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Third Order Sandwich Results for Analytic Multivalent Functions Defined by Darweesh-Atshan-Battor Operator

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Abstract
In the present work, by employing the differential subordination notion and Darweesh-Atshan-Battor operator, we investigate third-order of sandwich-type theorems for the p-valent of analytic functions.

Keywords: Analytic functions, Darweesh-Atshan-Battor operator, Differential subordination, Superordination, Sandwich theorems, Third-order.

نتائج ساندويتش من الدرجة الثالثة للوظائف التحليلية متعددة التكافؤ المحددة بواسطة مشغل درويش-عطشان-باتور

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الخلاصة
في العمل الحالي، من خلال استخدام فكرة التبعية التفاضلية ومشغل درويش–عطشان–باتور، قمنا بدراسة نظريات الدرجة الثالثة من نوع الساندويتش للدوال التحليلية متعددة التكافؤ.

1. Introduction

Antonino with Miller [1] expanded the scope for second-order differential subordinations, which were initially formulated by Mocanu and Miller [2], to encompass third-order differential subordinations. Approaches suggested through Miller with Antonino offer a possibility of acquiring intriguing novel findings. Furthermore, some authors have commenced their work in this specific series of investigation [3, 4]. This concept of expanding the pair theory of differential superordination [5] to third-order differential superordination was introduced within 2014 [6], also novel intriguing outcomes soon following [7, 8]. The next symbols and concepts serve as the fundamental framework in this study.

The family of analytic functions is denoted by $\mathcal{H}(U)$, when $U = \{z \in \mathbb{C} : |z| < 1\}$, and $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$, also $\partial U = \{z \in \mathbb{C} : |z| = 1\}$. Let n will be a positive integer as well a will be a complex number, the next major subfamils of $\mathcal{H}(U)$ are defined:

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U): f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U\}.$$

Such that $\mathcal{H}_0 = \mathcal{H}[0,1]$ and $\mathcal{H}_1 = \mathcal{H}[1,1]$.

Let $\mathcal{K} \subset \mathcal{H}(U)$ denoted the collection of functions that are analytic within U, and possess the normalised Taylor-Maclaurin series in the format:

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$$f(z) = z^p + \sum_{n=1+p}^{\infty} a_n z^n, \ (z \in U). \tag{1.1}$$

Assume the functions f and g are within $\mathcal{H}(U)$, the function f is said to be subordinant to g ($f \prec g$) if there is a Schwarz function $w \in \mathcal{H}(U)$, which is analytic within U with $w(0) = 0$, also $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$. When the function g is univalent within U, thus we obtain the next equivalence relationship [5]:
$f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ with $f(U) \subset g(U)$.

Definition 1.1. [9]: Let $f \in \mathcal{K}$ and $y_1, y_2, \dots, y_n \in \mathbb{N}$, the operator of Darweesh-Atshan-Battor is given by

$$T^m_{y_1+y_2+\cdots+y_n} f(z) = z^p + \sum_{n=1+p}^{\infty} \left(\frac{n+y_1+y_2+\cdots+y_n}{p+y_1+y_2+\cdots+y_n}\right)^m a_n z^n, \ (m \in N \cup \{0\}). \tag{1.2}$$

By simple calculation, we obtain

$$z\left(T^m_{y_1+y_2+\cdots+y_n} f(z)\right)' = (p + y_1 + y_2 + \cdots + y_n)\, T^{m+1}_{y_1+y_2+\cdots+y_n} f(z) - (y_1 + y_2 + \cdots + y_n)\, T^m_{y_1+y_2+\cdots+y_n} f(z). \tag{1.3}$$

The idea of third-order differential subordination is discussed of the study conducted by Juneja and Ponnusamy [10]. The recent works by some authors (for instance, [6, 8]). The second as well third-order differential subordination has garnered significant attention from authors in this field. (for instance, [3, 11]).
In this study, we examine a specific family of admissible functions involved to the differential operator and establish adequate criteria for the normalised analytic function known as the sandwich condition.

2-Preliminaries

The acquisition of the next definitions and lemmas is important to fulfil our outcomes.

Definition 2.1. [1]: Letting $\psi: \mathbb{C}^4 \times U \to \mathbb{C}$ with $h(z)$ be univalent within U. When $p(z)$ is analytic within U as well fulfils the third-order differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{2.1}$$

therefore, the function $p(z)$ is referred to as a solution of the differential subordination (2.1). Additionally, a specified the univalent function $q(z)$ is referred to as a dominant of the solution of the differential subordination (2.1), or more clearly dominant when $p(z) \prec q(z)$ for each $p(z)$ fulfilling (2.1). A dominant $\tilde{q}(z)$ that fulfils $\tilde{q}(z) \prec q(z)$ for each dominants $q(z)$ of (2.1) is claimed to be the best dominant.

Definition 2.2. [12]**:** Suppose that $\mathbb{Q}$ the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, when $\overline{U} = U \cup \{z \in \partial U\}$, with

$$E(f) = \left\{\zeta \in \partial U: \lim_{z \to \zeta} f(z) = \infty\right\}, \tag{2.2}$$

where $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. Additionally, let the subclass of $\mathbb{Q}$ for which $f(0) = a$ denoted as $\mathbb{Q}(a)$, and $\mathbb{Q}(0) = \mathbb{Q}_0$, $\mathbb{Q}(1) = \mathbb{Q}_1 = \{f \in \mathbb{Q}: f(0) = 1\}$.

Definition 2.3 [1]: Letting Ω denoted a set within $\mathbb{C}$, $q \in \mathbb{Q}$ also $n \in N \backslash \{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ comprises the functions $\psi: \mathbb{C}^4 \times U \to \mathbb{C}$ that fulfil the next admissibility condition:
$\psi(r, s, t, u; z) \notin \Omega$,
when
$r = q(\zeta), \ s = k\zeta q'(\zeta), \ \mathcal{R}e\left(\frac{t}{s} + 1\right) \geq k\mathcal{R}e\left(1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right),$

$\mathcal{R}e\left(\frac{u}{s}\right) \geq k\mathcal{R}e\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right)$, when $z \in U, \zeta \in \partial\mathrm{U} \setminus \mathrm{E(q)}$ and $k \geq n$.

Lemma 2.4. [1]: Letting $p \in \mathcal{H}[a,n]$, where $n \geq 2$. Additionally, assume that $q \in \mathbb{Q}(a)$, where fulfils the next condition:
$\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq k,$
such that $z \in U, \zeta \in \partial\mathrm{U} \setminus \mathrm{E(q)}$ with $k \geq n$. When Ω be a set within $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and $\psi(\mathrm{p(z)}, \mathrm{zp'(z)}, \mathrm{z^2p''(z)}, \mathrm{z^3p'''(z)}; \mathrm{z}) \in \Omega$, thus $p(z) \prec q(z), (z \in U)$.

Definition 2.5. [2]: Letting $\psi: \mathbb{C}^4 \times \mathrm{U} \to \mathbb{C}$ with h(z) denoted analytic within U. When the function p(z) with $\psi(\mathrm{p(z)}, \mathrm{zp'(z)}, \mathrm{z^2p''(z)}, \mathrm{z^3p'''(z)}; \mathrm{z})$ are univalent within U, where fulfils the third-order differential superordnation:

$$h(z) \prec \psi(\mathrm{p(z)}, \mathrm{zp'(z)}, \mathrm{z^2p''(z)}, \mathrm{z^3p'''(z)}; \mathrm{z}), \quad (2.3)$$

therefore, the function p(z) is referred to as a solution of the differential superordination (2.3). The analytic function q(z) is referred to as a subordinant of the solution of the differential superordination (2.3), or more clearly subordinant when $\mathrm{q(z)} \prec \mathrm{p(z)}$ for each p(z) fulfilling (2.3). A univalent subordinant $\tilde{\mathrm{q}}(\mathrm{z})$ that fulfils $\mathrm{q(z)} \prec \tilde{\mathrm{q}}(\mathrm{z})$ for each subordinant*s* q(z) of (2.3) is claimed to be the best subordnant. We observe that the best dominant as well best subordinant are unique up to rotation of U.

Definition 2.6. [2]: Letting Ω denoted a set within $\mathbb{C}$, $q \in \mathcal{H}[a,n]$ with $q'(z) \neq 0$ as well $n \in N\backslash\{1\}$. The class of admissible functions $\Psi'_n[\Omega, q]$ comprises the functions $\psi: \mathbb{C}^4 \times \overline{\mathrm{U}} \to \mathbb{C}$ that fulfil the next admissibility condition:

$$\psi(r, s, t, u; z) \in \Omega,$$

when
$r = q(z),\ s = \frac{zq'(z)}{m},\ \mathcal{R}e\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\mathcal{R}e\left(1 + \frac{zq''(z)}{q'(z)}\right),$
$\mathcal{R}e\left(\frac{u}{s}\right) \leq \frac{1}{m^2}\mathcal{R}e\left(\frac{z^2 q'''(z)}{q'(z)}\right),$
where $z \in U, \zeta \in \partial\mathrm{U} \setminus \mathrm{E(q)}$, and $m \geq n \geq 2$.

Lemma 2.7. [2]: Let $\psi \in \Psi_n[\Omega, q]$. If

$$\psi\ (\mathrm{p(z)}, \mathrm{zp'(z)}, \mathrm{z^2p''(z)}, \mathrm{z^3p'''(z)}; \mathrm{z}) \in \Omega,$$

is univalent within U, $p \in \mathbb{Q}(a)$ with $q \in \mathcal{H}[a,n]$ fulfil the next condition:
$\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq m$, when $z \in U, \zeta \in \partial\mathrm{U}$ with $m \geq n \geq 2$, thus
$\Omega \subset \{\psi\ (\mathrm{p(z)}, \mathrm{zp'(z)}, \mathrm{z^2p''(z)}, \mathrm{z^3p'''(z)}; \mathrm{z})\}$,
we get
$q(z) \prec p(z), (z \in U)$.

An essential method within the research of third-order differential superordination involves utilising a fundamental notion of admissible function as presented within reference [13]. Utilising that approach, notable outcomes were attained by several authors investigating suitable classes of admissible functions including generalised Bessel functions [8], some operators [7, 14], the Srivastave-Attiya operator [15], the linear operators [16, 17], the meromorphic functions [18] or Mittag-Leffler functions [19]. The two pair hypotheses of third-order differential subordination with superordination are developing well. Very recent outcomes acquired utilising this approach can be found in papers such as [20, 21, 22]. A novel approach for third-order differential subordination has been obtained within modern study taking another essential notion within the theory of differential subordination, that is the best dominant of the differential subordination. From [23, 24], approaches to determine the dominant of a third-order differential subordination's best dominant will be provided.

This study centers on the application of the dual theory of third-order differential superordination within the same academic domain. The findings obtained here provide an alternative to the approach that takes into account the concept of the class of admissible functions. The aim of this work is to discover novel findings regarding the establishment of a subordinate for specific third-order differential superordinations. Additionally, we aim to identify the best subordinate for third-order differential superordinations within situations where they support such functions.

3- Third-order differential subordination results

In this context, we present a set of differential subordination outcomes utilising the Darweesh-Atshan-Battor operator.

Definition 3.1. Letting Ω denoted a set within $\mathbb{C}$ as well $q \in \mathbb{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\mathcal{M}_{j}[\Omega, q]$ comprises the functions $\psi: \mathbb{C}^4 \times \mathrm{U} \to \mathbb{C}$, that fulfil the subsequent admissibility conditions:

$$\psi(a, b, c, d; z) \notin \Omega,$$

when

$$a = q(\zeta), \qquad b = \frac{\zeta k q'(\zeta) + (1 + y_1 + y_2 + \cdots + y_n) q(\zeta)}{(p + y_1 + y_2 + \cdots + y_n)},$$

$$\mathcal{R}e\left(\frac{(p + y_1 + \cdots + y_n)^2 c - (1 + y_1 + y_2 + \cdots + y_n)[2b(p + y_1 + y_2 + \cdots + y_n) - (y_1 + y_2 + \cdots + y_n)a]}{(p + y_1 + y_2 + \cdots + y_n)b - (y_1 + y_2 + \cdots + y_n)a}\right) \geq k\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\mathcal{R}e\left(\frac{(p + y_1 + y_2 + \cdots + y_n)^2[(p + y_1 + y_2 + \cdots + y_n)d - 3c(1 + y_1 + y_2 + \cdots + y_n)]}{(p + y_1 + y_2 + \cdots + y_n)b - (y_1 + y_2 + \cdots + y_n)a} + \frac{(p + y_1 + y_2 + \cdots + y_n)b[2 + 3(y_1 + y_2 + \cdots + y_n)(2 + y_1 + y_2 + \cdots + y_n)]}{(p + y_1 + y_2 + \cdots + y_n)b - (y_1 + y_2 + \cdots + y_n)a} - \frac{(y_1 + y_2 + \cdots + y_n)a[2 + (y_1 + y_2 + \cdots + y_n)(3 + y_1 + y_2 + \cdots + y_n)]}{(p + y_1 + y_2 + \cdots + y_n)b - (y_1 + y_2 + \cdots + y_n)a}\right) \geq k^2\mathcal{R}e\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

when $z \in U, \zeta \in \partial\mathrm{U} \setminus \mathrm{E(q)}$, with $k \geq 2$.

Theorem 3.2. Letting $\psi \in \mathcal{M}_{j}[\Omega, q]$. When the functions $f \in \mathcal{K}$ with $q \in \mathbb{Q}_0 \cap \mathcal{H}_0$ fulfil the next conditions:

$$\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)}{q'(z)}\right| \leq k, \tag{3.1}$$

and

$$\left\{\psi\left(\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z); z\right): z \in U\right\} \subset \Omega, \tag{3.2}$$

then

$\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z) \prec q(z), (z \in U)$.

Proof: Suppose that $p(z)$ be analytic function within U denoted by

$$p(z) = \mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z). \tag{3.3}$$

From Equations (1.3) and (3.3), we have

$$\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z) = \frac{zp'(z) + (y_1 + y_2 + \cdots + y_n)p(z)}{(p + y_1 + y_2 + \cdots + y_n)}. \tag{3.4}$$

By similar argument, yields

$$\mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z) = \frac{z^2 p''(z) + (1 + 2(y_1 + y_2 + \cdots + y_n))zp'(z) + (y_1 + y_2 + \cdots + y_n)^2 p(z)}{(p + y_1 + y_2 + \cdots + y_n)^2}, \tag{3.5}$$

and

$$\mathrm{T}^{\mathrm{m}+3}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n}f(z)=\frac{(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^3p(z)}{(p+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^3}$$

$$+\frac{z^3p'''(z)+3(1+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)\,z^2p''(z)+[1+3(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)(1+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)]zp'(z)}{(p+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^3} \quad (3.6)$$

defined the transformation form $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u)=r,\qquad b=\frac{s+(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)r}{(p+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)},$$

$$c(r,s,t,u)=\frac{t+\big(1+2(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)\big)s+(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^2r}{(p+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^2} \quad (3.7)$$

and

$$d(r,s,t,u)=\frac{u+3(1+\mathrm{y}_1+\cdots+\mathrm{y}_n)\,t+[1+3(\mathrm{y}_1+\cdots+\mathrm{y}_n)(1+\mathrm{y}_1+\cdots+\mathrm{y}_n)]s+(\mathrm{y}_1+\cdots+\mathrm{y}_n)^3r}{(p+\mathrm{y}_1+\cdots+\mathrm{y}_n)^3}. \quad (3.8)$$

Let

$$\varphi(r,s,t,u)=\psi(a,b,c,d;z)=$$

$$\psi\left(\begin{array}{c} r,\frac{s+(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)r}{(p+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)},\frac{t+\big(1+2(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)\big)s+(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^2r}{(p+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^2}, \\ \frac{u+3(1+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)\,t+[1+3(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)(1+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)]s+(\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^3r}{(p+\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n)^3}\end{array}\right) \quad (3.9)$$

The proof will utilise the Lemma 2.4. Applying (3.2) through (3.5), and by (3.8), we acquire

$$\varphi\,(\mathrm{p}(z),z\mathrm{p}'(z),z^2\mathrm{p}''(z),z^3\mathrm{p}'''(z);z)=$$
$$\psi\big(\mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n}f(z),\mathrm{T}^{\mathrm{m}+1}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n}f(z),\mathrm{T}^{\mathrm{m}+2}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n}f(z),\mathrm{T}^{\mathrm{m}+3}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n}f(z);z\big). \quad (3.10)$$

Hence, (3.2) leads to

$$\varphi\,(\mathrm{p}(z),z\mathrm{p}'(z),z^2\mathrm{p}''(z),z^3\mathrm{p}'''(z);z)\in\Omega.$$

We observed this

$$1+\frac{t}{s}=\frac{(p+y_1+y_2+\cdots+y_n)^2c-(y_1+y_2+\cdots+y_n)[2b(p+y_1+y_2+\cdots+y_n)-(y_1+y_2+\cdots+y_n)a]}{(p+y_1+y_2+\cdots+y_n)b-(y_1+y_2+\cdots+y_n)a},$$

and

$$\frac{u}{s}=\frac{(p+y_1+y_2+\cdots+y_n)^2[(y_1+y_2+\cdots+y_n)d-3c(1+y_1+y_2+\cdots+y_n)]+(p+y_1+y_2+\cdots+y_n)b[2+3(y_1+y_2+\cdots+y_n)(2+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b-(y_1+y_2+\cdots+y_n)a}-\frac{(y_1+y_2+\cdots+y_n)a[2+(y_1+y_2+\cdots+y_n)(3+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b-(y_1+y_2+\cdots+y_n)a}.$$

Thus, the admissibility conditions of $\psi\in\mathcal{M}_{\jmath}[\Omega,q]$ on Definition 3.1 is equivalent to admissiblity conditions of $\varphi\in\Psi_2[\Omega,q]$ that given on Definition 2.3 where $n=2$. Thus, by utilising Equation (3.1) with Lemma 2.4, we obtain

$$\mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n}f(z)\prec q(z).$$

Then the proof of Theorem 3.2 has been completed.
The subsequent outcome is a continuation of Theorem 3.2 for the situation when the conduct of $q(z)$ on ∂U is unknown.

Corollary 3.3. Letting $\Omega\subset\mathbb{C}$, the function q is univalent within U, and $q(0)=1$. Assume $\psi\in\mathcal{M}_{\jmath}[\Omega,q_\rho]$ where $\rho\in(0,1)$, when $q_\rho(z)=q(z\rho)$. When the function $f\in\mathcal{K}$ with q_ρ fulfil the next conditions:

$\mathcal{Re}\left(\frac{\zeta q''_{\rho}(\zeta)}{q'_{\rho}(\zeta)}\right) \geq 0, \left|\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{q'_{\rho}(z)}\right| \leq k, (z \in U, \zeta \in \partial U \setminus E(q_{\rho}),\ k \geq 2)$
and
$\psi\left(T^{m}_{y_1+y_2+\cdots+y_n}f(z), T^{m+1}_{y_1+y_2+\cdots+y_n}f(z), T^{m+2}_{y_1+y_2+\cdots+y_n}f(z), T^{m+3}_{y_1+y_2+\cdots+y_n}f(z); z\right) \in \Omega,$
then

$$T^{m}_{y_1+y_2+\cdots+y_n}f(z) \prec q_{\rho}(z), \quad (z \in U).$$

Proof: By utilising Theorem 3.2, we obtain

$$T^{m}_{y_1+y_2+\cdots+y_n}f(z) \prec q_{\rho}(z), \quad (z \in U).$$

The conclusion established by outcome 3.3 is now obtained from the subsequent subordination property.
$q_{\rho}(z) \prec q(z), \ (z \in U).$
This complete the proof of Corollary 3.3.
When $\Omega \neq \mathbb{C}$ be a simple connected domain, then $\Omega = h(U)$ for certain conformal mappings $h(z)$ of U onto Ω. In this situation, the class $\mathcal{M}_{j}[h(U), q]$ is expressed $\mathcal{M}_{j}[h, q]$. This follows is derived immediately from of Theorem 3.2.

Theorem 3.4. Letting $\psi \in \mathcal{M}_{j}[h, q]$. When the function $f \in \mathcal{K}$ with $q \in \mathbb{Q}_0 \cap \mathcal{H}_0$ fulfil the next conditions:

$$\mathcal{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{q'(z)}\right| \leq k, \tag{3.11}$$

and
$\psi\left(T^{m}_{y_1+y_2+\cdots+y_n}f(z), T^{m+1}_{y_1+y_2+\cdots+y_n}f(z), T^{m+2}_{y_1+y_2+\cdots+y_n}f(z), T^{m+3}_{y_1+y_2+\cdots+y_n}f(z); z\right) \prec h(z).$ (3.12)
Then
$T^{m}_{y_1+y_2+\cdots+y_n}f(z) \prec q(z), \quad (z \in U).$
The following corollary is an immediate consequence of outcome 3.3.

Corollary 3.5. Letting $\Omega \subset \mathbb{C}$, and the function q be univalent within U as well $q(0) = 1$. Assume that $\psi \in \mathcal{M}_{j}[h, q_{\rho}]$ where $\rho \in (0,1)$, when $q_{\rho}(z) = q(z\rho)$. When the function $f \in \mathcal{K}$ with q_{ρ} fulfil the next conditions:
$\mathcal{Re}\left(\frac{\zeta q''_{\rho}(\zeta)}{q'_{\rho}(\zeta)}\right) \geq 0, \left|\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{q'_{\rho}(z)}\right| \leq k, (z \in U, \zeta \in \partial U \setminus E(q_{\rho}),\ k \geq 2)$
and
$\psi\left(T^{m}_{y_1+y_2+\cdots+y_n}f(z), T^{m+1}_{y_1+y_2+\cdots+y_n}f(z), T^{m+2}_{y_1+y_2+\cdots+y_n}f(z), T^{m+3}_{y_1+y_2+\cdots+y_n}f(z); z\right) \prec h(z),$
then

$$T^{m}_{y_1+y_2+\cdots+y_n}f(z) \prec q_{\rho}(z), \quad (z \in U).$$

The subsequent outcome produces to best dominant of differential subordination (3.12).

Theorem 3.6. Letting h be univalent function within U. Also Assume that $\psi: \mathbb{C}^4 \times U \to \mathbb{C}$ with φ be given in (3.9). Consider that Subsequent differential equation:

$$\varphi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z), \tag{3.13}$$

has a solution $q(z)$ and $q(0) = 1$, which fulfil condition (3.1). When $f \in \mathcal{K}$ fulfils the condition (3.12), and
$\psi\left(T^{m}_{y_1+y_2+\cdots+y_n}f(z), T^{m+1}_{y_1+y_2+\cdots+y_n}f(z), T^{m+2}_{y_1+y_2+\cdots+y_n}f(z), T^{m+3}_{y_1+y_2+\cdots+y_n}f(z); z\right),$
is analytic within U, thus

$$T^{m}_{y_1+y_2+\cdots+y_n}f(z) \prec q(z), \quad (z \in U),$$

and q(z) is the best dominant.

Proof. By Theorem 3.2, it is clear that q is a dominant of (3.12). Since q fulfils (3.13), it is also a solution of (3.12). Consequently, q becomes dominated by each dominant. Therefore, q is the best dominant. So, the proof of Theorem 3.6 has been completed.
Considering Definition 3.1, within special case where $q(z) = Mz, M > 0$, the class of admissible functions $\mathcal{M}_{j}[\Omega, q(z)]$, given as $\mathcal{M}_{j}[\Omega, M]$, is represented as following.

Definition 3.7. Letting Ω denoted a set within $\mathbb{C}$ with $M > 0$. The class $\mathcal{M}_{j}[\Omega, M]$ of admissible functions comprises the functions $\psi: \mathbb{C}^4 \times \mathrm{U} \to \mathbb{C}$ such that

$$\psi\left(\begin{array}{c} Me^{i\vartheta}, \frac{(k+(y_1+y_2+\cdots+y_n))Me^{i\vartheta}}{(p+y_1+y_2+\cdots+y_n)}, \frac{L+[(1+2(y_1+y_2+\cdots+y_n))k+(y_1+y_2+\cdots+y_n)^2]Me^{i\vartheta}}{(p+y_1+y_2+\cdots+y_n)^2} \\ \frac{N+3(1+y_1+y_2+\cdots+y_n)\,L+[(1+3(y_1+y_2+\cdots+y_n)(1+y_1+y_2+\cdots+y_n))k+(y_1+y_2+\cdots+y_n)^3]Me^{i\vartheta}}{(p+y_1+y_2+\cdots+y_n)^3}; z \end{array}\right) \notin \Omega, \quad (3.14)$$

where $z \in U$,
$\mathcal{Re}\left(Le^{-i\vartheta}\right) \geq (k-1)Mk$, also $\mathcal{Re}\left(Ne^{-i\vartheta}\right) \geq 0, k \geq 2$, for all $\vartheta \in \mathbb{R}$.

Corollary 3.8. Letting $\psi \in \mathcal{M}_{j}[\Omega, M]$. When the function $f \in \mathcal{K}$ fulfils the following conditions:
$\left|\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)\right| \leq Mk, \ (z \in U; k \geq 2; M > 0)$,
and

$$\psi\left(\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z); z\right) \in \Omega,$$

then

$$\left|\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)\right| < M.$$

If $\Omega = q(U) = \{w: |w| < M\}$, the class $\mathcal{M}_{j}[\Omega, M]$ is simple denoted as $\mathcal{M}_{j}[M]$. A consequence (3.8) will be changed in the next format.

Corollary 3.9. Letting $\psi \in \mathcal{M}_{j}[M]$. When the function $f \in \mathcal{K}$ fulfils the next conditions:
$\left|\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)\right| \leq Mk, \ (z \in U; k \geq 2; M > 0)$,
and

$$\left|\psi\left(\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z); z\right)\right| < M,$$

then

$$\left|\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)\right| < M.$$

Corollary 3.10. Letting $k \geq 2$, and $M > 0$. When the function $f \in \mathcal{K}$ fulfils the next condition:
$\left|\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)\right| \leq Mk$,
and
$\left|\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z) - \mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)\right| \leq \frac{(2-p)M}{(p+y_1+y_2+\cdots+y_n)}$,
then

$$\left|\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)\right| \leq M.$$

Proof. Letting $\psi(a, b, c, d; z) = b - a, \Omega = h(U)$, when $h(z) = \frac{[k-p]Mz}{p+y_1+y_2+\cdots+y_n}, z \in U, M > 0$.
By using Corollary 3.8, we must demonstrate that $\psi \in \mathcal{M}_{j}[\Omega, M]$, that is the admissibility condition (3.14) is fulfilled. This can be easily understood, as it is clear that
$|\psi(a, b, c, d; z)| = \left|\frac{(k-p)Me^{i\vartheta}}{(p+y_1+\cdots+y_n)}\right| = \frac{(k-p)M}{(p+y_1+\cdots+y_n)} \geq \frac{(2-p)M}{(p+y_1+\cdots+y_n)}$, whenever $z \in U$, $\vartheta \in \mathbb{R}, k \geq 2$.

Definition 3.11. Letting Ω denoted a set within $\mathbb{C}$ with $q \in \mathbb{Q}_1 \cap \mathcal{H}_1$. The class of admissible function $\mathcal{M}_{j,1}[\Omega, q]$ comprises the functions $\psi: \mathbb{C}^4 \times \mathrm{U} \to \mathbb{C}$ that fulfil the subsequent admissibility conditions:

$$\psi(a, b, c, d; z) \notin \Omega,$$

when

$$a = q(\zeta), \qquad b = \frac{\zeta k q'(\zeta) + (1+y_1+y_2+\cdots+y_n)q(\zeta)}{(p+y_1+y_2+\cdots+y_n)},$$

$$\mathcal{R}e\left(\frac{[(p+y_1+y_2+\cdots+y_n)c - 2b(1+y_1+y_2+\cdots+y_n)](p+y_1+y_2+\cdots+y_n) + (1+y_1+y_2+\cdots+y_n)^2 a}{(p+y_1+y_2+\cdots+y_n)b - (1+y_1+\cdots+y_n)a}\right)$$

$$\geq k\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\mathcal{R}e\left(\frac{(p+y_1+y_2+\cdots+y_n)^2[d - 3c(2+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b - (1+y_1+y_2+\cdots+y_n)a}\right.$$
$$+\frac{(p+y_1+y_2+\cdots+y_n)b[11+3(y_1+y_2+\cdots+y_n)(4+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b - (1+y_1+y_2+\cdots+y_n)a}$$
$$\left.-\frac{(1+y_1+y_2+\cdots+y_n)a[6+(y_1+y_2+\cdots+y_n)(5+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b - (1+y_1+y_2+\cdots+y_n)a}\right) \geq k^2\mathcal{R}e\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

when $z \in U, \zeta \in \partial\mathrm{U} \setminus \mathrm{E(q)}$, and $k \geq 2$.

Theorem 3.12. Letting $\psi \in \mathcal{M}_{j,1}[\Omega, q]$. When the functions $f \in \mathcal{K}$ with $q \in \mathbb{Q}_1 \cap \mathcal{H}_1$ fulfil the following conditions:

$$\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)}{z\, q'(z)}\right| \leq k, \tag{3.15}$$

and

$$\left\{\psi\left(\frac{\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)}{z}, \frac{\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)}{z}, \frac{\mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z)}{z}, \frac{\mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z)}{z}; z\right) : z \in U\right\}$$

$\subset \Omega$, then (3.16)

$$\frac{\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)}{z} \prec q(z), \ (z \in U).$$

Proof: Suppose that $p(z)$ be analytic function within U denoted by

$$p(z) = \frac{\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)}{z}. \tag{3.17}$$

From Equations (1.3) and (3.17), we have

$$\frac{\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)}{z} = \frac{zp'(z) + (1+y_1+y_2+\cdots+y_n)p(z)}{(p+y_1+y_2+\cdots+y_n)}. \tag{3.18}$$

By similar argument, we have

$$\frac{\mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z)}{z} = \frac{z^2 p''(z) + \left(3+2(y_1+y_2+\cdots+y_n)\right)zp'(z) + (1+y_1+y_2+\cdots+y_n)^2 p(z)}{(p+y_1+y_2+\cdots+y_n)^2}, \tag{3.19}$$

and

$$\frac{\mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z)}{z} = \frac{(1+y_1+y_2+\cdots+y_n)^3 p(z)}{(p+y_1+y_2+\cdots+y_n)^3} +$$
$$\frac{z^3 p'''(z) + 3(2+y_1+y_2+\cdots+y_n)\, z^2 p''(z) + [7+3(3+y_1+y_2+\cdots+y_n)(1+y_1+y_2+\cdots+y_n)]zp'(z)}{(p+y_1+y_2+\cdots+y_n)^3} \tag{3.20}$$

Define the transformation form $\mathbb{C}^4$ to $\mathbb{C}$ as

$$a(r,s,t,u)=r,\quad b=\frac{s+(1+y_1+y_2+\cdots+y_n)r}{(p+y_1+y_2+\cdots+y_n)},$$
$$c(r,s,t,u)=\frac{t+\big(3+2(y_1+y_2+\cdots+y_n)\big)s+(1+y_1+y_2+\cdots+y_n)^2r}{(p+y_1+y_2+\cdots+y_n)^2}, \tag{3.21}$$
and
$$d(r,s,t,u)=\frac{u+3(2+y_1+y_2+\cdots+y_n)t+[7+3(y_1+y_2+\cdots+y_n)(3+y_1+y_2+\cdots+y_n)]s+(1+y_1+y_2+\cdots+y_n)^3r}{(p+y_1+y_2+\cdots+y_n)^3}. \tag{3.22}$$
Let
$$\varphi(r,s,t,u)=\psi(a,b,c,d;z)=$$
$$\psi\left(\begin{array}{c} r,\frac{s+(1+y_1+y_2+\cdots+y_n)r}{(p+y_1+y_2+\cdots+y_n)},\frac{t+\big(3+2(y_1+y_2+\cdots+y_n)\big)s+(1+y_1+y_2+\cdots+y_n)^2r}{(p+y_1+y_2+\cdots+y_n)^2},\\ \frac{u+3(2+y_1+y_2+\cdots+y_n)t+[7+3(y_1+y_2+\cdots+y_n)(3+y_1+y_2+\cdots+y_n)]s+(1+y_1+y_2+\cdots+y_n)^3r}{(p+y_1+y_2+\cdots+y_n)^3}\end{array}\right). \tag{3.23}$$
The proof will utilise the Lemma 2.4. Applying (3.17) to (3.20), and by (3.23), we acquire
$$\varphi\,(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)=$$
$$\psi\left(\frac{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}{z},\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{z},\frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{z},\frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{z};z\right). \tag{3.24}$$
Thus, (3.16) becomes
$\varphi\,(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\in\Omega$,
we observed this
$$1+\frac{t}{s}=\frac{[(p+y_1+y_2+\cdots+y_n)c-2b(1+y_1+y_2+\cdots+y_n)](p+y_1+y_2+\cdots+y_n)+(1+y_1+y_2+\cdots+y_n)^2a}{(p+y_1+y_2+\cdots+y_n)b-(1+y_1+y_2+\cdots+y_n)a},$$
and
$$\frac{u}{s}=\frac{(p+y_1+y_2+\cdots+y_n)^2[d-3c(2+y_1+y_2+\cdots+y_n)]-(1+y_1+y_2+\cdots+y_n)a[6+(y_1+y_2+\cdots+y_n)(5+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b-(1+y_1+y_2+\cdots+y_n)a}+\frac{(p+y_1+y_2+\cdots+y_n)b[11+3(y_1+y_2+\cdots+y_n)(4+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b-(1+y_1+y_2+\cdots+y_n)a}.$$

Thus, the admissibility conditions of $\psi\in\mathcal{M}_{j,1}[\Omega,q]$ in Definition 3.11 is equivalent to admissibility conditions of $\varphi\in\Psi_2[\Omega,q]$ that given in Definition 2.3 where $n=2$. So, by employing Equation (3.15) with Lemma 2.4, we get
$$\frac{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}{z}\prec q(z).$$

This complete the proof of Theorem 3.12.
When $\Omega\notin\mathbb{C}$ is a simple connected domain, thus $\Omega=h(U)$ for certain conformal mappings $h(z)$ of U onto Ω. In this situation, the class $\mathcal{M}_{j,1}[h(U),q]$ is rewritten by $\mathcal{M}_{j,1}[h,q]$. This follows is derived immediately from of Theorem 3.12 is provided beneath.

Theorem 3.13. Letting $\psi\in\mathcal{M}_{j,1}[h,q]$. If the functions $f\in\mathcal{K}$ with $q\in\mathbb{Q}_1$ fulfil the next conditions:
$$\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)\geq 0,\left|\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{z\,q'(z)}\right|\leq k, \tag{3.25}$$
and
$$\left\{\psi\left(\frac{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}{z},\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{z},\frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{z},\frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{z};z\right):\ z\in U\right\}$$
$\prec h(z)$, then (3.26)
$$\frac{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}{z}\prec q(z),\quad (z\in U).$$

Considering Definition 3.11 and in the particular situation $q(z) = Mz, M > 0$, the class of admissiblity functions $\mathcal{M}_{j,1}[\Omega, q]$, denoted as $\mathcal{M}_{j,1}[\Omega, M]$, is represented as following.

Definition 3.14. Letting Ω denoted a set within $\mathbb{C}$ and $M > 0$. The class admissible functions $\mathcal{M}_{j,1}[\Omega, M]$ comprises the functions.$\psi: \mathbb{C}^4 \times U \to \mathbb{C}$, thus

$$\psi\left(\begin{array}{c} Me^{i\vartheta}, \frac{(k+(1+y_1+y_2+\cdots+y_n))Me^{i\vartheta}}{(p+y_1+y_2+\cdots+y_n)}, \frac{L+[(3+2(y_1+y_2+\cdots+y_n))k+(1+y_1+y_2+\cdots+y_n)^2]Me^{i\vartheta}}{(p+y_1+y_2+\cdots+y_n)^2} \\ \frac{N+3(2+y_1+y_2+\cdots+y_n)\,L+[(7+3(y_1+y_2+\cdots+y_n)(3+y_1+y_2+\cdots+y_n))k+(1+y_1+y_2+\cdots+y_n)^3]Me^{i\vartheta}}{(p+y_1+y_2+\cdots+y_n)^3}; z \end{array}\right)$$

$$\notin \Omega, \tag{3.27}$$

whenever

$\mathcal{R}e(Le^{-i\vartheta}) \geq (k-1)Mk,\ z \in U,$

with

$\mathcal{R}e(Ne^{-i\vartheta}) \geq 0,$ for all $\vartheta \in \mathbb{R}, k \geq 2.$

Corollary 3.15. Letting $\psi \in \mathcal{M}_{j,1}[\Omega, M]$. When the function $f \in \mathcal{K}$ fulfils the next condition:

$$\left|\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{z}\right| \leq Mk,\ (z \in U;\ k \geq 2;\ M > 0),$$

and

$$\psi\left(\frac{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}{z}, \frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{z}, \frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{z}, \frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{z}; z\right) \in \Omega,$$

then

$$\left|\frac{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}{z}\right| < M.$$

In this particular situation $\Omega = q(U) = \{w: |w| < M\}$, the class $\mathcal{M}_{j,1}[\Omega, M]$ is simple denoted as $\mathcal{M}_{j,1}[M]$. The consequence (3.15) will be changed in the next format.

Corollary 3.16. Letting $\psi \in \mathcal{M}_{j}[M]$. If the function $f \in \mathcal{K}$ fulfils the next conditions:

$$\left|\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{z}\right| \leq Mk,\ (z \in U; k \geq 2; M > 0),$$

and

$$\left|\psi\left(\frac{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}{z}, \frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{z}, \frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{z}, \frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{z}; z\right)\right| < M,$$

then

$$\left|\frac{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}{z}\right| < M.$$

Definition 3.17. Letting Ω denoted a set within $\mathbb{C}$ with $q \in \mathbb{Q}_1 \cap \mathcal{H}_1$. The class of admissiblity function $\mathcal{M}_{j,2}[\Omega, q]$ comprises the functions $\psi: \mathbb{C}^4 \times U \to \mathbb{C}$ that fulfil the subsequent admissibility conditions:

$$\psi(a, b, c, d; z) \notin \Omega,$$

when

$$a = q(\zeta),\quad b = \frac{1}{(p+y_1+y_2+\cdots+y_n)}\left[\frac{\zeta k q'(\zeta)+(p+y_1+y_2+\cdots+y_n)q^2(\zeta)}{q(\zeta)}\right],$$

$$\mathcal{R}e\left(\frac{(p+y_1+y_2+\cdots+y_n)[b(c-3a)+2a^2]}{b-a}\right) \geq k\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\mathcal{R}e\Big([bc(p+y_1+y_2+\cdots+y_n)^2(d-c) - b(p+y_1+y_2+\cdots+y_n)(c-b)(1-c-b+3a) - 3b(p+y_1+y_2+\cdots+y_n)(c-b)(b-a)+2(b-a) + 3a(p+y_1+y_2+\cdots+y_n)(b-a) + (p+y_1+\cdots+y_n)(b-a)^2\big((p+y_1+y_2+\cdots+y_n)(b-5a)-3\big) + a^2(p+y_1+y_2+\cdots+y_n)^2(b-a)\,]\times(b-a)^{-1}\Big) \geq k^2\mathcal{R}e\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

when $z \in U, \zeta \in \partial U \setminus E(q)$, and $k \geq 2$.

Theorem 3.18. Letting $\psi \in \mathcal{M}_{j,2}[\Omega, q]$. If the functions $f \in \mathcal{K}$ with $q \in \mathbb{Q}_1 \cap \mathcal{H}_1$ fulfil the next conditions:

$$\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \left|\frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+1}_{y_1+y_2+\cdots+y_n}q'(z)}\right| \leq k, \tag{3.28}$$

and

$$\left\{\psi\left(\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+4}_{y_1+y_2+\cdots+y_n}}{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}f(z); z\right) : z \in U\right\} \subset \Omega, \tag{3.29}$$

then

$$\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)} \prec q(z), (z \in U).$$

Proof: Assume that the analytic function $p(z)$ within U denoted as

$$p(z) = \frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}. \tag{3.30}$$

From Equations (1.3) and (3.30), we have

$$\frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)} = \frac{1}{(p+y_1+y_2+\cdots+y_n)}\left[\frac{zp'(z)+(p+y_1+y_2+\cdots+y_n)p^2(z)}{p(z)}\right] = \frac{A}{(p+y_1+y_2+\cdots+y_n)}. \tag{3.31}$$

By a similar argument, we get

$$\frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)} = \frac{B}{(p+y_1+y_2+\cdots+y_n)} \tag{3.32}$$

and

$$\frac{T^{m+4}_{y_1+y_2+\cdots+y_n}}{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}f(z) = \frac{1}{(p+y_1+y_2+\cdots+y_n)}[B + B^{-1}(C + A^{-1}D - A^{-2}C^2)], \tag{3.33}$$

where

$$A = \frac{zp'(z)+(p+y_1+y_2+\cdots+y_n)p^2(z)}{p(z)},\ A^{-2} = \left(\frac{p(z)}{zp'(z)+(p+y_1+y_2+\cdots+y_n)p^2(z)}\right)^2,$$

$$B = \frac{zp'(z)+(p+y_1+y_2+\cdots+y_n)p^2(z)}{p(z)} + \frac{\frac{z^2p''(z)+zp'(z)}{p(z)}+(p+y_1+y_2+\cdots+y_n)zp'(z)-(\frac{zp'(z)}{p(z)})^2}{\frac{zp'(z)+(p+y_1+y_2+\cdots+y_n)p^2(z)}{p(z)}},$$

$$B^{-1} = \frac{p(z)}{zp'(z)+(p+y_1+y_2+\cdots+y_n)p^2(z)} + \frac{\frac{zp'(z)+(p+y_1+y_2+\cdots+y_n)p^2(z)}{p(z)}}{\frac{z^2p''(z)+zp'(z)}{p(z)}+(p+y_1+y_2+\cdots+y_n)zp'(z)-(\frac{zp'(z)}{p(z)})^2},$$

$$C = \frac{z^2p''(z)+zp'(z)}{p(z)} + (p+y_1+y_2+\cdots+y_n)zp'(z) - \left(\frac{zp'(z)}{p(z)}\right)^2,$$

and

$$D = \frac{z^3p'''(z)+3z^2p''(z)+zp'(z)}{p(z)} - \frac{3z^3p''(z)\,p'(z)+3\big(zp'(z)\big)^2}{p(z)} + (p+y_1+y_2+\cdots+y_n)[z^2p''(z) + zp'(z)] + 2\left(\frac{zp'(z)}{p(z)}\right)^3.$$

Now, we define the transformation form $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u) = r,$$

$$b = \frac{1}{(p+y_1+y_2+\cdots+y_n)}\left[\frac{s+(p+y_1+y_2+\cdots+y_n)\,r^2}{r}\right] = \frac{E}{(p+y_1+y_2+\cdots+y_n)},$$

$$c(r,s,t,u) = \frac{F}{(p+y_1+y_2+\cdots+y_n)} =$$

$$\frac{1}{(p+y_1+y_2+\cdots+y_n)}\left[\frac{s+(p+y_1+y_2+\cdots+y_n)\,r^2}{r} + \frac{\frac{t+s}{r}+(p+y_1+y_2+\cdots+y_n)s-\left(\frac{s}{r}\right)^2}{\frac{s+(p+y_1+y_2+\cdots+y_n)\,r^2}{r}}\right], \tag{3.34}$$

and

$$d(r,s,t,u) = \frac{1}{(p+y_1+y_2+\cdots+y_n)}\left[F + F^{-1}(L + HE^{-1} - E^{-2}L^2)\right], \tag{3.35}$$

where

$$E = \frac{s+(p+y_1+y_2+\cdots+y_n)\,r^2}{r},$$

$$L = \frac{t+s}{r} + (p+y_1+y_2+\cdots+y_n)s - \left(\frac{s}{r}\right)^2,$$

$$F = \frac{s+(p+y_1+y_2+\cdots+y_n)\,r^2}{r} + \frac{\frac{t+s}{r}+(p+y_1+y_2+\cdots+y_n)s-\left(\frac{s}{r}\right)^2}{\frac{s+(p+y_1+y_2+\cdots+y_n)\,r^2}{r}},$$

and

$$H = \frac{u+3t}{r} - \frac{3ts+3(s)^2}{r^2} + \frac{s}{r} + (p+y_1+y_2+\cdots+y_n)(t+s) + 2\frac{s^3}{r^3}.$$

Let

$$\varphi(r,s,t,u) = \psi(a,b,c,d) = \psi\left(\begin{array}{c} r, \frac{E}{(p+y_1+y_2+\cdots+y_n)}, \frac{F}{(p+y_1+y_2+\cdots+y_n)}, \\ \frac{1}{(p+y_1+y_2+\cdots+y_n)}\left[F + F^{-1}(L + HE^{-1} - E^{-2}L^2)\right] \end{array}\right). \tag{3.36}$$

The proof will utilise the Lemma 2.4. Applying (3.30) to (3.33), and from (3.36), we acquire

$$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) =$$

$$\psi\left(\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+4}_{y_1+y_2+\cdots+y_n}}{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}f(z); z\right). \tag{3.37}$$

Hence, clearly (3.29) leads to

$$\varphi\,(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$

We observe that

$$\frac{t}{s} + 1 = \frac{(p+y_1+y_2+\cdots+y_n)[b(c-3a)+2a^2]}{b-a}$$

and

$$\begin{aligned} \frac{u}{s} = \big[&bc(p+y_1+y_2+\cdots+y_n)^2(d-c) \\ &- b(p+y_1+y_2+\cdots+y_n)(c-b)(1-c-b+3a) \\ &- 3b(y_1+y_2+\cdots+y_n)(c-b)(b-a) + 2(b-a) \\ &+ 3a(p+y_1+y_2+\cdots+y_n)(b-a) \\ &+ (p+y_1+y_2+\cdots+y_n)(b-a)^2\big((p+y_1+y_2+\cdots+y_n)(b-5a)-3\big) \\ &+ a^2(p+y_1+y_2+\cdots+y_n)^2(b-a)\,\big] \times (b-a)^{-1}. \end{aligned}$$

Then, the admissibility conditions of $\psi \in \mathcal{M}_{j,2}[\Omega, q]$ within Definition 3.17 is equivalent to admissibility conditions of $\varphi \in \Psi_2[\Omega, q]$ that given within Definition (2.3) as well $n = 2$. So, by utilising Equation (3.30) with Lemma 2.4, we obtain

$$\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)} \prec q(z).$$

This complete the proof of Theorem 3.18.

When $\Omega \neq \mathbb{C}$ is a simple connected domain, then $\Omega = h(U)$ for certain conformal mappings $h(z)$ of U onto Ω. In this situation, the class $\mathcal{M}_{j,2}[h(U), q]$ is written by $\mathcal{M}_{j,2}[h, q]$. The immediated implication of Theorem (3.18) is given here without proof.

Theorem 3.19. Letting $\psi \in \mathcal{M}_{j,2}[h, q]$. If the function $f \in \mathcal{K}$ with $q \in \mathbb{Q}_1 \cap \mathcal{H}_1$ fulfil the conditions (3.28) and

$$\psi\left(\frac{\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)}{\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)}, \frac{\mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z)}{\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)}, \frac{\mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z)}{\mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z)}, \frac{\mathrm{T}^{m+4}_{y_1+y_2+\cdots+y_n}}{\mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z)} f(z); z\right) \prec h(z), \quad (3.38)$$

then

$$\frac{\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)}{\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z)} \prec q(z), \qquad (z \in U).$$

4- Third-order differential superordination results

In the next part, we establish certain third-order differential superordination outcomes. To achieve that objective, the class of admissible functions is established in the next manner:

Definition 4.1. Letting Ω denoted a set within $\mathbb{C}$, also $q \in \mathbb{Q}_0 \cap \mathcal{H}_0$ as well $q'(z) \neq 0$. The class of admissible function $\mathcal{M}'_j[\Omega, q]$ comprises the functions $\psi: \mathbb{C}^4 \times \overline{\mathrm{U}} \to \mathbb{C}$ that fulfil the subsequent admissibility conditions:

$$\psi(a, b, c, d; \zeta) \in \Omega,$$

when

$$a = q(z), \qquad b = \frac{zq'(z) + (y_1+y_2+\cdots+y_n)mq(z)}{(p+y_1+y_2+\cdots+y_n)m},$$

$$\mathcal{Re}\left(\frac{(p+y_1+y_2+\cdots+y_n)^2 c - (y_1+y_2+\cdots+y_n)[2b(p+y_1+y_2+\cdots+y_n) - (y_1+y_2+\cdots+y_n)a]}{(p+y_1+y_2+\cdots+y_n)b - (y_1+y_2+\cdots+y_n)a}\right) \leq \frac{1}{m}\mathcal{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\mathcal{Re}\left(\frac{(p+y_1+y_2+\cdots+y_n)^2[(p+y_1+y_2+\cdots+y_n)d - 3c(1+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b - (y_1+y_2+\cdots+y_n)a}\right.$$
$$+\frac{(p+y_1+y_2+\cdots+y_n)b[2+3(y_1+y_2+\cdots+y_n)(2+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b - (y_1+y_2+\cdots+y_n)a}$$
$$\left.-\frac{(y_1+y_2+\cdots+y_n)a[2+(y_1+y_2+\cdots+y_n)(3+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b - (y_1+y_2+\cdots+y_n)a}\right) \leq \frac{1}{m^2}\mathcal{Re}\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

Where $m \geq 2$, and $z \in U, \zeta \in \partial\mathrm{U} \setminus \mathrm{E(q)}$.

Theorem 4.2. Letting $\psi \in \mathcal{M}'_j[\Omega, q]$. If the functions $f \in \mathcal{K}$ with $\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z) \in \mathbb{Q}_0$ and $q \in \mathcal{H}_0$ also $q'(z) \neq 0$, satisfy the next conditions:

$$\mathcal{Re}\left(\frac{\zeta q''(z)}{q'(z)}\right) \geq 0, \quad \left|\frac{\mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z)}{q'(z)}\right| \leq m, \qquad (4.1)$$

and

$$\psi\left(\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z); z\right),$$

is univalent function within U, thus

$$\Omega \subset \left\{\psi\left(\mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+1}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+2}_{y_1+y_2+\cdots+y_n} f(z), \mathrm{T}^{m+3}_{y_1+y_2+\cdots+y_n} f(z); z\right); z \in U\right\}, \qquad (4.2)$$

implies

$$q(z) \prec \mathrm{T}^{m}_{y_1+y_2+\cdots+y_n} f(z), \qquad (z \in U).$$

Proof. Letting the function $p(z)$ be defined in (3.3) and φ in (3.9). Since $\psi \in \mathcal{M}'_j[\Omega, q]$. By (3.10) with (4.2), we get

$$\Omega \subset \{\psi(\mathrm{p(z)}, \mathrm{zp'(z)}, \mathrm{z^2 p''(z)}, \mathrm{z^3 p'''(z)}; z); z \in U\}.$$

By (3.7) and (3.8), it is clear that the admissibility condition of $\psi \in \mathcal{M}'_{j}[\Omega, q]$ within Definition 4.1 is equivalent to the admissbility of $\varphi \in \Psi'_n[\Omega, q]$ that given within Definition 2.3 where $n = 2$. Thus $\varphi \in \Psi'_2[\Omega, q]$ and by utilising (4.2) with Lemma 2.7, we find this

$$q(z) \prec \mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \qquad (z \in U).$$

The proof has been completed.

When $\Omega \neq \mathbb{C}$ is a simple connected domain, thus $\Omega = h(U)$ for certain conformal mappings $h(z)$ of U, onto Ω. In this situation, the class $\mathcal{M}'_{j}[h(U), q]$ is rewritten by $\mathcal{M}'_{j}[h, q]$. The following assertion is an immediate outcome of Theorem 4.2.

Theorem 4.3. Letting $\psi \in \mathcal{M}'_{j}[h, q]$ and h be analytic within U. When the functions $f \in \mathcal{K}$ with $\mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z) \in \mathbb{Q}_0$ and $q \in \mathcal{H}_0$ also $q'(z) \neq 0$, fulfill the next conditions (4.1) and
$\psi\big(\mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+1}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+2}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+3}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z); z\big)$,
is univalent function within U, thus

$$\mathrm{h}(\mathrm{z}) \prec \psi\big(\mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+1}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+2}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+3}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z); z\big), \qquad (4.3)$$

implies

$$q(z) \prec \mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \qquad (z \in U).$$

Theorems 4.2 and 4.3 can solely be utilised to acquire subordination for the third-order differential superordination of the form (4.2) or (4.3). The subsequent theorem establishes the presence of the best subordinant for Equation (4.3) for an appropriate ψ.

Theorem 4.4. Letting the function h be univalent within U. Also, assume $\psi: \mathbb{C}^4 \times \overline{\mathrm{U}} \to \mathbb{C}$ and φ be given in (3.9). Assume that the following differential equation:

$$\varphi(\mathrm{q}(\mathrm{z}), \mathrm{zq}'(\mathrm{z}), \mathrm{z}^2\mathrm{q}''(\mathrm{z}), \mathrm{z}^3\mathrm{q}'''(\mathrm{z}); z) = h(z), \qquad (4.4)$$

has a solution $q(z) \in \mathbb{Q}_0$. If the functions $f \in \mathcal{K}$, and $\mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z) \in \mathbb{Q}_0$ and if $q \in \mathcal{H}_0$ with $q'(z) \neq 0$, satisfy the conditions (4.1) and the function
$\psi\big(\mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+1}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+2}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+3}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z); z\big)$,
is univalent within U, thus

$$\mathrm{h}(\mathrm{z}) \prec \psi\big(\mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+1}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+2}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \mathrm{T}^{\mathrm{m}+3}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z); z\big),$$

implies that

$$q(z) \prec \mathrm{T}^{\mathrm{m}}_{\mathrm{y}_1+\mathrm{y}_2+\cdots+\mathrm{y}_n} f(z), \qquad (z \in U),$$

where $q(z)$ is the best subordinant.

Proof: By utilising Theorems 4.2 and 4.3,we deduce that q is a subordinant of (4.3). because q fulfils (4.4), it is additionally the solution to (4.3). As well, q is to be subordinant for all subordinants. Thus, q is the best subordinant, and the proof has been completed.

Definition 4.5. Letting Ω denoted a set within $\mathbb{C}$ and $q \in \mathcal{H}_1$ as well $q'(z) \neq 0$. The class of admissible function $\mathcal{M}'_{j,1}[\Omega, q]$ comprises the functions $\psi: \mathbb{C}^4 \times \overline{\mathrm{U}} \to \mathbb{C}$ that fulfil the subsequent admissibility conditions:

$$\psi(a, b, c, d; \zeta) \in \Omega,$$

when

$$a = q(z), \qquad b = \frac{zq'(z) + (1+y_1+y_2+\cdots+y_n)mq(z)}{(p+y_1+y_2+\cdots+y_n)m},$$

$$\mathcal{R}e\left(\frac{[(p+y_1+y_2+\cdots+y_n)c - 2b(1+y_1+y_2+\cdots+y_n)](p+y_1+y_2+\cdots+y_n) + (1+y_1+y_2+\cdots+y_n)^2 a}{(p+y_1+y_2+\cdots+y_n)b - (1+y_1+y_2+\cdots+y_n)a}\right) \leq \frac{1}{m}\mathcal{R}e\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\mathcal{R}e\left(\frac{(p+y_1+y_2+\cdots+y_n)^2[(p+y_1+y_2+\cdots+y_n)d-3c(1+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b-(y_1+y_2+\cdots+y_n)a} + \frac{(p+y_1+y_2+\cdots+y_n)b[2+3(y_1+y_2+\cdots+y_n)(2+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b-(y_1+y_2+\cdots+y_n)a} - \frac{(y_1+y_2+\cdots+y_n)a[2+(y_1+y_2+\cdots+y_n)(3+y_1+y_2+\cdots+y_n)]}{(p+y_1+y_2+\cdots+y_n)b-(y_1+y_2+\cdots+y_n)a}\right) \le \frac{1}{m^2}\mathcal{R}e\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in U, \zeta \in \partial\mathrm{U}$, and $m \ge 2$.

Theorem 4.6. Letting $\psi \in \mathcal{M}'_{j,1}[\Omega, q]$. When the function $f \in \mathcal{K}$ with $\frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z} \in \mathbb{Q}_1$ and $q \in \mathcal{H}_1$ where $q'(z) \neq 0$, fulfil the following conditions :

$$\mathcal{R}e\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \ge 0, \left|\frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z\,q'(z)}\right| \le m, \tag{4.5}$$

and

$$\psi\left(\frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}; z\right),$$

is univalent in U, then

$$\Omega \subset \left\{\psi\left(\frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}; z\right): z \in U\right\}, \tag{4.6}$$

implies that

$$q(z) \prec \frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \ (z \in U).$$

Proof. Letting the function $p(z)$ be defined in (3.17), also φ in (3.23). Since $\psi \in \mathcal{M}'_{j,1}[\Omega, q]$, by (3.24) and (4.6) that

$$\Omega \subset \{\varphi(\mathrm{p(z)}, \mathrm{zp'(z)}, \mathrm{z^2p''(z)}, \mathrm{z^3p'''(z)}; z): z \in U\}.$$

By the Equations (3.21) and (3.22), it is clear that the admissible condition of $\psi \in \mathcal{M}'_{j,1}[\Omega, q]$ within Definition 4.1 is equivalent to the admissibility condition of φ that given within Definition 2.3 where n=2. Thus, $\varphi \in \Psi'_2[\Omega, q]$ and applying (4.6) with Lemma 2.7, we obtain

$$q(z) \prec \frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \ (z \in U).$$

The proof of Theorem 4.6 has been complete.

When $\Omega \notin \mathbb{C}$ is a simple connected domain, thus $\Omega = \mathrm{h(U)}$ for certain conformal mappings $h(z)$ of U onto Ω. In this situation, the class $\mathcal{M}'_{j,1}[h(U), q]$ is written as $\mathcal{M}'_{j,1}[h, q]$. The subsequent immediate outcome of Theorem 4.6 is given beneath.

Theorem 4.7. Letting $\in \mathcal{M}'_{j,1}[h, q]$, and h be analytic within U. When the functions $f \in \mathcal{K}$ and $q \in \mathcal{H}_1$ where $q'(z) \neq 0$, fulfil the next conditions (4.5) and the function

$$\psi\left(\frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}; z\right),$$

is univalent in U, then

$$h(z) \prec \psi\left(\frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}; z\right),$$

implies that

$$q(z) \prec \frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y}_n}f(z)}{z}, \ \ (z \in U).$$

Definition 4.8. Letting Ω denoted a set within $\mathbb{C}$, also $q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible functions $\mathcal{M}'_{j,2}[\Omega, q]$ comprises the functions $\psi: \mathbb{C}^4 \times \overline{\mathrm{U}} \to \mathbb{C}$ that fulfil the subsequent admissibility conditions:

$$\psi(a, b, c, d; \zeta) \in \Omega,$$

when
$a = q(z), \quad b = \frac{1}{(p+y_1+y_2+\cdots+y_n)}\left[\frac{zq'(z)+m(p+y_1+y_2+\cdots+y_n)q^2(z)}{mq(z)}\right],$
$\mathcal{R}e\left(\frac{(p+y_1+y_2+\cdots+y_n)[b(c-3a)+2a^2]}{b-a}\right) \le \frac{1}{m}\mathcal{R}e\left(\frac{zq''(z)}{q'(z)}+1\right),$
and

$$\begin{aligned}\mathcal{R}e\big([&bc(p+y_1+y_2+\cdots+y_n)^2(d-c)\\ &- b(p+y_1+y_2+\cdots+y_n)(c-b)(1-c-b+3a)\\ &- 3b(p+y_1+y_2+\cdots+y_n)(c-b)(b-a)+2(b-a)\\ &+ 3a(p+y_1+y_2+\cdots+y_n)(b-a)\\ &+ (p+y_1+y_2+\cdots+y_n)(b-a)^2\big((p+y_1+y_2+\cdots+y_n)(b-5a)-3\big)\\ &+ a^2(p+y_1+y_2+\cdots+y_n)^2(b-a)\,]\times(b-a)^{-1}\big) \ge \frac{1}{m^2}\mathcal{R}e\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),\end{aligned}$$

such that $m \ge 2$, and $z \in U, \zeta \in \partial U \setminus E(q)$.

Theorem 4.9. Letting $\psi \in \mathcal{M}'_{j,2}[\Omega, q]$. When the functions $f \in \mathcal{K}$ and $q \in \mathcal{H}_1$ and $\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)} \in \mathbb{Q}_1$ with $q'(z) \neq 0$, satisfy the following conditions:

$$\mathcal{R}e\left(\frac{zq''(z)}{q'(z)}\right) \ge 0, \left|\frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+1}_{y_1+y_2+\cdots+y_n}q'(z)}\right| \le m, \tag{4.7}$$

and
$\psi\left(\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+4}_{y_1+y_2+\cdots+y_n}}{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}f(z); z\right),$
in U, it is univalent, then
$\Omega \subset \left\{\psi\left(\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+4}_{y_1+y_2+\cdots+y_n}}{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}f(z); z\right): z \in U\right\}, \quad (4.8)$
implies that
$q(z) \prec \frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}, (z \in U).$

Proof. Letting the function $p(z)$ be define in (3.30) also φ in (3.36). Because $\psi \in \mathcal{M}'_{j,2}[\Omega, q]$, we find from (3.37) and (4.8) that

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z): z \in U\}.$$

By the Equations (3.34) and (3.35), it is clear that the admissible condition of $\psi \in \mathcal{M}'_{j,2}[\Omega, q]$ within Definition 4.8 is equivalent to the admissibility condition of φ that given within Definition 2.6 where $n = 2$. Thus, $\varphi \in \Psi'_2[\Omega, q]$ and applying (4.7) and Lemma 2.7, we obtain
$q(z) \prec \frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}, \ (z \in U).$
This completes the proof of Theorem 4.6.

Theorem 4.10. Letting $\psi \in \mathcal{M}'_{j,2}[h, q]$. When the functions $f \in \mathcal{K}$ and $q \in \mathcal{H}_1$ and $\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)} \in \mathbb{Q}_1$ with $q'(z) \neq 0$, fulfil the next conditions (4.7) and the function
$\psi\left(\frac{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+1}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}{T^{m+2}_{y_1+y_2+\cdots+y_n}f(z)}, \frac{T^{m+4}_{y_1+y_2+\cdots+y_n}}{T^{m+3}_{y_1+y_2+\cdots+y_n}f(z)}f(z); z\right),$
in U, it is univalent, then

$\mathrm{h}(\mathrm{z}) \prec \psi\left(\frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+4}}_{\mathrm{y_1+y_2+\cdots+y_n}}}{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}f(z); z\right),$

implies that

$q(z) \prec \frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, (z \in U).$

5- Sandwich results

By utilising Theorem 3.4 with (4.3), the sandwich-type theorem can be obtained as follows.

Theorem 5.1. Letting h_1 and q_1 are analytic functions within U. Also, let h_2 be univalent function within U and $q_2 \in \mathbb{Q}_0$ with $q_1(0) = q_2(0) = 1$ and $\psi \in \mathcal{M}_{\jmath}[h_2, q_2] \cap \mathcal{M}'_{\jmath}[h_1, q_1]$. If the function $f \in \mathcal{K}$ with $\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z) \in \mathbb{Q}_0 \cap \mathcal{H}_0$ and the function

$$\psi\left(\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z), \mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z), \mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z), \mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z); z\right),$$

is univalent within U, when the conditions (3.1) and (4.1) are fulfilled, thus

$h_1(z) \prec \psi\left(\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z), \mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z), \mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z), \mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z); z\right) \prec h_2(z),$

implies that

$$q_1(z) \prec \mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z) \prec q_2(z), (z \in U). \quad (5.1)$$

Combining Theorems 3.13 and 4.7, we have the next sandwich-type theorem.

Theorem 5.2. Letting h_1 and q_1 are analytic functions within U. Also, let h_2 be univalent function within U and $q_2 \in \mathbb{Q}_1$ and $q_1(0) = q_2(0) = 1$ with $\psi \in \mathcal{M}_{\jmath,1}[h_2, q_2] \cap \mathcal{M}'_{\jmath,1}[h_1, q_1]$. When the function $f \in \mathcal{K}$ with $\frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z} \in \mathbb{Q}_1 \cap \mathcal{H}_1$ and the function

$\psi\left(\frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z}; z\right),$

is univalent within U, when the conditions (3.15) and (4.5) are fulfilled, thus

$h_1(z) \prec \psi\left(\frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z}; z\right) \prec h_2(z),$

implies that

$$q_1(z) \prec \frac{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{z} \prec q_2(z), (z \in U). \quad (5.2)$$

Theorem 5.3. Letting h_1 and q_1 are analytic functions within U. Also, let h_2 be univalent function within U and $q_2 \in \mathbb{Q}_1$ and $q_1(0) = q_2(0) = 1$ with $\psi \in \mathcal{M}_{\jmath,2}[h_2, q_2] \cap \mathcal{M}'_{\jmath,2}[h_1, q_1]$. When the function $f \in \mathcal{K}$ with $\frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)} \in \mathbb{Q}_1 \cap \mathcal{H}_1$ and the function

$\psi\left(\frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+4}}_{\mathrm{y_1+y_2+\cdots+y_n}}}{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}f(z); z\right),$

is univalent within U, when the conditions (3.28) and (4.7) are fulfilled, thus

$h_1(z) \prec \psi\left(\frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m+2}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}, \frac{\mathrm{T}^{\mathrm{m+4}}_{\mathrm{y_1+y_2+\cdots+y_n}}}{\mathrm{T}^{\mathrm{m+3}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}f(z); z\right) \prec h_2(z),$

implies that

$$q_1(z) \prec \frac{\mathrm{T}^{\mathrm{m+1}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)}{\mathrm{T}^{\mathrm{m}}_{\mathrm{y_1+y_2+\cdots+y_n}}f(z)} \prec q_2(z), (z \in U). \quad (5.3)$$

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