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On higher derivatives of multivalent analytic functions with negative coefficients associated with (r,q) - Calculus

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ABSTRACT

In this paper, we review established concepts and fundamental results of (r,q) Calculus. Throughout this paper, let r,q be constants with $0 < q < r \leq 1$. We recall some definitions and theorems related to (r,q)-Calculus that are needed in the subsequent sections of this paper.

Keywords: *Analytic function, New subclass, p-valent function, Quantum or (r,q)-Calculus, (r,q)-Derivative operator, Unit disk*

1 INTRODUCTION

In the nineteenth century, geometric function theory emerged within complex analysis as a systematic study of the geometric properties of analytic functions. This field revealed a wealth of elegant results that remain highly relevant and continue to motivate active research. Mathematicians have found these ideas both deep and accessible, which has helped sustain broad interest in their development. The present work lies within this framework of geometric function theory and focuses specifically on univalent and multivalent functions. One of the central questions in the study of univalent functions concerns the existence of a univalent mapping from a simply connected domain. This is a natural problem that invites deeper investigation. For a given simply connected domain, the Riemann mapping theorem provides a powerful tool that reduces the difficulty of constructing such mappings by relating the domain to the open unit disk. In this paper, we consider the class $\hat{A}$ of all analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on the open unit disk $U = z \in \mathbb{C} : |z| < 1$. Also, let $A(p)$, for $p \in \mathbb{N} = 1, 2, 3, \ldots$, be the class consisting of all analytic functions f of the form $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$, which are called p-valent functions. It is important to observe that $A(1) \equiv A$. The subclass of all univalent functions in the open disk U is denoted by $S(p)$, which is a subclass of $A(p)$. Furthermore, let $S_p^(\alpha)$ and $C_p(\alpha)$ denote the classes of starlike and convex functions of order α, respectively, for $0 \leq \alpha < p$. In particular, $S_p^(0)$ coincides with S_p^* and $C_p(0)$ coincides with C_p, the well-known classes of starlike and convex p-valent functions in U, respectively.

Next, let us assume that $T(p)(p \in N = \{1, 2, 3, \ldots\})$ denote the subclass of $S(p)$ of analytic functions having the structure:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, a_n > 0 \qquad (1)$$

As it is defined on the open unit disk $U = \{z \in C : |z| < 1\}$. A function $f \in T(p)$ is denoted as a p -valent function with negative coefficients. Moreover, it is evident that $S^*_{T,p}(\alpha)$ and $C_{T,p}(a)$ for $0 \le \alpha < p$, that are p -valent functions, respectively starlike of order α and convex of order α which is subclasses of $T(p)$. Clearly, the class $T(1) = T$ in [1], he derived and investigated the subclasses of $T(1)$ denoted by $S^*_{T,1}(\alpha) = S^*_T(\alpha)$ and $C_{T,1}(\alpha) = C_T(\alpha)$, for $0 \le \alpha < 1$ that are respectively starlike and convex of order α. Many authors such as [2–7] study related paper. For $0 < q < r \le 1$, the Jackson's (r, q)-derivative of a function $f \in A(p)$ is given as follow:

$$D_{r,q}f(z) := \begin{cases} \frac{f(rz)-f(qz)}{(r-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{2}$$

From (2), we have

$$D_{r,q}f(z) = [p]_{r,q}z^{p-1} + \sum_{n=p+1}^{\infty} [n]_{r,q}a_n z^{n-1}$$

where:

$$[p]_{r,q} = \frac{r^n - q^n}{r-q}, \quad [n]_{r,q} = \frac{r^n - q^n}{r-q}$$

Notice that for $r = 1$, the Jackson (r, q)-derivative reduces to the Jackson q-derivative operator of the function $f, D_q f(z)$ [8–10]. Clearly for a function $g(z) = z^n$, we obtain:

$$D_{r,q}g(z) = D_{r,q}z^n = \frac{r^n - q^n}{r-q}z^{n-1} = [n]_{r,q}z^{n-1}$$

and $\lim_{q\to 1^-} D_{1,q}g(z) = \lim_{q\to 1^-} \frac{1-q^n}{1-q}z^{n-1} = nz^{n-1} = g'(z)$, where g' is the ordinary derivative.

The theory of q-calculus has been extensively applied to a wide range of problems in the applied sciences, including ordinary and fractional calculus, quantum physics, optimal control, hypergeometric series, operator theory, q-difference and q-integral equations, as well as geometric function theory in complex analysis. The initial application of q-calculus was introduced in [11]. In [12], the authors employed fractional q-calculus operators to investigate specific classes of functions that are analytic in the open unit disk U. For more comprehensive discussions on q-calculus, readers may consult [13–20], along with the references cited therein.

Now, let $M(A, B, C)$ be the subclass of $A(1)$ consisting of functions $f \in A(1)$ which satisfy the inequality:

$$\left|\frac{f'(z) - 1}{Af'(z) + (1-B)}\right| < C$$

where $0 \le A \le 1, 0 \le B < 1$ and $0 < C \le 1$ for all $z \in u$. This class of functions was studied by, [21].

In, [21], defined the class $M^T(A, B, C)$ by $M^T(A, B, C) = M(A, B, C) \cap T$.

Alharayzeh, Darus and Alzboon [17–20] present a class of multivalent analytic function associated with (r, q) derivative operator as follows.

$$\operatorname{Re}\left(\frac{(D_{r,q}f(z)) - 1}{\alpha\left(D_{r,q}f(z)\right) + (1-\gamma)}\right) > k\left|\frac{(D_{r,q}f(z)) - 1}{\alpha\left(D_{r,q}f(z)\right) + (1-\gamma)} - 1\right| + \beta$$

For $0 \le \alpha \le 1, 0 \le \beta < 1, 0 \le \gamma < 1, k \ge 0, 0 < q < r \le 1$ and $p \in N = \{1, 2, 3, \ldots\}$. Now, we present generalizations of above definition by using higher derivatives as follows.

Definition 1.1 For $0 \le \alpha \le 1, 0 \le \beta < 1, 0 \le \gamma < 1, k \ge 0, 0 < q < r \le 1, p > m$ and $p \in N = \{1, 2, 3, \ldots\}$ and $m \in N \cup \{0\}$.

$$\operatorname{Re}\left(\frac{\left(D_{r,q}f(z)\right)^m - 1}{\alpha\left(D_{r,q}f(z)\right)^m + (1-\gamma)}\right) > k\left|\frac{\left(D_{r,q}f(z)\right)^m - 1}{\alpha\left(D_{r,q}f(z)\right)^m + (1-\gamma)} - 1\right| + \beta \tag{3}$$

This study focuses on a particular class of analytic functions denoted by $f \in \Upsilon(\alpha, \beta, \gamma, k, r, q, p, m)$.

Following this, the radii of starlikeness and convexity for functions in $Y(\alpha, \beta, \gamma, k, r, q, p, m)$ are determined. These results help delineate the geometric domains within which the functions exhibit starlike or convex properties, thus contributing to a deeper understanding of their classification and nature.

First and foremost, let us consider the coefficient inequalities, using the method discussed in [22] and also in [23–26].

2 COEFFICIENT ESTIMATE INEQUALITIES

In this section we present a fundamental and sufficient condition for the function f in the subclass $Y(\alpha, \beta, \gamma, k, r, q, p, m)$.

Theorem 2.1: The function f given by (1) is in the subclass $Y(\alpha,\beta,\gamma,k,r,q,p,m)$ if and only if

$$\sum_{n=p+1}^{\infty}\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n \leq \left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1 \quad (4)$$

Where, $0\leq\alpha\leq 1, 0\leq\beta<1, 0\leq\gamma<1, k\geq 0, 0<q<r\leq 1, p>m$ and $p\in N=\{1,2,3,\ldots\ldots\}$.

Proof. Let $f\in Y(\alpha,\beta,\gamma,k,r,q,p,m)$ if and only if the condition (3) is satisfied assume

$$w=\frac{\left(D_{r,q}f(z)\right)^m-1}{\alpha\left(D_{r,q}f(z)\right)^m+(1+\gamma)}$$

Recall that,

$$R(w)\geq k|w-1|+\beta \text{ iff } (k+w)|w-1|\leq 1-\beta$$

Now

$$(k+1)|\omega-1|=(k+1)=(k+1)\left|\sum_{n=p+1}^{\infty}[(\alpha-1)][n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n z^{n-1-m} -(\alpha-1)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}z^{p-1-m}-(2-\gamma)\right| \quad (5)$$

$$/\left|\alpha[p]_{r,q}\frac{(p-1)!}{(p-1)!}z^{p-1-m}-\alpha \sum_{n=p+1}^{\infty}[n]_{r,q}\frac{(n-1)!}{(n-1)!}a_n z^{n-1-m}-(\gamma-1)\right|\leq 1-\beta$$

$$\leq 1-\beta \quad (6)$$

The above inequality reduces to

$$(k+1)\left(\left|\sum_{n=p+1}^{\infty}(\alpha-1)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n z^{n-1-q}\right| -\left|(\alpha-1)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}z^{p-1-m}\right|-|2-\gamma|\right) /\left(\left|\alpha[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}z^{p-1-m}\right| -\alpha\left|\sum_{n=p+1}^{\infty}[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n z^{n-1-m}\right|-|\gamma-1|\right)\leq 1-\beta \quad (7)$$

After that, Then, we get

$$\sum_{n=p+1}^{\infty}[(k+1)(1-\alpha)+\alpha(1-\beta)][n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n \leq[\alpha(1-\beta)+(k+1)(1-\alpha)][p]_{r,q}\frac{(p-1)!}{(p-1-m)!}-(1-\gamma)(1-\beta)+(k+1)(2-\gamma)$$

Thus

$$\sum_{n=p+1}^{\infty}[(k+1)(1-\alpha)+\alpha(1-\beta)]\,[n_o]_{r,q}\,\frac{(n-1)!}{(n-1-m)!}a_n \leq[\alpha(1-\beta)+(k+1)(1-\alpha)][p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+(1-\gamma)(k+\beta)+(k+1)$$

Divide by $(1-\gamma)(k+\beta)+(k+1)$ for both side which

yield to (4)

$$\sum_{n=p+1}^{\infty}\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q} \leq\left(\frac{\alpha(1-\beta)+(n-1)!}{(n-1-m)!}a_n+1 \quad (8)$$

Suppose that (4) holds and we have to show (6) holds. Here the in equality (4) is equivalent to (7). So it suffices to show that,

$$\left|\left\{\frac{\left(\sum_{n=p+1}^{\infty}(\alpha-1)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n z^{p-1-m}\right)-(\alpha-1)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}z^{p-1-m}-(2-\gamma)\}}{\left\{\alpha[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}z^{p-1-m}-\alpha\left(\sum_{n=p+1}^{\infty}[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n z^{p-1-m}\right)-(\gamma-1)\right\}}\right|$$

$$\leq\frac{\left\{\left(\sum_{n=p+1}^{\infty}(1-\alpha)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n\right)-(1-\alpha)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}-(2-\gamma)\right\}}{\left\{\alpha[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}-\alpha\left(\sum_{n=p+1}^{\infty}[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n\right)-(\gamma-1)\right\}}$$

Since

$$\alpha[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}z^{p-1-m}-\left(\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n z^{p-1-m}\right)$$

$$\geq\left|\alpha[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}z^{p-1-m}\right|-\left|\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n z^{p-1-m}\right|-|\gamma-1|$$

We have

$\alpha[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}z^{p-1-m}-\left(\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n z^{p-1-m}\right)-(\gamma-1)\Big|,$
$\geq\alpha[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}-\alpha\sum_{n=p+1}^{\infty}[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n-(1-\gamma),$

Where $|z|<1$, and hence, we obtain (8).

Theorem 2.2: Let $0\leq\alpha\leq1, 0\leq\beta<1, 0\leq\gamma<1, k\geq0, 0<q<r\leq1, p>m, p\in N=\{1,2,3,\ldots\ldots\}$

if the function f given by (1) is in the subclass $Y(\alpha,\beta,\gamma,k,r,q,p,m)$ then

$$a_n\leq\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}$$

$$n=p+1,p+2,p+3,\ldots \quad (9)$$

Equality holds for the function f given by

$$f(z)=z^p-\frac{\left[\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1\right]}{\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}z^n \quad (10)$$

Proof. Since $f\in Y(\alpha,\beta,\gamma,k,r,q,p,m)$ by Theorem 2.1 holds.

Now
$\sum_{n=p+1}^{\infty}\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(\beta+k)+(k+1)}[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n\leq\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(\beta+k)+(k+1)}[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1.$

We have,

$$a_n\leq\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}$$

The function given by (10) satisfies (9) and therefor f given by (10) in $\Upsilon(\alpha,\beta,\gamma,k,r,q,p,m)$ for this function, the result is sharp.

3 GROWTH AND DISTORTION THEOREMS FOR THE SUBCLASS $Y(\alpha,\beta,\gamma,k,r,q,p,m)$

Theorem 3.1: Let $0\leq\alpha\leq1, 0\leq\beta<1, 0\leq\gamma<1, k\geq0, 0<q<r\leq1, p>m, p\in N=\{1,2,3,\ldots\ldots\}$. if the function f given by (1) be in the subclass $Y(\alpha,\beta,\gamma,k,r,q,p,m)$ then for $0<|z|=r<1$, we get

$$r^p-\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}}r^{p+1}\leq|f(z)|$$

$$\leq r^p+\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}}r^{p+1} \quad (11)$$

Equality holds for the function,

$f(z)$

$= z^p - \frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}} z^{p+1}$

Proof We only prove the right hand side inequality in (11), since the other inequality can be justified using similar arguments.

Since $f \in \Upsilon(\alpha,\beta,\gamma,k,r,q,p,m)$ by Theorem 2.1 we have,

$\sum_{n=p+1}^{\infty}\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n$

$\le \left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1$

Now,

$\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(k+\beta)(1-\gamma)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}$

$\sum_{n=p+1}^{\infty}a_n = \sum_{n=p+1}^{\infty}\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(k+\beta)(1-\gamma)+(k+1)}\right)$

$$[p+1]_{r,q}\frac{p!}{(p-m)!}a_n$$

$\le \sum_{n=p+1}^{\infty}\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(k+\beta)(1-\gamma)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n \le |\grave{f}(z)|$

$$\le \left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(k+\beta)(1-\gamma)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}$$

$+1$

Therefor,

$\sum_{n=p+1}^{\infty}a_n \le \frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}}$

If $f(z) = z^p - \sum_{n=p+1}^{\infty}a_n z^n$ we get,

$|f(z)| = \left|z^p - \sum_{n=p+1}^{\infty}a_n z^n\right|$

$\le |z|^p + |z|^{p+1}\sum_{n=p+1}^{\infty}a_n|z|^{n-(p+1)}$,

$\le r^p + r^{p+1}\sum_{n=p+1}^{\infty}a_n$

By the inequality (12), yields the right hand side inequality of (11).

Theorem 3.2: The function f given by (1) belongs to the subclass $\Upsilon(\alpha,\beta,\gamma,k,r,q,p,m)$ then for $0<|z|=$ $l<1$ we have

$pr^p - \frac{(p+1)\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}} r^p$

$\le pr^{p-1} + \frac{(p+1)\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}} r^p$

Equality holds for the function f given by

$f(z)$

$= z^p - \frac{\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}} z^{p+1}$

Proof. Since $f \in Y(\alpha,\beta,\gamma,k,r,q,p,m)$ by Theorem 2.1

we have

$\sum_{n=p+1}^{\infty}\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n$

$\le \left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1$

Now,

$\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}$

$\sum_{n=p+1}^{\infty}na_n \le$

$(p+1)\sum_{n=p+1}^{\infty}\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n$

$\le (p+1)\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1$

Hence

$\sum_{n=p+1}^{\infty}na_n \le \frac{(p+1)\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{(k+1)(1-\alpha)+\alpha(1-\beta)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}}$,

If $\grave{f}(z) = pz^{p-1} - \sum_{n=p+1}^{\infty}na_n z^{n-1}$, then

$p|z|^{p-1} - |z|^p\sum_{n=p+1}^{\infty}na_n|z|^{n-1-p}$

$\le |\grave{f}(z)| \le p|z|^{p-1} - |z|^p\sum_{n=p+1}^{\infty}na_n|z|^{n-1-p}$,

Where $|z|<1$ by using the inequality (14), this Completes the proof.

Theorem 3.3: The function f given by (1) belongs to the subclass $Y(\alpha,\beta,\gamma,k,r,q,p,m)$ then f is starlike of order δ and this is sharp for the function

$f(z) = z^p - \frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p+1]_{r,q}\frac{p!}{(p-m)!}} z^{p+1}$

Proof. It is sufficing to show that (4) implies $\sum_{n=p+1}^{\infty}a_n(n-\delta) \le 1-\delta$.

That is,

$\frac{n-\delta}{1-\delta} \le \frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}$,

$$n \ge p+1$$

and $\varphi(n) \ge \varphi(p+1)$, (16) holds true for any $0 \le \alpha \le 1, 0 \le \beta < 1, 0 \le \gamma < 1, k \ge 0, 0 < q < r \le 1$ and $p \in N = \{1,2,3,\ldots\ldots\}$. this Completes the proof of Theorem 3.3.

4 EXTREME POINT OF THE CLASS $\Upsilon(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},k,\boldsymbol{r},\boldsymbol{q},p,m)$

Theorem 4.1: Let $f_p(z)=z^p$, and $f_n(z)=z^p$
$-\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}z^n,$
$n=p+1,p+2,p+3,\ldots\ldots\ldots\ldots,$
Then, $f\in Y(\alpha,\beta,\gamma,k,r,q,p,m)$ if and only if it can be represented in the form

$$f(z)=\sum_{n=p}^{\infty}y_nf_n(z) \qquad (12)$$

Where $y_n\geq 0$ and $\sum_{n=p}^{\infty}y_n=1$
Proof. Consider that f can be expressed as in (17), we must show that $f\in\Upsilon(\alpha,\beta,\gamma,k,r,q,p,m)$ by (17) we get,
$f(z)=\sum_{n=p}^{\infty}y_nf_n(z)=y_pf_p(z)+\sum_{n=p+1}^{\infty}y_nf_n(z),$
$=y_pz^p+$
$\sum_{n=p+1}^{\infty}y_n\left(z^p-\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}z^n\right),$
$=y_pz^p+\sum_{n=p+1}^{\infty}y_nz^p$
$-\sum_{n=p+1}^{\infty}y_n\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}z^n,$
$=z^p$
$-\sum_{n=p+1}^{\infty}\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}z^n.$

After that, $f(z)=z^p-\sum_{n=p+1}^{\infty}a_nz^n$ we see
$a_n=\frac{\left[\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1\right]y_n}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}},$
$n\geq p+1.$
Now, we have $\sum_{n=p}^{\infty}y_n=y_p+\sum_{n=p+1}^{\infty}y_n=1$ Then, $\sum_{n=p+1}^{\infty}y_n=1-y_p\leq 1$
Setting
$\sum_{n=p+1}^{\infty}y_n\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}$
$\times\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}$
$=\sum_{n=p+1}^{\infty}y_n=1-y_p\leq 1$
It follows that from Theorem 2.1 the function $f\in Y(\alpha,\beta,\gamma,k,r,q,p,m)$.

Conversely, it suffices to show that
$a_n=\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}y_n.$
Now we get $f\in Y(\alpha,\beta,\gamma,k,r,q,p,m)$ then by previous
Theorem 2.2
$a_n\leq\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}},$
$n\geq p+1$
That is
$\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}\leq 1,$
but $y_n\leq 1$. Setting,
$y_n=\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}a_n}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1},$
$n\geq p+1$
Thus, the proof is complete.

Corollary 4.2: The extreme point of the subclass $\gamma(\alpha,\beta,\gamma,k,\gamma,q,p,m)$ are the function $f_p(z)=z^p$, and $f_n(z)$
$=p-\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}z^n,$
$n=p+1,p+2,p+3,\ldots\ldots$

5 RADIUS OF STAR LIKENESS AND CONVEXITY

The radius of star likeness and convexity for the function in the subclass $Y(\alpha,\beta,\gamma,k,r,q,p,m)$ will also be considered.

Theorem 5.1: Let the function f given by (1) is in the subclass $Y(\alpha,\beta,\gamma,k,r,q,p,m)$ then f is star like of order $\delta(0\leq\delta<p)$, in the disk $|z|<R$ where $R=$
$\inf\left[\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}\times\left(\frac{p-\delta}{n-\delta}\right)\right]^{\frac{1}{n-p}},$
Where $n=p+1,p+2,p+3,\ldots\ldots\ldots,$
Proof : Here (18) implies
$\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1(n-\delta)|z|^{n-p}$
$\leq\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}(p-\delta)$
It suffices to show that
$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\delta$

For $|z| < R$, we get:

$$\left|\frac{z\grave{f}(z)}{f(z)} - p\right| \leq \frac{\sum_{n=p+1}^{\infty}(n-p)a_n|z|^{n-p}}{1-\sum_{n=p+1}^{\infty}a_n|z|^{n-p}} \tag{13}$$

By aid of (9), we get

$$\left|\frac{z\grave{f}(z)}{f(z)} - p\right| \leq \frac{\sum_{n=p+1}^{\infty}\frac{\left(\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1\right)(n-p)|z|^{n-p}}{\left(\frac{\alpha(1-\beta)+k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}}{1-\sum_{n=p+1}^{\infty}\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}(n-1)!}}$$

which is bounded above by $p-\delta$ if.

$$\sum_{n=p+1}^{\infty}\frac{\left[\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1\right](n-p)|z|^{n-p}}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}$$

$$\leq \left[1-\sum_{n=p+1}^{\infty}\frac{\left[\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1\right]|z|^{n-p}}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}\right](p-\delta),$$

And it follows that

$$|z|^{n-p} \leq \left[\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1}\left(\frac{p-\delta}{n-\delta}\right)\right], \quad n \geq p+1$$

Which corresponds to condition (18) given in the Theorem.

Theorem 5.2: The function f given by (1) belongs to the subclass $Y(\alpha,\beta,\gamma,k,r,q,p,m)$ then f is convex of order $\in$ $(0 \leq \epsilon < p)$, in the disk $|z| < w$ where

$$w = \inf\left[\frac{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[n]_{r,q}\frac{(n-1)!}{(n-1-m)!}}{\left(\frac{\alpha(1-\beta)+(k+1)(1-\alpha)}{(1-\gamma)(k+\beta)+(k+1)}\right)[p]_{r,q}\frac{(p-1)!}{(p-1-m)!}+1} \times \left(\frac{p(p-\epsilon)}{n(n-\epsilon)}\right)\right]^{\frac{1}{n-p}}$$

Where $n = p+1, p+2, p+3, \ldots$

Proof: By applying the same method used in the proof of Theorem 5.1, It follows that

$$\left|\frac{z\grave{f}(z)}{f(z)} - (p-1)\right| \leq p-\epsilon, \text{ for } \quad |z| \leq w,$$

Using (9). Hence, Theorem 5.2 is established

6 CONCLUSION

This paper is devoted to the study of a newly introduced subclass of multivalent analytic functions defined in the open unit disk, characterized by Jackson's derivative operator. We begin by establishing the fundamental conditions that functions must satisfy to belong to this subclass, using coefficient characterization as the main tool. Within this framework, we derive several properties of the class, including coefficient bounds, growth and distortion theorems, identification of extreme points, determination of the radius of starlikeness, and convexity results. The findings indicate that the usefulness of Jackson's derivative operator extends beyond this particular subclass, providing a basis for defining and analyzing more general families of multivalent analytic functions. Throughout, we emphasize the role of the involved parameters in shaping broader classes, thereby clarifying the underlying structure of these functions and their geometric behavior.

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