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An Intensive Study of Finite Dimensional Hesitant Fuzzy Normed Linear Spaces

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Abstract

The author's previous covered the concept of fuzzy normed linear space into hesitant fuzzy normed linear space. This work introduces hesitant fuzzy n-normed linear space as a generalization of n-normed space in addition to the generic t-norm, which is the major objective, and provides an expanding family of S*-n-norms that correspond to fuzzy n-norms. The decomposition theorem of a hesitant fuzzy norm into a set of crisp norms has been proven to fail when a t-norm other than "min" is used. This paper also addresses essential conclusions for fuzzy normed linear spaces of limited dimensions within a broad t-norm framework.

Keywords: t-norm, Hesitant fuzzy norm, Hesitant fuzzy normed linear space.

1. Introduction

In 1984 there is a discussion of certain relationships between the axioms of KM fuzzy metric spaces and KM fuzzy normed spaces [1]. The concept of fuzzy sets, along with other concepts that are developed, which is explored in 1992 [2], after that the study examines the proximity properties of Bag and Samanta fuzzy normed spaces as well as Felbin fuzzy normed spaces [3]. During the year 2006, a number of the characteristics of fuzzy semi-normed and fuzzy normed spaces are investigated [4], and the year 2007 the analyse intuitionistic fuzzy n-normed linear spaces using the Fibonacci lacunary statistical convergence approach found in [5]. In 2005, the authors demonstrate that, in the perspective of Bag and Samanta, every classical inner product on a linear space generates the fuzzy inner product as well as fuzzy norm [6], At the year 2020 the idea of a fuzzy metric space has been presented [7]. In 2023, a fuzzy normed linear space and considered the general t-norm proposed in [8] also an investigation is conducted into the relationship between convergence and Cauchy double sequence in fuzzy normed spaces [9], After that the idea of fuzzy normed linear space was first presented in 2019, as well as a fuzzy linear operators and fuzzy normed linear spaces were introduced in [10] . Further more fuzzy sets and logic are introduced in 2019 [11], more over in 1997 they described fuzzy soft Hilbert spaces and defined fuzzy soft symmetric operators as a particular kind of fuzzy soft linear operator [12]. In 1984 [13], there is an introduction to the notions of uniformly convex fuzzy normed linear space, fuzzy normal structure, fuzzy non-expansive mapping, and strongly and weakly fuzzy convergent sequences. Around 1975, the authors in [14] established fuzzy inner product and examine the characteristics of the associated

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fuzzy norm. In the year 2020 the definition and fundamental characteristics of fuzzy normed space are examined [15]. The discussion of intuitionistic fuzzy hyponormal operators can be found in [16]. Fuzzy n-normed linear space was first proposed in 2005 as a generalization of n-normed space [17]. During 2010 the ideas of fuzzy compact sets and sectional fuzzy continuous mappings are examined [18]. Further in 2020, insightistic fuzzy unitary operator on intuitionistic fuzzy Hilbert spaces is defined and discussed by [19], and the generalized quadratic functional equation's general solution is obtained and explained by [20]. Also, examination the idea of intuitionistic fuzzy n-normed linear spaces' Fibonacci lacunary statistical convergence can be found in [21]. The concept of inner product space in a neutrosophic vector space has been introduced in [22]. In 2007, the authors used intuitionistic fuzzy n-normed spaces to create lacunary Δ^m -statistic convergent for triple sequences [23]. Moreover, a discussion of neutrosophic complex inner product spaces can be found in [24]. The concept of an I3 convergent and I3 cauchy convergent sequence in a fuzzy normed linear space is examined in [25], In 2021 the topic of rough deferred statistical convergence is covered in general [26], also in 2020, the concept of a linear operator's boundedness from one fuzzy normed linear space to another is presented [27]. In 2024, fuzzy metric spaces and fuzzy normed spaces were defined as SP-convergent sequences [28].

The ideas of Ceasaro summability in an intuitionistic fuzzy n-normed linear space are presented and discussed in 2022 [29]. Furthermore, a discussion of the notion of refined neutrosophic metric spaces and symbolic neutrosophic metric spaces found in [30]. In 2025, the concept of hesitant fuzzy normed linear space was discussed in [31].

Although the decomposition theorem greatly aids in the investigation of functional analysis, it necessitates meticulous adherence to the specific selection between this "min" t-norm and the t-norm of the triangle inequality is essential to the decomposition theorem's development. Hence, the question of to what extent the results of linear spaces with fuzzy norms may be determined using the broad version of the fuzzy norm ,that is, removing the restricted "min" Triangle inequality using the t-norm, naturally emerges. This work endeavored to address this issue in this study and it was successful in establishing several significant findings about completeness and compactness of finite dimensional hesitant fuzzy normed linear spaces.

The paper is structured as follows:
Using the universal t-norm, we define a hesitant fuzzy norm and provide an example of when the decomposition theorem for hesitant fuzzy norms into a family of crisp norms is invalid and establishes several fundamental results of hesitant fuzzy normed linear spaces of small dimensions. The Riesz lemma in a broad t-norm situation is established in the final section.

2. Preliminaries

Within this part, the notion of hesitant fuzzy set and a few fundamental definitions are given.

Definition 2.1: [3]
On a linear space, a fuzzy set $\mathcal{N}$ in $\mathcal{M}\times[0,\infty)$ is known as a fuzzy norm on $\mathcal{M}$ if it fulfills,
(FN1) $\mathcal{N}(d, 0)=0, \forall d\in \mathcal{M}$.
(FN2) $\mathcal{N}(d, t)=1, \forall t>0 \Leftrightarrow d=0$.
(FN3) $\mathcal{N}(\lambda d,t)=\mathcal{N}(d, t/|\lambda|), \forall d\in \mathcal{M}, \forall t>0, \forall\lambda\in\mathbb{K}^*$ ($\mathbb{K}^*$ is non negative real numbers).
(FN4) $\mathcal{N}(d+b, t+s)\supseteq \mathcal{N}(d, t) * \mathcal{N}(b,s), \forall d,b\in \mathcal{M}, \forall t,s>0$.
(FN5) $\forall d\in \mathcal{M}$, $\mathcal{N}(d,\cdot)$ is left continuous and $\lim_{n\to\infty} \mathcal{N}(d,\cdot)=1$.
Thus, $(\mathcal{M}, \mathcal{N}, *)$ is known fuzzy normed linear space.

Definition 2.2: [31]

For a field $\mathbb{R}$ or $\mathbb{C}$. Consider the t-norm * to be continuous and $\mathbb{X}$ be a vector space. The Hesitant fuzzy set $\mathscr{H}$ in $\mathcal{M}\times[0,\infty)$ is known as hesitant fuzzy norm on in $\mathcal{M}$, if it fulfills,

HFN1. $\mathscr{H}(d,0)=\emptyset^*$,∀d∈ $\mathcal{M}$. (Where $\emptyset^*$ means the empty set).

HFN2. $\mathscr{H}(d,t)=U$, ∀t>0, if and only if d=0. (U: universal set).

HFN3. $\mathscr{H}(\lambda d,t)=\mathscr{H}(d,t|\lambda|)$,∀d∈ $\mathcal{M}$,∀t≥0,∀λ∈$\mathbb{R}$, (where $\mathbb{R}$ is non-negative real numbers).

HFN4. $\mathscr{H}(d+y, t+u)\supseteq\mathscr{H}(d, t)\cap\mathscr{H}(y, u)$,∀ d,∈ $\mathcal{M}$,∀ t, u ≥0.

HFN5. ∀d ∈ $\mathcal{M}$, $\mathscr{H}$(d, ·)is left continuous and lim $n\to\infty\mathscr{H}(d,t)=U$.

3. Hesitant Fuzzy n-Normed Linear Space

In this section, by extending Definition 2.2, the following concept of n-normed linear space with hesitant fuzzy is obtained.

Definition 3.1: [31]

We define Hesitant fuzzy n-normed linear space (HFnNLS) from the Definition 3.1 in [6].
Given a real field F, let **Y** be a linear space over it.
A hesitant fuzzy subset $\boldsymbol{H}$ of Y × Y × ... × Y(n −times)=$\boldsymbol{R}$ ($\boldsymbol{R}$, set of real numbers) referred to a hesitant fuzzy n-norm on **Y** if and only if

HFnN 1: For all $t_1\in R$ with $t_1=0, H(y_{11},y_{12}\dots y_{1n},0)=\emptyset^*$(Empty set).

HFnN 2: For all $t_1\in R$ with $t_1>0, H(y_{11},y_{12}\dots y_{1n},t_1)=U^*$(Universal set).

HFnN 3 : $H(y_{11},y_{12}\dots y_{1n},t_1)$ is invariant under any permutation of $y_{11},y_{12}\dots y_{1n}$.

HFnN 4 : For all $t_1\in R$ with $t_1>0$,

$H(y_{11},y_{12}\dots cy_{1n},t_1)=H\left(y_{11},y_{12}\dots y_{1n},\frac{t_1}{|c|}\right)$, if $c\neq 0, c\in F$.

HFnN 5 : For all $t_1,t_2\in R$ with $t_1,t_2>0$,

$$H(y_{11},y_{12}\dots y_{1n}+y_{1m},t_1+t_2)\supseteq H(y_{11},y_{12}\dots y_{1n},t_1)\cap H(y_{11},y_{12}\dots y_{1m},t_2).$$

HFnN 6: $H(y_{11},y_{12}\dots y_{1n},\bullet)$ is a non-decreasing function of R and
$\lim_{t_1\to\infty}H(y_{11},y_{12}\dots y_{1n},t_1)=U^*$.

Then $(\boldsymbol{Y},H)$ is known as Hesitant Fuzzy n-Normed Linear Space (HFnNLS).

Example 3.2:

Let (Y, || •,• ... • ||) be a linear space with n-norm.

Define $H(y_{11},y_{12}\dots y_{1n},t_1)=$

$$\begin{cases}\emptyset^*\ , & \text{if } t_1=0, (y_{11},y_{12}\dots y_{1n})\in Y\times Y\times\dots\times Y(n-times)\\ U^*\ , & \text{if and only if } y_{11},y_{12}\dots y_{1n} \text{ are linearly dependent}\\ S^*\in P[0,1] & \text{where } S^* \text{ is an arbitrary subset of } P[0,1]\text{ , Otherwise}\end{cases}$$

Then (X,H) is an HFnNLS.

Proof:

HFnN 1: For all $t_1\in R$ with $t_1=0$,
$H(y_{11},y_{12}\dots y_{1n},0)=\emptyset^*, (y_{11},y_{12}\dots y_{1n})\in Y\times Y\times\dots\times Y(n-\text{times})$.

HFnN 2: For all $t_1\in R$ with $t_1>0, H(y_{11},y_{12}\dots y_{1n},t_1)=U^*$, if and only if $y_{11},y_{12}\dots y_{1n}$ are linearly independent.

HFnN 3: $H(y_{11},y_{12}\dots y_{1n},t_1)$ is unaffected by any combination of $y_{11},y_{12}\dots y_{1n}$.

HFnN 4: By the definition of For all $t_1>0$,

$H(y_{11},y_{12}\dots y_{1n},t_1)=S^*$ *and* $H\left(y_{11},y_{12}\dots y_{1n},\frac{t_1}{|c|}\right)=S^*$

so that

$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots c\mathcal{y}_{1n}, t_1) = H\left(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1n}, \frac{t_1}{|c|}\right)$.

HFnN 5 :

CASE (A) :

If $\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i}$ are linearly dependent, and $\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1j}$ are not linearly dependent, then

(i) $t_1 = 0, t_2 = 0$;
(ii) $t_1 = 0, t_2 \neq 0$;
(iii) $t_1 \neq 0, t_2 = 0$;
(iv) $t_1 \neq 0, t_2 \neq 0$.

Subcase (i):

If $t_1 = 0, t_2 = 0$, then

$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i} + \mathcal{y}_{1j}, 0) = \emptyset^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i}, 0) = \emptyset^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1j}, 0) = \emptyset^*$$

$\Rightarrow \emptyset^* \supseteq \emptyset^* \cap \emptyset^*$.

Subcase (ii):

If $t_1 = 0, t_2 \neq 0$, then

$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i} + \mathcal{y}_{1j}, t_1) = S^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i}, t_1) = \emptyset^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1j}, t_2) = S^*$$

$\Rightarrow S^* \supseteq \emptyset^* \cap S^*$.

Subcase (iii):

If $t_1 \neq 0, t_2 = 0$, then

$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i} + \mathcal{y}_{1j}, t_1) = S^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i}, t_1) = U^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1j}, t_2) = \emptyset^*$$

$\Rightarrow S^* \supseteq U^* \cap \emptyset^*$.

Subcase (iv):

If $t_1 \neq 0, t_2 \neq 0$, then

$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i} + \mathcal{y}_{1j}, t_1 + t_2) = S^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i}, t_1) = U^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1j}, t_2) = S^*$$
$$\Rightarrow S^* \supseteq U^* \cap S^*.$$

CASE (B):

If $\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i}$ are not linearly dependent, and $\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1j}$ are linearly dependent, then

(i) $t_1 = 0, t_2 = 0$;
(ii) $t_1 = 0, t_2 \neq 0$;
(iii) $t_1 \neq 0, t_2 = 0$;
(iv) $t_1 \neq 0, t_2 \neq 0$.

Subcase (i):

If $t_1 = 0, t_2 = 0$, then

$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i} + \mathcal{y}_{1j}, 0) = \emptyset^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1i}, 0) = \emptyset^*$$
$$H(\mathcal{y}_{11}, \mathcal{y}_{12} \dots \mathcal{y}_{1j}, 0) = \emptyset^*$$

$\Rightarrow \emptyset^* \supseteq \emptyset^* \cap \emptyset^*$.

Subcase (ii):

If $t_1 = 0, t_2 \neq 0$, then

$$H(y_{11}, y_{12} \dots y_{1i} + y_{1j}, t_1) = S^*,$$
$$H(y_{11}, y_{12} \dots y_{1i}, t_1) = \emptyset^*$$
$$H(y_{11}, y_{12} \dots y_{1j}, t_2) = U^*$$

$\Rightarrow S^* \supseteq \emptyset^* \cap U^*$.

Subcase (iii):

If $t_1 \neq 0, t_2 = 0$, then

$$H(y_{11}, y_{12} \dots y_{1i} + y_{1j}, t_1) = S^*$$
$$H(y_{11}, y_{12} \dots y_{1i}, t_1) = S^*$$
$$H(y_{11}, y_{12} \dots y_{1j}, t_2) = \emptyset^*$$

$\Rightarrow S^* \supseteq S^* \cap \emptyset^*$.

Subcase (iv):

If $t_1 \neq 0, t_2 \neq 0$, then

$$H(y_{11}, y_{12} \dots y_{1i} + y_{1j}, t_1 + t_2) = S^*$$
$$H(y_{11}, y_{12} \dots y_{1i}, t_1) = S^*$$
$$H(y_{11}, y_{12} \dots y_{1j}, t_2) = U^*$$

$\Rightarrow S^* \supseteq S^* \cap U^*$.

HFnN 6: Suppose,
$t_2 > t_1 > 0$, then,
$H(y_{11}, y_{12} \dots y_{1n}, t_2) \supseteq H(y_{11}, y_{12} \dots y_{1n}, t_1)$.
Thus, $H(y_{11}, y_{12} \dots y_{1n}, t_1)$ is a non-decreasing function, also
$\lim_{t_1 \to \infty} H(y_{11}, y_{12} \dots y_{1n}, t_1) = U^*$.
Hence, then (Y, H) is known as hesitant fuzzy n-normed linear space.

Remark 3.3:

We extend decomposition theorem (Fuzzy normed linear space) from the Theorem 2.2 in [3] to Hesitant fuzzy normed linear space.
Consider the Hesitant fuzzy normed linear space (Y,H).Assume that, for all $t_1 > 0$,
$H(y_1, t_1) = \emptyset^* \Rightarrow y_1 = 0$.
Define
$\| y_1 \|_{S^*} = \wedge \{t_1 > 0 : H(y_1, t_1) = S^*, S^* \in P[0,1]$,
then $\{\| \ \|_{S^*} : S^* \in P[0,1]\}$ refers to the hesitant fuzzy norm (S^*-norm)on Y and is an ascending family of norms on $S^* \in P[0,1]$.

Definition 3.4:

We define a convergent sequence in Hesitant fuzzy normed linear space from the Definition 2.3 in [3].
Consider the Hesitant fuzzy normed linear space (Y, H), and consider the sequence $\{y_n\}$ in Y. Then $\{y_n\}$ is called convergent if there exists $y \in Y$, so that $\lim_{n \to \infty} H(y_n - y, t_1) = U^*$ for every $t_1 > 0$,then y is said to be $\lim_{n \to \infty} y_n$ which represent the limit of the sequence $\{y_n\}$.

Definition 3.5: [3]

We define closure of a set in Hesitant fuzzy normed linear space from the Definition 2.6 in [3].
A Hesitant fuzzy normed linear space is (Y,H). The closure of F is a subset B of Y if for any $y \in B$, then there exists a sequence $\{y_n\}$ in F,so that

$$\lim_{n \to \infty} H(y_n - y, t_1) = U^*.$$

Definition 3.6: [12]
A binary operation $*: [0,1]\ X\ [0,1] \to [0,1]$
is continuous t −norm, if it meets the requirements listed below,
A. $*$ is commutative and associative;
B. $b * 1 = b\ \forall\ b \in [0,1]$;
C. $b * c \leq d * e$ whenever $b \leq d$ and $c \leq e$, $\forall b, c, d, e \in [0,1]$.
When $*$ is continuous, it is referred to be continuous $t -$ norm.

Lemma 3.7:
The underlying t-norm † is continuous at (1,1) in the Hesitant fuzzy normed linear space (Y, H), with $\{\mathscr{y}_{11}, \mathscr{y}_{12} \ldots \mathscr{y}_{1n}\}$ representing a set of vectors in Y that are linearly independent. Then, for every set of scalars $\{\beta_1, \beta_2 \ldots \beta_n\}$, there exists $a > 0$ and
$S^* \in P[0,1]$, $H(\beta_1\mathscr{y}_{11} + \beta_2\mathscr{y}_{12} \ldots + \beta_n\mathscr{y}_{1n}, a\sum_{l=1}^{n} \mid \beta_l \mid) \subset U^* - S^*$.

Proof:
Let $t_1 = \mid \beta_1 \mid + \mid \beta_2 \mid + \cdots \mid \beta_n \mid$.
If $t_1 = 0$ then $\beta_j = 0, \forall l = 1,2 \ldots n$ and the relation
$H(\beta_1\mathscr{y}_{11} + \beta_2\mathscr{y}_{12} \ldots + \beta_n\mathscr{y}_{1n}, a\sum_{l=1}^{n} \mid \beta_l \mid) \subset U^* - S^*$
is equivalent to

$$H(\gamma_1\mathscr{y}_{11} + \gamma_2\mathscr{y}_{12} \ldots + \gamma_n\mathscr{y}_{1n}, a) \subset U^* - S^*, \tag{1}$$

for some $a > 0$, and $S^* \in P[0,1]$ and for all scalars $\gamma's$ with
$\sum_{l=1}^{n} \mid \gamma_l \mid = 1$, where $\gamma_l = \frac{\beta_l}{\sum|\beta_l|}$.
If possible, suppose that Equation (1) is not true.
Thus, for each $a > 0$ and $S^* \in P[0,1]$, there exists a set of scalars $\{\gamma_1, \gamma_2 \ldots \gamma_n\}$ with
$\sum_{l=1}^{n} \mid \gamma_l \mid = 1$,for which
$H(\gamma_1\mathscr{y}_{11} + \gamma_2\mathscr{y}_{12} \ldots + \gamma_n\mathscr{y}_{1n}, a) \supseteq U^* - S^*$,
then for $a = \frac{1}{q}, q = 1,2 \ldots$,there exists a set of scalars, $\{\gamma_1^{(q)}, \gamma_2^{(q)} \ldots \gamma_n^{(q)}\}$ with
$\sum_{l=1}^{n} \mid \gamma_l^{(q)} \mid = 1$,such that $H\left(z_q, \frac{1}{q}\right) \supseteq U^* \setminus \left\{\frac{1}{q}\right\}$, where
$z_m = \gamma_1^{(q)}\mathscr{y}_{11} + \gamma_2^{(q)}\mathscr{y}_{12} + \cdots + \gamma_n^{(q)}\mathscr{y}_{1n}$,
since,$\sum_{l=1}^{n} \mid \gamma_l^{(q)} \mid = 1$ and $0 \leq \mid \gamma_l^{(q)} \mid \leq 1$, for $l = 1,2, \ldots n$,
so, for each l the sequence $\{\gamma_l^{(q)}\}$ the subsequence of $\{\gamma_1^{(q)}\}$ is convergent since it is bounded. Let the limit of that subsequence be represented by γ_1, and the equivalent subsequence of $\{z_q\}$ by $\{z_{1,q}\}$. According to the same reasoning, $\{z_{1,q}\}$ has a subsequence $\{z_{2,q}\}$, for which γ_2 is the matching subsequence of scalars$\{\gamma_1^{(q)}\}$. Following this process, we get the subsequence $\{z_{n,q}\}$ after n steps.
Since $z_{n,q} = \sum_{l=1}^{n} \delta_l^{(q)}\mathscr{y}_l$ with $\sum_{l=1}^{n} \mid \delta_l^{(q)} \mid = 1$ and $\delta_l^{(q)} \to \gamma_j$ as $q \to \infty$.
Let $z = \gamma_1\mathscr{y}_1 + \gamma_2\mathscr{y}_2 \ldots + \gamma_n\mathscr{y}_n$, thus we have

$$\lim_{n\to\infty} H(z_{n,q} - z, t_1) = U^*, \forall t_1 > 0.$$

Now ,for $\mathscr{k}_{11} > 0$, choose m such that $\frac{1}{q} < \mathscr{k}_{11}$, we have

$$H(z_{n,q}, \mathscr{k}_{11}) = H\left(z_{n,q} + 0, \frac{1}{q} + \mathscr{k}_{11} - \frac{1}{q}\right)$$

$$\supseteq H\left(z_{n,m}, \frac{1}{q}\right) \cap H\left(0, \mathscr{k}_{11} - \frac{1}{q}\right)$$

$\supseteq U^* \setminus \left\{\frac{1}{q}\right\} \cap H\left(0, \mathscr{k}_{11} - \frac{1}{q}\right)$.

That is

$$\lim_{n\to\infty} H(z_{n,q}, k_{11}) \supseteq U^*$$

$\Rightarrow \lim_{n\to\infty} H(z_{n,q}, k_{11}) = U^*$.

Now,

$$H(z.2k_{11}) = H(z - z_{n,q} + z_{n,q}, k_{11} + k_{11}) \supseteq H(z - z_{n,q}, k_{11}) \cap H(z_{n,q}, k_{11})$$
$$\supseteq \lim_{n\to\infty} H(z - z_{n,q}, k_{11}) \cap \lim_{n\to\infty} H(z_{n,q}, k_{11})$$

$\supseteq U^* \cap U^*$

$\Rightarrow H(z, 2k_{11}) = U^*$,

since k_{11} is random, by HFN, consequently $z = 0$. Once more, considering that $\sum_{l=1}^{n} |\, \delta_l^{(q)} \,| = 1$ and since the set of vectors $\{y_1, y_2 \dots y_n\}$ is linearly independent, so

$$z = \gamma_1 y_{11} + \gamma_2 y_{12} \dots + \gamma_n y_{1n} \neq 0.$$

As a result, the lemma is proved and we reach a contradiction.

Theorem 3.8:

At (1,1) the continuity of the underlying t-norm * makes all finite dimensional Hesitant fuzzy normed linear spaces (Y, H) complete.

Proof:

Consider a Hesitant fuzzy normed linear space (Y,H), and $\dim Y = l$.

Consider $\{y_n\}$ to be a Cauchy sequence in Y and $\{b_1, b_2 \dots b_l\}$ to be a basis for Y.

Let $y_n = \delta_1^{(n)} b_1 + \delta_2^{(n)} b_2 + \dots + \delta_l^{(n)} b_l$ where $\delta_1^{(n)}, \delta_2^{(n)}, \dots \delta_l^{(n)}$ are suitable scalars.

So,

$$\lim_{q,n\to\infty} H(y_q - y_n, t_1) = U^*, \forall\, t_1 > 0. \quad (2)$$

Now, from the Lemma 3.7, consequently, there is $a > 0$ and $S^* \in P[0,1]$, so that

$H\left(\sum_{k=1}^{l} (\delta_k^{(q)} - \delta_k^{(n)}) b_k, c \sum_{k=1}^{l} |\, \delta_k^{(q)} - \delta_k^{(n)} \,|\right) \subset U^* - S^*$. (3)

Again, for $U^* \supset S^* \supset \emptyset^*$ from Equation (2), consequently, there exists an integer that is positive

p_0, so

$H\left(\sum_{k=1}^{l} (\delta_k^{(q)} - \delta_k^{(n)}) b_k, t_1\right) \supset U^* - S^*, \forall\, q, n \geq p_0$. (4)

Now, from Equation (3) and Equation (4) we have, $H\left(\sum_{k=1}^{l} (\delta_k^{(q)} - \delta_k^{(n)}) b_k, t_1\right) \supset U^* - S^*$

$\supset H\left(\sum_{k=1}^{l} (\delta_k^{(q)} - \delta_k^{(n)}) b_k, c \sum_{k=1}^{l} |\, \delta_k^{(q)} - \delta_k^{(n)} \,|\right), \forall\, q, n \geq p_0$.

$\Rightarrow c \sum_{k=1}^{l} |\, \delta_k^{(q)} - \delta_k^{(n)} \,| < t_1, \forall\, q, n \geq p_0$.

$\Rightarrow \sum_{k=1}^{l} |\, \delta_k^{(q)} - \delta_k^{(n)} \,| < \frac{t_1}{c}, \forall q, n \geq p_0$.

$\Rightarrow |\, \delta_k^{(q)} - \delta_k^{(n)} \,| < \frac{t_1}{c}, \forall\, q, n \geq p_0$ and $k = 1,2,3 \dots l$.

Since, $t_1 > 0$ is arbitrary ,from above we have $\lim_{m,n\to\infty} |\, \delta_k^{(q)} - \delta_k^{(n)} \,| = 0$ for $k = 1,2 \dots l$

$\Rightarrow \{\delta_k^{(n)}\}$ is a cauchy scalar sequence for every $k = 1,2 \dots l$.

So, each sequence $\{\delta_k^{(n)}\}$ converges, and $y = \sum_{k=1}^{l} \delta_k b_k$,clearly $y \in Y$,

Now, $\forall\ t_1 > 0$,

$$H(y_n - y,\ t_1) = H\left(\sum_{k=1}^{l} (\delta_k^{(n)} b_k - \sum_{k=1}^{l} \delta_k b_k,\ t_1\right)$$

$= H\left(\sum_{k=1}^{l} (\delta_k^{(n)} - \delta_k) b_k,\ t_1\right)$.

That is,

$H(y_n - y,\ t_1) \supseteq H\left(b_1, \frac{t_1}{l|\delta_1^{(n)} - \delta_1|}\right) \cap H\left(b_2, \frac{t_1}{l|\delta_2^{(n)} - \delta_2|}\right) \cap \dots H\left(b_l, \frac{t_1}{l|\delta_l^{(n)} - \delta_l|}\right)$. (5)

When $n \to \infty$ then $\frac{t_1}{l|\delta_k{}^{(n)}-\delta_k|} \to \infty$, since $\delta_k{}^{(n)} \to \delta_k$ for $k = 1,2, \dots l$ and $t_1 > 0$.
From Equation (5) we obtain, utilizing the t-norm's continuity at (1, 1),
$\lim_{n\to\infty} H(y_n - y,\ t_1) \supseteq U^* \cap U^* \cap \dots \cap U^* = U^*,\ \forall\ t_1 > 0.$
$\Rightarrow \lim_{n\to\infty} H(y_n - y,\ t_1) = U^*, \forall\ t_1 > 0$
$\Rightarrow y_n \to y \ \Rightarrow Y$ is complete.

Definition 3.9:
We define Hesitant fuzzy bounded in Hesitant fuzzy normed linear space from the Definition 4.3 in [3].
Let $A \subset Y$ and (Y, H) be a Hesitant fuzzy normed linear space. If for every $U^*, \emptyset^* \subset S^* \subset U^*$, there exists $t_1 > 0$ such that $H(y, t) \supset U^* - S^*$, $\forall\ t_1 > 0$. Then A is considered hesitant fuzzy bounded.

Theorem 3.10:
In the finite dimensional Hesitant fuzzy normed linear space (Y,H), Assume that at (1,1) the underlying t-norm * is continuous. A subset A must be bounded and closed in order to be considered compact.

Proof :
The first assumption is that A is compact. We must demonstrate that A is bounded and closed. Assume that y. A sequence $\{y_n\}$ in A then exists such that $\lim_{n\to\infty} y_n = y$
because $\boldsymbol{A}$ is compact.
A subsequence of $\{y_n\}$, $\{y_{n_l}\}$ exists and converges in A to a point. Since $\{y_n\} \to y$, once more,
$\{y_{n_l}\} \to y$ and $\boldsymbol{y} \in$ A. Thus, A is closed.
Assuming, if at all feasible, that A is unbounded, there exists $S^* = S_0^*,\ \emptyset^* \subset S_0^* \subset U^*$,
In this way, for every positive integer j, there exists $y_j \in A$, such that $H(y_j, j) \subset U^* - S_0^*$.
Since A is compact, there exists a subsequence $\{\boldsymbol{y}_{\mathbf{n_l}}\}$ of $\{\boldsymbol{y_n}\}$ converging to some element $y \in A$. Thus

$$\lim_{l\to\infty} H(y_{n_l} - y, t_1) = U^* \ , \forall t_1 > 0.$$

Also,

$$H(y_{n_l}, n_l) \subseteq U^* - S_0^*.$$

Now,

$$U^* - S_0^* \supseteq H(y_{n_l}, n_l) = H(y_{n_l} - y + y, n_l - t_1 + t_1)\ , where\ t_1 > 0.$$

$\Rightarrow U^* - S_0^* \supseteq H(y_{n_l} - y, t_1) \cap H(y, n_l - t_1)$
$\Rightarrow U^* - S_0^* \supseteq \lim_{l\to\infty} H(y_{n_l} - y, t_1) \cap \lim_{l\to\infty} H(y, n_l - t_1)$
$\Rightarrow U^* - S_0^* \supseteq U^* \cap U^* \Rightarrow S_0^* \subseteq \emptyset^*$.
Because of this contradiction, A is bounded. Assume, on the other hand, that A is bounded and closed, then demonstrate that A is compact. Consider, $\dim Y = m$, where Y's basis is $\{b_{11}, b_{12} \dots b_{1m}\}$. Select a sequence in A called $\{y_l\}$ and assume
$y_l = \delta_1{}^{(l)} b_{11} + \delta_2{}^{(l)} b_{12} + \cdots + \delta_m{}^{(l)} b1_m\ , where\ \delta_1{}^{(l)}, \delta_2{}^{(l)}, \dots \delta_m{}^{(l)}$ are scalars.
Now, from Lemma 3.12, there exists $d > 0$, and $S_2^* \in P[0,1]$, such that ,
$H\left(\sum_{j=1}^{m} \delta_j{}^{(l)} b_{1j}\ \ , d \sum_{j=1}^{m} |\ \delta_j{}^{(l)}\ |\right) \subset U^* - S_2^*$. (6)
Again, since A is bounded, for $S_2^* \in P[0,1]$ such that $H(y, t_1) \supset U^* - S_2^*$, $\forall y \in A$.
So,

$$H\left(\sum_{j=1}^{m}\delta_j{}^{(l)}b_{1j}\ ,t_1\right)\supset\ U^*-S_2^*. \tag{7}$$

From Equation (6) and Equation (7) we obtain,

$$H\left(\sum_{j=1}^{m}\delta_j{}^{(l)}b_{1j}\ \ ,d\sum_{j=1}^{m}\mid\delta_j{}^{(l)}\mid\right)\subset\ U^*-S_2^*\subset H\left(\sum_{j=1}^{m}\delta_j{}^{(l)}b_{1j}\ ,t_1\right).$$

$\Rightarrow H\left(\sum_{j=1}^{m}\delta_j{}^{(l)}b_{1j}\ \ ,d\sum_{j=1}^{m}\mid\delta_j{}^{(l)}\mid\right)\ \subset\ H\left(\sum_{j=1}^{m}\delta_j{}^{(l)}b_{1j}\ ,t_1\right)$

$\Rightarrow d\sum_{j=1}^{m}\mid\delta_j{}^{(l)}\mid<t_1$ (because $H(y,\bullet)$ is non-decreasing)

$\Rightarrow\mid\delta_j{}^{(l)}\mid\leq\frac{t_1}{d}$ for $l,j=1,2\ldots m$.

So, each sequence $\{\delta_j{}^{(l)}\}$ $(j=1,2\ldots m)$ is bounded.

The Bolzano-Weierstrass theorem can be applied repeatedly to show that every sequence $\{\delta_j{}^{(l)}\}$ has a convergent subsequence, say $\left\{\delta_j^{(l_j)}\right\}$, $\forall\, j=1,2\ldots m$.

Let $\mathcal{y}_{l_1}=\delta_1^{(l_1)}b_{11}+\delta_2^{(l_1)}b_{12}+\cdots+\delta_m^{(l_1)}b1_m$ and $\{\delta_1^{(l_1)}\},\{\delta_2^{(l_2)}\},\ldots\{\delta_m^{(l_m)}\}$ are all convergent.

Let $\delta_j=\lim_{l\to\infty}\delta_j^{(l_p)}$ $,j=1,2\ldots m$ and $\mathcal{y}=\delta_1b_{11}+\delta_2b_{12}+\cdots+\delta_mb_{1m}$ now for $t_1>0$, we have,

$$H\left(\mathcal{y}_{l_p}-\mathcal{y},t_1\right)=H\left(\sum_{j=1}^{m}(\,\delta_j^{(l_p)}-\delta_j)b_{11},t_1)\right)$$

$\supseteq H(\,b_{11},\frac{t_1}{m|\delta_1^{(l_p)}-\delta_1|})\cap H(\,b_{12},\frac{t_1}{m|\delta_2^{(l_p)}-\delta_2|})\cap\ldots\ H(\,b_{1l},\frac{t_1}{m|\delta_m^{(l_p)}-\delta_l|})$

$\Rightarrow\lim_{p\to\infty}H\left(\mathcal{y}_{l_p}-\mathcal{y},t_1\right)\supseteq\lim_{p\to\infty}H(\,b_{11},\frac{t_1}{m|\delta_1^{(l_p)}-\delta_1|})\cap\ldots.\cap\lim_{p\to\infty}H(b_{1l},\frac{t_1}{m|\delta_m^{(l_p)}-\delta_l|})$

$$\Rightarrow\lim_{p\to\infty}H\left(\mathcal{y}_{l_p}-\mathcal{y},t_1\right)\supseteq U^*\cap U^*\ldots\cap U^*,\qquad(\delta_j^{(l_p)})\to\delta_j\ _{as}\ p\to\infty$$

(using the t-norm $*$ continuity at (1,1)

$$\Rightarrow\lim_{p\to\infty}H\left(\mathcal{y}_{l_p}-\mathcal{y},t_1\right)=U^*.$$

Since $t_1>0$ is arbitrary, it follows that $\lim_{p\to\infty}\mathcal{y}_{l_p}=U^*$

That is, $\{\mathcal{y}_{l_p}\}$ is a subsequence that convergent to $\{\mathcal{y}_{l_p}\}$ belongs to $\{\mathcal{y}_l\}$ and converges to $\boldsymbol{\mathcal{y}}$ as a convergent subsequence. Since $\{\mathcal{y}_l\}$ is a sequence in A and A is closed, y must be a member of A. Every sequence in A therefore converges to an element of A and possesses a convergent subsequence. Thus, A is compact.

4. Riesz Lemma

In this section Riesz lemma is extended to Hesitant Fuzzy Normed Linear Space from the Lemma 5.1 in [3] from Fuzzy normed linear space.

In the Hesitant fuzzy normed linear space (Y, H), let X and Z be subspaces. Let X be a closed and proper subset of Z. After that, for each real number $\tau\in(0,1)$, there exists $z\in Z$. Such that $H(z,1)\neq\emptyset^*$ and $H(z-x,\tau)=\emptyset^*,\ \forall x\in X$.

Proof:

Since X is a proper subset of Z,there exists $u\in Z-X$.

Denote $e=\wedge_{x\in X}\wedge\{t_1>0\colon H(u-x,t_1)\neq\emptyset^*\}$

We claim that $e>0$, if $e=0$, that is

$$\wedge_{x\in X} \wedge \{t_1 > 0: H(u - x, t_1) = 0. \quad (8)$$

This implies for a given $\mu > 0$, there exists $x(\mu) \in X$ such that
$\wedge \{t_1 > 0: H(u - \mathrm{x}, \mathrm{t}_1) \neq \emptyset^*\} < \mu$. Obtaining $H(\mathrm{u} - \mathrm{x}, \mu) \neq \emptyset^*$.
Choose $S^* \in P[0,1]$ such that, $H(\mathrm{u} - \mathrm{x}, \mu) \supset \mathrm{U}^* - S^*$, that is , $\mathrm{x} \in \mathrm{B}(\mathrm{u}, \mathrm{U}^* - \mathrm{S}^*, \mu)$.
Given that $\mu > 0$ is arbitrary, u must be in the closure of X. Given that X is closed, it follows that $u \in X$ which is contradictory, thus $e > 0$. Taking $\vartheta \in (0,1)$ now, thus $\frac{e}{\vartheta} > e$.
Consequently, for some $x_0 \in X$, we have

$$e \leq \{t_1 > 0: H(u - x_0, t_1) \neq \emptyset^*\} < l < \frac{e}{\vartheta}. \quad (9)$$

Let $= \frac{u-x_0}{l}$, now $H(z, 1) = H\left(\frac{u-x_0}{l}, 1\right)$. That is

$$H(z, 1) = H(u - x_0, l) \quad (10)$$

Now, $\wedge \{\mathrm{t}_1 > 0: \mathrm{H}(\mathrm{u} - \mathrm{x}_0, \mathrm{t}_1) \neq \emptyset^*\} < \mathrm{l}$ this implies $H(\mathrm{u} - \mathrm{x}_0, t_1) \neq \emptyset^*$
From Equation (9) ,we have $H(z, 1) \neq \emptyset^*$.
Now, for $\mathrm{x} \in X$, $\wedge \{\mathrm{t}_1 > 0: \mathrm{H}(\mathrm{z} - \mathrm{x}, \mathrm{t}_1) \neq \emptyset^*\} = \wedge \{\mathrm{t}_1 > 0: \mathrm{H}(\mathrm{u} - \mathrm{x}_0 - \mathrm{lx}, \mathrm{lt}_1) \neq \emptyset^*\}$
$= \frac{1}{l} \wedge \{t_1 > 0: H(u - x_0 - \mathrm{lx}, lt_1) \neq \emptyset^*\} \wedge \{\mathrm{t}_1 > 0: H(\mathrm{z} - \mathrm{x}, \mathrm{t}_1) \neq \emptyset^*\} \geq \frac{e}{l}$ (since $x_0 + lx \in X$).
$\Rightarrow \wedge \{\mathrm{t}_1 > 0: H(\mathrm{z} - \mathrm{x}, \mathrm{t}_1) \neq \emptyset^*\} > \vartheta$ through Equation (8).

$\Rightarrow H(\mathrm{z} - \mathrm{x}, \vartheta) \subseteq \emptyset^*$
$\Rightarrow H(\mathrm{z} - \mathrm{x}, \vartheta) = \emptyset^*, \forall x \in X$.

Theorem 4.1:
Given a hesitant normed linear space (Y, H) and $H(y, \bullet)$ $(x \neq 0)$, Y is finite dimensional if the set R={y: $H(y, 1) \neq \emptyset^*$} is compact.

Proof:
Assume $\dim Y = \infty$,if at all possible. Let $H(y_1, 1) \neq \emptyset^*$ for every $y_1 \in Y$.
Assume that the subspace of Y that y_1 generates is Y_1. Given that $\dim Y_1 = 1$ .This subset of Y is proper and closed. Consequently, by Riesz Lemma, there exists $y_2 \in Y$ such that
$H(y_2, 1) \neq \emptyset^*$ and $H\left(y_2 - y_1, \frac{1}{2}\right) = \emptyset^*$.
The elements y_1, y_2 create a proper closed two-dimensional subspace of Y.
Through the Lemma 3.7, there exists $y_3 \in Y$ with $H(y_3, 1) \neq \emptyset^*$ such that,
$H\left(y_3 - y_1, \frac{1}{2}\right) = \emptyset^*$, $H\left(y_3 - y_2, \frac{1}{2}\right) = \emptyset^*$.
We get a sequence $\{y_n\}$ of elements $y_n \in R$ by continuing in the same manner, such that
$H(y_n, 1) \neq \emptyset^*$ and $H\left(y_n - y_m, \frac{1}{2}\right) = \emptyset^*$, $(m \neq n)$. (11)
From Equation (11), consequently, neither the sequence $\{y_n\}$ nor any of its subsequences converge.
Since this defies the compactness of, $\dim Y$ is finite.

5. Conclusions

We considered tackling substantial issues when dealing with finite dimensions. Linear spaces with fuzzy norms, this study extended certain fundamental principles by significantly relaxing the t-norm's "min" constraint, and fuzzy norms in their general form can be utilized to extend the findings of Hesitant fuzzy normed linear spaces. In the broad t-norm context, it is assumed that there will be plenty of room to draw more ambiguous functional analysis results.

Conflict of Interest

The authors declare that they have no conflicts of interest.

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