g \in G} x(g)\overline{y^G(g)}$
   #Compute the scalar product of x and y^G
7. If s_p > 1: # Criteria of multiplicity-free
8. Return False
9. end if
10. end for
11. end for

12. Return True

Proof. Since the groups G and its subgroup H are finite, their character tables, in particular their irreducible characters, exist and are well defined. The induced character is computed effectively by applying the formula above. The inner product of class functions, specifically irreducible characters, is well defined, and therefore the algorithm satisfies both correctness and completeness.

3.2 Strong gelfand pairs in wreath products $(G \wr S_{n-1}, G \wr S_n)$

To construct $G \wr S_{n-1}$ as a subgroup of $G \wr S_n$, first observe that the symmetric group S_n acts on an n-element set by permutation. The subgroup S_{n-1} of S_n is the stabilizer of a single element under this action. By the definition of the wreath product, the action of S_n extends to its cosets with respect to S_{n-1}. We embed G into the base group G^n, ensuring that it remains a subgroup of the wreath product. Similarly, we embed S_{n-1} into the wreath product by mapping it to the stabilizer subgroup of a single point. The group $G \wr S_{n-1}$ is then generated by the images of these two embeddings. Applying this procedure in the computer algebra system GAP [12], we obtain the results listed in Table 1.

While [9] shows that the pair $(G \wr S_{n-1}, G \wr S_n)$ is a Gelfand pair if and only if G is abelian, Table 1 demonstrates that both abelian and non-abelian groups can form strong Gelfand pairs and also provides examples of non-abelian groups that form Gelfand pairs.

3.3 Strong gelfand for the pairs ($G \wr S_n, E_n \times S_n$)

While [11] defines a positive integer N(G) that measures the ability of the pair (W_n, C_n) to be a Gelfand pair, where, $2 \le N(G) < |G|$, it is shown that (W_n, C_n) is a Gelfand pair for $n < N(G)$ but fails for $n \ge N(G)$. Since a strong Gelfand pair is a generalization of a Gelfand pair, we introduce a positive integer $0 \le SN(G) \le N(G) < |G|$, such that (W_n, C_n) is a strong Gelfand pair for $n < SN(G)$ but fails for $n \ge SN(G)$.

Table 1 Implementation for the pair $(G \wr S_{n-1}, G \wr S_n)$

G	n	Description of $G \wr S_n$	Description of $G \wr S_{n-1}$	Gelfand	Strong Gelfand
C_2	2	D_8	C_2	yes	yes
C_2	3	$C_2 \times S_4$	$C_2 \times C_2$	yes	yes
C_2	4	$((((C_2 \times C_2 \times C_2 \times C_2) : C_2) : C_2) : C$	D_{12}	yes	no
C_3	2	$C_3 \times S_3$	C_3	yes	yes
C_3	3	$((C_3 \times C_3 \times C_3) : C_3) : C_2$	C_6	yes	yes
C_3	4	$C_3 \times ((((C_3 \times C_3 \times C_3) : (C_2 \times C_2)) : C_3) : C_2$	$C_3 \times S_3$	yes	no
S_2	2	D_8	C_2	yes	yes
S_2	3	$C_2 \times S_4$	$C_2 \times C_2$	yes	yes
S_2	4	$((((C_2 \times C_2 \times C_2 \times C_2) : C_2) : C_2) : C$	D_{12}	yes	no
S_3	2	$(S_3 \times S_3) : C_2$	S_3	no	no
A_3	2	$C_3 \times S_3$	C_3	yes	yes
A_3	3	$((C_3 \times C_3 \times C_3) : C_3) : C_2$	C_6	yes	yes
A_3	4	$C_3 \times ((((C_3 \times C_3 \times C_3) : (C_2 \times C_2)) : C_3) : C_2)$	$C_3 \times S_3$	yes	no
$GL(2,3)$ is not strong for any n					
$SL(3,2)$ is not strong for any n					

Following the same approach, we construct $E_n \times S_n$ as a subgroup of $G \wr S_n$, where E_n is the diagonal subgroup of the Cartesian product of n copies of G and implementing this in GAP, we get:

Table 2 Implementation to the pair $(G \wr S_n, E_n S_n)$

n	G	Description $E_n \times S_n$	Description $G \wr S_n = G^n \rtimes S_n$	Gelfand pair $/N(G)$	Strong Gelfand pair $/SN(G)$
2	D_6	D_{12}	$(S_3 \times S_3): C_2$	Yes/	Yes/
3	D_6	$S_3 \times S_3$	$((S_3 \times S_3 \times S_3) : C_3) : C_2$	Yes/	No/ 3
4	D_6	$S_4 \times S_3$	$(((C_3 \times C_3 \times C_3 \times C_3) : (((C_2 \times C_2 \times C_2 \times C_2) : C_2) : C_2)) : C_3) : C_2$	Yes/	No/
5	D_6	$S_5 \times S_3$	$(C_3 \times C_3 \times C_3 \times C_3 \times C_3) : (C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) : S_5))$	Yes/	No/
6	D_6	$S_6 \times S_3$	$(C_3 \times C_3 \times C_3 \times C_3 \times C_3 \times C_3) : (((C_2 \times C_2 \times C_2 \times C_2 \times C_2) : A_6) : (C_2 \times C_2))$	No/6	No/
2	GL(2,3	$C_2 \times GL(2,3)$	$(GL(2,3) \times GL(2,3)) : C_2$	Yes/ 3*	No/ 2
2	$SL(3,2)$	$C_2 \times \mathrm{PSL}(3,2$	$(\mathrm{PSL}(3,2) \times \mathrm{PSL}(3,2)) : C_2$	Yes/ 3*	No/ 2

2	$SL(2,3)$	$C_2 \times SL(2,3)$	$(((Q8 \times Q8)$ $:C_3) : C_2) : C_3$	Yes/	Yes/
3	$SL(2,3)$	$SL(2,3) \times S_3$	$(((Q8xQ8 \times Q8)$ $: ((C_3$ $\times C_3) : (C_3)) :$ $C_2 : C_3$	Yes/	No/ 3
4	$SL(2,3)$	$S_4 \times SL(2,3)$	((((((($(Q_8 \times Q_8)$// : C3) $x((Q8x$: C3)) : C2) : C3) : C2) : C3) : C2) : C3	No/4	No/
2	A_3	C_6	$C_3 \times S_3$	Yes/	Yes/
3	A_3	$C_3 \times S_3$	$((C_3 \times C_3 \times C_3) : C_3) : C_2$	Yes/	No/ 3
4	A_3	$C_3 \times S_4$	C_3 $\times (((C_3 \times C_3$ $\times C_3) : (C_2$ $\times C_2)) : C_3) : C_2)$	Yes/	No/
5	A_3	$C_3 \times S_5$	C_3 $\times ((C_3 \times C_3$ $\times C_3$ $\times C_3) : S_5)$	Yes/	No/
6	A_3	$C_3 \times S_6$	$((C_3 \times C_3 \times C_3$ $\times C_3$ $\times C_3) : A_6) : C_6$	Yes/	No/
7	A_3	$C_3 \times S_7$	C_3 $\times ((C_3 \times C_3$ $\times C_3 \times C_3$ $\times C_3$ $\times C_3) : S_7)$	Yes/	No/
8	A_3	$C_3 \times S_8$	C_3 $\times ((C_3 \times C_3$ $\times C_3 \times C_3$ $\times C_3 \times C_3$ $\times C_3) : S_8$	Yes/	No/
9	A_3	$C_3 \times S_9$	$((C_3 \times C_3 \times C_3$ $\times C_3 \times C_3$ $\times C_3 \times C_3$ $\times C_3) : A_9) : C_6$	Yes/	No/
P = 2 , m= 3, 3	C_9	$C_8 \times S_3$	C_8 $\times (((C_8$ $\times C_8) : C_3) : C_2)$	Yes/	No/

Table 2. demonstrates that $SN(D_6) = 3, N(D_6) = 6$, (this result is with [11] for $p = 3$.), $SN(GL(2,3) = 2$, $SN(SL(3,2) = 2$, $SN(SL(2,3) = 3, N(SL(2,3) = 4$, $SN(A_3) = 3$, n^* means this result coincides with [10].

4 CONCLUSION

In this work, we develop a new method to determine whether a given pair (G, H), consisting of group G and its subgroup H, forms a strong Gelfand pair. We focus on the following cases:

1. The wreath product $G \wr S_n$ with its subgroup $G \wr S_{n-1}$.
2. The group $W_n = G \wr S_n = G^n \rtimes S_n$ with its subgroup $C_n = E_n \times S_n$. For the first case, we show that both abelian and non-abelian groups can form strong Gelfand pairs and provide examples of non-abelian groups that form Gelfand pairs.

   For the second case, we introduce a positive integer $0 \leq SN(G) \leq N(G) < |G|$ such that (W_n, C_n) is a strong Gelfand pair for $n < SN(G)$ but fails for $n \geq SN(G)$. We obtain the following results:

- $SN(D_6) = 3N(D_6) = 6$ (the second coincides with [11] for p = 3).
- SN($GL(2,3) = 2$.
- SN(SL(3, 2) = 2.
- SN($SL(2,3) = 3, N(SL(2,3) = 4$.
- SN $(A_3) = 3$, n^* means this result coincides with [10].

ACKNOWLEDGEMENT

We express our gratitude to Alexander Hulpke for suggesting the use of the construction of the diagonal subgroup instead of the IsomorphicSubgroups command, which is computationally expensive. This approach enabled a more efficient algorithm and the investigation of larger groups. We also thank Mustansiriyah University for providing a comfortable environment and necessary facilities.

FUNDING SOURCE

No funds received.

DATA AVAILABILITY

N/A

DECLARATIONS

Conflict of interest

No known competing financial interests.

Consent to publish

All authors consent to the publication of this work.

Ethical approval

N/A

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How to cite this article

Bedaiwi SO. Exploring strong Gelfand pairs in wreath products of specific finite groups. Journal of University of Anbar for Pure Science. 2025; 19(2):216-220. doi:10.37652/juaps.2025.157548.1358