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Applications of Partial Differential Equations in Heat Transfer Modelling

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ABSTRACT

Heat transfer plays a critical role in energy, and environmental systems, where modelling is important to monitor the performance and efficiency of the system. With the rise of complex physical processes, partial differential equations were the corner stone to describe them in detail. In this paper, we will highlight the role of PDEs for numerous applications of heat transfer, including conduction, convection, radiation, and phase-change processes. Out paper examines recent case studies from many engineering domains such as solar thermal systems, phase-change solar collectors, industrial drying ovens, and electrical transformer cooling systems. The paper concludes by emphasizing future directions in physics-informed machine learning, fractional modelling, and digital twin integration for real-time thermal management.

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1. INTRODUCTION

Heat transfer is a very important process to study in engineering and science, and its applications extends from the cooling of transformers to the design of solar energy systems. As these applications grow in complexity and size, the accurate prediction of heat transfer is becoming a must. Therefore, engineers use heat transfer modelling can solve these problems and give us feedback better than the costly and complex running of experiments [1]. In order to model the heat transfer, partial differential equations (PDEs) are extensively used to capture how the temperature is changing through time and space. PDEs are suitable to represent different types of heat transfer, from conduction in solids, to convection in fluids, and radiation between surfaces. In real world applications, these different types are usually coupled together, making the model hard to solve analytically. Thus, we can apply approximate methods using numerical methods to find a good-enough solution [2]. In this paper, we aim to review the role of PDEs in heat transfer modelling by delving into the details of mathematical foundations (Section 2), followed be case studies across many engineering domains (Section 3), and finally we conclude the paper with main results and future directions in Section 4.

The study of heat transfer requires understanding of theory of partial differential equations (PDE), because the describe how physical quantities change in space and time. Unlike ordinary differential equations, which describes change with respect to one-dimension, partial differential equations can capture multidimensional relationships and dynamics. That is why PDEs are used to model heat transfer such as conduction, convection, and radiation. In this section we will explain the theory and principal of PDEs, and how they relate to basic heat transfer models.

a. Partial Differential Equations (PDEs)

Depending on the physical behaviour of the underlying phenomenon, PDEs are classified into three main classes: elliptic, parabolic, and hyperbolic. Elliptic equations describe steady-state phenomena, and they are used when the system reaches equilibrium and doesn't further change with respect to time dimension. An example of elliptic PDEs is the heating of a metal plate with fixed flow. Parabolic PDEs, describe processes related to diffusion, such as transient conduction in solids. In these types of equations, we notice that irregularities "smooth out" over time and the disturbance at one point gradually diffuse outward. Finally, hyperbolic PDEs are the choice to model wave-like propagation. This class of equations is relevant in thermoelastic problems where we should consider thermal waves, such as very fast heating of small structures. We need to choose the appropriate classification because they affect the boundary conditions that are required to solve the problem[3].

To solve problems related to heat transfer, researcher utilize different methods of PDEs. For example, Analytical methods can be used to find a closed-form solutions to simple heat transfer problems such as heat flow through a rod in one dimension. While analytical methods only work in such idealized cases, they often provide insight and are used as benchmarks to assess more complex methods [4]

Nonetheless, most of realistic problems involve nonlinearities, or coupled physics the can't be solved analytically. In these cases, we can use numerical methods such as Finite Difference Method (FDM), which discretizes the PDEs into algebraic equations by replacing derivatives with differences. However, FDM fails in complex geometries. Finite Element Method (FEM) is a more developed method, in which we divide the domains into meshes, and approximate the solution with basis functions. They are widely used in electronic packaging and turbine colling. Another method is the Finite Volume Method (FVM) which preserves conservation of fluxed and is extensively used in computation fluid dynamics (CFD).

In addition to traditional methods, new innovative methods are used to solve heat transfer problems. Average Radial Particle Method (ARPM) is used to approximate derivatives using particle distributions without generating meshes in three-dimensional conduction problems [5]Moreover, machine learning is used in hybrid techniques such as Physics-Informed Neural Networks (PINNs), where the physical nature of the phenomenon is imposed into the network, making its predictions more respect to underlying physics even when training on small amount of data. These methods are promising in modelling conductive and convective heat transfer, especially in real-time simulations

b. Heat Transfer Modelling

Heat transfer phenomena can be divided into four main mechanisms: conduction, convection, radiation, and phase change. These modes appear in combinations that demand coupled PDE formulations.

i. Conduction

Conduction is considered the fundamental mode of heat transfer. It is described with Fourier's Law, stating that the flux is proportional to the negative gradient of temperature. Classical heat equation is described as follows:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

where $\alpha = \frac{k}{(\rho c_p)}$ is the thermal diffusivity, k is the thermal conductivity, ρ is the density, and c_p the specific heat capacity.

This parabolic PDE describes how heat diffuses through a material over time, and it is used in solids and stationary fluids. Analytical solutions can be used in simple geometries to solve conduction problems, but most problems require FEM or meshfree methods [4]

ii. Convection

While conduction is used to model stationary fluids, convection includes the effects of bulk fluid motion, and it is described using the convection-diffusion equation:

$$\rho c_p \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T\right) = k\nabla^2 T + q$$

where $\vec{u}$ is the velocity field of the fluid and q represents volumetric heat sources.

The convective term $\vec{u}.\nabla T$ introduces couples heat transfer to the fluid's momentum equations (Navier–Stokes PDEs), which make the analytical solution challenging.

Convection heat transfer is crucial in HVAC turbine cooling, blood flow analysis, and solar collector, where fluid motion affects temperature fields [6].

iii. Radiation

Unlike thermal conduction and convection, radiation involves electromagnetic energy transport. It is modelled using Radiative Transfer Equation (RTE):

$$\frac{dI}{ds} = -\kappa I + \kappa I_b + \sigma_s \int_{4\pi} \mathrm{I}(\vec{s'})\Phi(\vec{s},\vec{s'})\mathrm{d}\Omega$$

where I is radiative intensity, κ is the absorption coefficient, σ_s is the scattering coefficient, and Φ is the scattering phase function.

The RTE is an integro-differential PDE that takes into account emission, absorbption, and scattering at the same time. Radiation equations are dominant in high-temperature systems such as combustion chambers and solar receivers. We notice that radiation is nonlinear -from the Stefan-Boltzman law-which make them only solvable with approximation method such as Monte Carlo ray tracing.

iv. Phase Change

Phase change processes include transition of material from one state to another, processes like melting, solidification, boiling and condensation can be harder to model because the boundaries are also changing. These processes are modelled as Stefan problems, where the location of the phase-change boundary evolves over time based on energy balance.

$$\rho L \frac{dS}{dt} = k \frac{\partial T}{\partial n}$$

where L is the latent heat, S is the boundary position, and $\frac{\partial T}{\partial n}$ is the temperature gradient at the boundary.

These PDEs are used in modelling welding of metals, food preservation, and thermal energy storage systems. In boiling and condensation, PDEs are coupled with the Navier-Stokes equations, which requires advanced methods such as Lattice Boltzamann Method (LBM) [7]

v. Complex Media and Coupled Phenomena

At last, in many practical applications, more than one heat transfer mode is apparent, and coupled with additional physics, such as Anisotropic conduction, Magnetohydrodynamic flows, electro-thermal coupling, and thermos-chemical processes.

Such coupled PDE systems are increasingly addressed using fractional calculus (to model memory effects), fractal PDEs (to capture anomalous transport), and machine learning-accelerated solvers for efficiency.

2. METHODOLOGY

In this section, we will go through practical applications of PDE-based modelling for heat transfer, including solar thermal collectors, multitubular reactors, MHD porous flows, industrial drying systems, nanofluids, and advanced numerical solution strategies.

a. Solar Energy Systems

i. Multitubular Solar Reactors

In multitubular Solar Reactors, the radiation emitted from solar flux interacts with tube walls and inner fluids, making the mathematical model combine all three modes of heat transfer as follows:

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 \mathrm{T} - \nabla.(\mathrm{q_{rad}}) + \mathrm{hA}(\mathrm{T_f} - \mathrm{T_w})$$

This equation combines conduction through walls, convection in the gas, and radiative flux. In order to solve this equation, FVM method was proposed along with Discrete Ordinates Method (DOM) . The governor equation was discretized in time and space dimensions and then iterative solvers handled nonlinear terms until the solution converged. To validate the results, they were compared to experimental measurements [8].

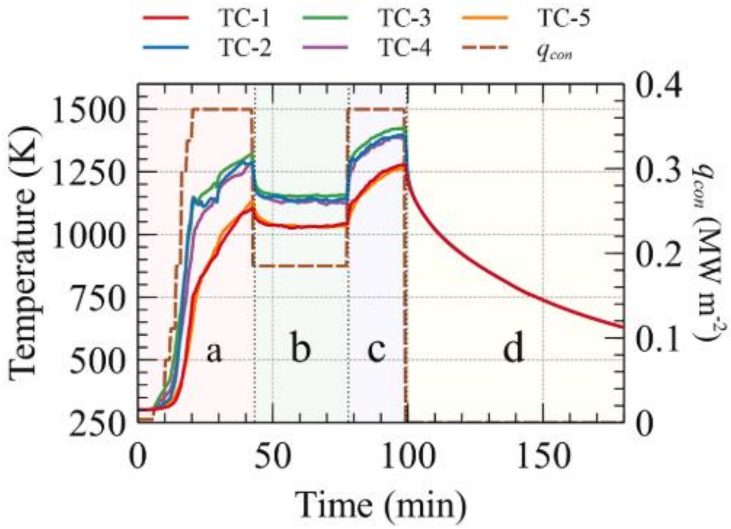

Figure 1 Temperature profiles for the experiment [8]

ii. Linear Fresnel Reflector Receivers

To focus the solar radiation into cavity receivers, Fresnel systems use arrays of mirrors, which require thermal modelling to reduce losses while trying to absorb energy as much as possible. In this case, the PDE governing the state include radiation balance term between cavity and wall:

$$-k\frac{\partial T_s}{\partial n} = h\left(T_s - T_f\right) + \varepsilon\sigma(T_s^4 - T_{sur}^4)$$

The solution of the previous PDE used FDM and Monte Carlo ray-tracing, and also they used successive under-relaxation scheme to deal with nonlinearity. Consequentially, the simulated results showed thermal efficiency close to experimental measurements by <3.4%, and they stated the importance of geometry optimization of the mirrors to enhance efficiency [9].

iii. Two-Phase Solar Collectors

The working fluid of evacuated tube collectors are being boiled and condensed repeatedly; Thus, the modelling of phase change is used to monitor vapor fractions since they can reduce efficiency. The following PDE is incorporated to describe this physical phenomenon:

$$\frac{\partial(\rho h)}{\partial t} + \nabla \cdot (\rho h\vec{u}) = \nabla \cdot (k\nabla T) + S_{phase}.$$

The **Stefan condition** governs the moving liquid–vapor interface:

$$\rho L \frac{dS}{dt} = k \frac{\partial T}{\partial n}$$

In this problem, FVM was used to solve the equation and obtain temperature profile of the working fluid. To ensure numerical stability, a semi-implicit scheme was used to handle phase transitions. As a result of the simulations, it was concluded that the collector efficiency dropped as the amount of vapour increased. This was due to the reduction of convective coefficients [10].

b. Magneto hydro dynamic (MHD) and Porous Media

i. Classical Porous MHD Flow

As an example of convection and electromagnetic damping, we can model the behaviour of fluids in porous channels under magnetic fields, which exists in metallurgical processes and nuclear cooling. The PDE of this system can be described as follows:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho} u, \partial t \partial u + u \partial x \partial u$$

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T + D_T \nabla^2 C + Q_{rad}.$$

To solve these equations, researchers [11] reduced the PDEs to ODEs using similarity transformations, then they solved them with a boundary-value problem solver. This approach resulted in high accuracy near the boundaries of the wall. As a result of the simulations, it was shown that magnetic fields reduce velocity profiles and heat transfer. At the same time, cross-diffusion effects led to the enhancement of thermal coupling in the channels.

ii. Casson Nanofluid with Darcy–Forchheimer Effects

When using nanofluids, it was noted that they improve heat transfer but cause non-Newtonian behaviour, which makes the mathematical model different from other types of fluids. PDEs for Casson nanofluids are described as follows:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\sigma B^2}{\rho c_p} u^2$$

The solution of this PDE included similarity variables to reduce PDEs to nonlinear ODEs. Then they can be solved with bvp4c [12].

c. Industrial Drying

Another application of PDEs is in the drying of porous products, which involves heat and moisture transfer. The modelling of heat transfer can be described as:

$$\rho V_p c_p \frac{\partial T}{\partial t} = hA(T_{air} - T) - h_{vap} k_m \rho_{air} (Y_e - Y),$$

$$\rho V_p \frac{\partial X}{\partial t} = -A k_m \rho_{air} (Y_e - Y)$$

This equation was solved using Neural-PDE famework using machine learning, which approximated PDE solutions in real time. The model reproduced experimental results [13].

3. Conclusion

The study of heat transfer through the lens of partial differential equations (PDEs) reveals not only the universality of mathematical frameworks across scientific domains but also their adaptability to increasingly complex problems. From simple conduction in solids to the coupled dynamics of magnetohydrodynamic (MHD) flows, multiphase boiling and condensation, and radiation–convection interactions in solar receivers, PDEs emerge as the central modeling tool that unifies these diverse phenomena.

The case studies examined illustrate this versatility. In solar thermal systems, PDE-based models captured conduction through tube walls, convection in working fluids, and radiation from concentrated solar flux, enabling predictive simulations of multitubular reactors and cavity receivers with excellent agreement to experiments. In solar collectors, the integration of phase-change PDEs and Stefan conditions allowed accurate representation of efficiency losses as vapor fractions increased. For MHD flows, the addition of Lorentz force terms, fractional derivatives, and fuzzy coefficients extended the classical energy equation to capture damping, memory effects, and parametric uncertainties. In industrial drying, coupled PDEs for heat and mass transfer were solved using neural–PDE hybrids, showing how machine learning can accelerate classical discretization methods without sacrificing accuracy. Finally, in the domain of numerical advances, meshfree methods such as the Average Radial Particle Method (ARPM) demonstrated how alternative discretization strategies can reduce computational costs while maintaining fidelity.

Taken together, these applications underscore several overarching insights:

1. PDEs as a unifying framework: Regardless of system complexity, the general energy conservation PDE serves as the backbone for thermal modeling. By adding or modifying terms, researchers adapt this equation to incorporate radiation, multiphase dynamics, electromagnetic effects, or uncertain parameters.

2. Solution diversity: Analytical solutions retain value for simple cases, but most real-world problems require numerical approaches. Finite volume and finite element methods dominate industrial practice, while bvp4c solvers, Laplace transforms, fuzzy exponential integrators, and neural surrogates represent specialized methods tailored to new challenges.

3. Validation and experimental coupling: The strongest models are those validated against experiments—whether thermocouple data in solar reactors, efficiency curves in collectors, or drying curves in ovens. PDEs not only predict but also explain measured data.

4. Adaptability to scale: PDE frameworks seamlessly extend from microfluidic flows and nanofluid suspensions to industrial-scale reactors and power transformers, providing continuity across scales.

Looking ahead, several trends will shape the future of PDE-based heat transfer modeling:

- Physics-Informed Machine Learning: PINNs and related frameworks will increasingly augment traditional solvers, embedding PDE constraints into neural networks to enable fast, real-time predictions.

- Fractional and Nonlocal Models: To capture anomalous transport and memory effects, fractional PDEs will continue to gain prominence in porous and heterogeneous systems.

- Uncertainty Quantification: Fuzzy PDEs and stochastic extensions will be crucial for designing systems under uncertain conditions (e.g., material variability, fluctuating boundary conditions).

- High-Performance and Green Computing: As models become more detailed, efficient algorithms, GPU acceleration, and reduced-order modeling will be necessary to balance accuracy with energy cost.

- Integration into Digital Twins: Real-time PDE-based simulations, validated by sensors, will serve as the computational engines of digital twins in renewable energy, manufacturing, and biomedical systems.

In conclusion, PDEs remain indispensable in advancing the science and engineering of heat transfer. Far from static, this field continues to evolve by incorporating new mathematics (fractional and fuzzy calculus), computational innovations (meshfree solvers, neural surrogates), and validation frameworks. The works reviewed demonstrate that the future of heat transfer modeling lies in hybrid methods that combine the rigor of PDEs with the flexibility of data-driven techniques, ensuring models are both physically grounded and computationally efficient.

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