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Theoretical Approach For Solving Scheduling On Unrelated Parallel Machines With Release Dates To Minimize Four Criterions

Hadi Hassan Hadi(1)
Department of Mathematics, College of Science, University of Mustansiriyah
hadi.h.hadi@uomustansiriyah.edu.iq

Marwa B. kadhem(2)
Department of Mathematics, College of Science, University of Mustansiriyah
Marwa.b.kadhim@uomustansiriyah.edu.iq

Hanan Ali chachan(3)
Department of Mathematics, College of Science, University of Mustansiriyah
hanan_altaai@yahoo.com

Abstract:
The challenge of scheduling a multi objective of unrelated parallel processors is a considered. During this environment, we scheduling a set of n jobs have to be on m unrelated parallel machines. A case wording is considered to schedule jobs in a cutting workshop to minimize the aggregate of the experiments' whole flow time, whole tardiness, whole earliness, and whole late work. The notion kernel beneath consideration with a chance release date. The dominant features and evaluate are proposed for substantial amount of randomly created.

Keywords: Dominance properties, Multi Objective Function, Release date, Unrelated parallel machine.

Introduction:

In this paper the literature have focused on parallel machine scheduling due to its diverse utilization. For instance, arranging truck deliveries, scheduling electronics, textiles, transportation, telecommunications, pharmaceutical, chemical, jobs wield diverse machines in a category technology, and all science filed (Fanjul-Peyro & Ruiz, 2010; Larroche et al, 2022). Machines are brought up as unrelated when their job processing times are arbitrary. Compared to uniform or identical machines, the unrelated parallel machine's theme is more global. Parallel machine scheduling is significant. Practically standpoint speaking, it is significant since, in a typical production setting, the majority of workstations have numerous machines. While there is a wealth of literary texts on parallel machine scheduling challenges, on the whole limited to condition where every part machines own the same processing time (Pm) or where machines operate at varied speeds but at a constant rate (Qm)

(Brucker, 2004; Pinedo, 2022). When a machine can behavior various jobs at various rates, the trajectory milieu is given vent to contain unrelated parallel machines Rm (Pinedo, 2022).

The majority of literary hold concentrated on parallel scheduling problems that are identical or uniform (Azizoglu & Kirca, 1999a; Della Croce et al, 2021; Dessouky et al, 1998; Dessouky, 1998; Edis et al, 2008; Liao & Lin, 2003; Mallek & Boudhar, 2024; Song et al, 2025). Inherent utilization of unrelated scheduling conundrum own be observed, although can been start about few outcomes. Horn creates (Horn, 1973) the unrelated scheduling issue and approved dominance properties to take down shorthand plight size, to diminish the flow time inside layout. Hariri and Potts (Hariri & Potts, 1991) remedy the unrelated schedule issue to diminish the completion time. Estates of the principal resolution for the unrelated schedule issue get going by Azizoglu and Kirca (Azizoglu & Kirca, 1999b), additionally lower limit of the harmonious cost action is accepted. build model to these characteristics, they find the optimum solution by branch and bound algorithm. Mokotoff and Chrétienne (Mokotoff & Chrétienne, 2002) operate with the polyhedral structure of the scheduling problem R//Cmax. Strong valid contrastare identified for the maximum completion time technique that serves as a foundation for the development of both a precise algorithm and an approximation algorithm. Liaw *et al*. (Liaw et al, 2003) consider the unrelated scheduling problem to minimize the total weighted tardiness. Efficient lower and upper bounds based on the solution of an assignment problem are developed and diverse dominance essentials are incorporated into a branch-and-bound algorithm.

In this paper, we deal with the unrelated parallel machine scheduling problem in which machine and job sequence dependent, with release dates and diverse performance measures with the purpose of lowering the aggregate of whole flow time, whole tardiness, whole earliness, and whole late work this problem is illustrate by $\mathbf{R_m / r_i / \sum_{i=1}^{m} \sum_{j=1}^{n} (C_{ij} + T_{ij} + E_{ij} + V_{ij})}$.

Problem Description:

In this section, we give some notations represent taken characterization to prove for $R_m / r_j / \sum_{i=1}^{m} \sum_{j=1}^{n} (C_{ij} + T_{ij} + E_{ij})$ problem.

The notations as follows:

J Set of jobs.

i Set of machines.

n Set of jobs (n = |J|).

m Set of the machines (m = |M|).

p_{ij} Processing time of job j on machine i where p_{ij}/s_i.
r_j Release date of job j
d_j Due date of job j.
s_i Relative processing speed of machine i
σ_i Partial schedule sequence of the jobs assigned to machine i.
C_{ij} (σ) Completion time of job j in partial schedule σ_i, $C_{i1}= r_1 + p_{ij}$, $C_{ij}=$ max { r_j, C_{ij-1}} + p_{ij}, j =2,…,n
T_{ij} (σ) Tardiness of job j in partial schedule σ_i, $T_{ij}=$ max { $C_{ij} - d_j$, 0}.
E_{ij} (σ) Earliness of job j in partial schedule σ_i, E_{ij} = max {$d_j - C_{ij}$, 0}.
V_{ij} (σ) Late work of job j in partial scheduleσ_i, V_{ij} = min {T_{ij}, p_{ij}}.

Integer Programming:

The $\boldsymbol{R_m / r_j / \sum_{i=1}^{m} \sum_{j=1}^{n} (C_{ij} + T_{ij} + E_{ij} + V_{ij})}$ the problem can be described as a hypothetical integer programming problem (P).

$$Z_{\mathcal{P}} = min \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{t=1}^{n} (C_{jk}^{t} X_{jk}^{t} + T_{jk}^{t} X_{jk}^{t} + E_{jk}^{t} X_{jk}^{t} + V_{jk}^{t} X_{jk}^{t} \quad (3.1)$$

Subject to

$$\sum_{k=1}^{m} \sum_{t=1}^{n} X_{jk}^{t} = 1 \qquad j = 1, 2, \dots, n\} \qquad (3.2)$$

$$\sum_{j=1}^{n} X_{jk}^{t} \leq 1 \qquad \left. \begin{matrix} t = 1, 2, \dots, n \\ k = 1, 2, \dots, m \end{matrix} \right\} \qquad (3.3)$$

$$C_{1k}^{t} = (r_1 + p_{lk}^{t})$$

$$C_{jk}^{t} = max \sum_{l=1}^{n} \sum_{s=1}^{t-1} (C_{j-1,k-1}^{t}, r_j) + p_{lk}^{t} \quad \left. \begin{matrix} j, t = 1, 2, \dots, n \\ k = 1, 2, \dots, m \end{matrix} \right\} \quad (3.4)$$

$$T_{jk}^{t} = \max(0, C_{jk}^{t} - d_j) \qquad \left. \begin{matrix} j, t = 1, 2, \dots, n \\ k = 1, 2, \dots, m \end{matrix} \right\} \qquad (3.5)$$

$$E_{jk}^{t} = \max(0, d_j - C_{jk}^{t}) \qquad \left. \begin{matrix} j, t = 1, 2, \dots, n \\ k = 1, 2, \dots, m \end{matrix} \right\} \qquad (3.6)$$

$$v_{jk}^{t} = \min\{ T_{jk}^{t}, p_{lk}^{t} \} \qquad \left. \begin{matrix} j, t = 1, 2, \dots, n \\ k = 1, 2, \dots, m \end{matrix} \right\} \qquad (3.7)$$

$$X_{jk}^{t} = 0 \text{ or } 1 \qquad \left. \begin{matrix} j, t = 1, 2, \dots, n \\ k = 1, 2, \dots, m \end{matrix} \right\} \qquad (3.8)$$

Where $\boldsymbol{C_{jk}^{t}}$ confirmed the completion time of job j is scheduling on machine k at $\boldsymbol{t^{th}}$ position and

$$X_{jk}^{t} = \begin{cases} 1 & if\ job\ j\ \ scheduled\ in\ t^{th} position\ on\ machine\ k \\ 0 & o.w \end{cases}$$

With regard to Constraint (3.2) status that every job is submitted to minutely one stance on machine, by contrast restriction (3.3) demands allocation of a maximum of one job to apiece stance on apiece machine. The completion time respecting job j whereas is scheduled at the $\boldsymbol{t^{th}}$ stance about machine k appears by restriction (3.4). restriction (3.5), (3.6) and(3.7)give the tardiness, earliness and late work of job j, when it is completion time C^t_{jk}in the th stance on machine k . Finally, restriction (3.8) is a variable that defines to show job j scheduling about machine k in stance t, then $X^t_{jk} = 1$ otherwise 0.

Dominance Rules:

We introduce some dominance rules for R_m $/r_j/\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij} + T_{ij} + E_{ij} + V_{ij})$ problem has been shown in theorem (4.1).

Let S_{1k}=($\beta_1 jk\ \beta_2$) and S_{2k}= ($\beta_1 kj\ \beta_2$) think about both sequences: β_1 and β_2, that infrequent subsequences of the remaining n-2 performances ,and the two jobs (j) in addition (k) which are adjacent jobs using identical device with $p_{ij} \le p_{ik}$ and $d_j \le d_k$.

allowing to Φ be completion time of β_1 and c be time when the first job in set β_2 starts, Let δ_{jki} be value of functions on which $\delta_{ikj}=\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij} + T_{ij} + E_{ij} + V_{ij})$ both jobs, subsequences (j), in addition (k) whereas (j) precedes (k) and δ_{jki} are functions value for the subsequences of both jobs (j) in addition (k) whereas (k) comes before (j).

Currently, while us analyze the adjustments in $\Delta_{ijk}=\delta_{ijk} - \delta_{ikj}$ and, with this subsequent instance $\Delta_{ijk} \le 0$ it displays that

- As to $\Delta_{ijk} < 0$ following this, at timeΦ, job *j* must come before job *k*.
- As to $\Delta_{ijk} > 0$ following this, at timeΦ, job *k* must come before job *j*.
- As to $\Delta_{ijk} = 0$ following this, there is no little change in schedule (j) or (k) first.

Theorem(4.1): About R_m $/r_j/\sum_{i=1}^{m}\sum_{j=1}^{n}(C_{ij} + T_{ij} + E_{ij} + V_{ij})$ problem if $p_{ij} \le p_{ik}$ for all *i*=1,2,…,m, , and $d_j \le d_k$where the jobs (*j*) in addition (*k*) are two adjacent jobs about the identical machine ,then the job (j) comes before the job (k) in optimal sequences.

First: We can categorize the situation in to the following categories if $r \le \Phi$ is true

Case 1: If $d_j \le \Phi + p_{ij}, d_k \le \Phi + p_{ik}$, jobs ($j$) and ($k$) at all times tardy.

Proof: Thus, we proved this one S_{1k} domains S_{2k}**,** all we have to complete are display $\Delta_{ijk}=\delta_{ijk} - \delta_{ikj}$ both jobs (j) in addition (k) are repeatedly tardy then $E_{ij}= E_{ik} = 0$ and $v_{ij} = p_{ij}, v_{ik} = p_{ik}$.

$\delta_{ijk}=$
$[(\Phi + p_{ij}) - r + (\Phi + p_{ij} - d_j) + 0 + p_{ij} + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik}) - d_k) + 0 + p_{ik})]$
$= [4\Phi + 5p_{ij} + 3p_{ik} - d_j - d_k - 2r]$
$\delta_{ikj} = [(\Phi + p_{ik}) - r + (\Phi + p_{ik}) - d_k) + 0 + p_{ik} + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij}) - d_j) + 0 + p_{ij}]$
$= [4\Phi + 5p_{ik} + 3p_{ij} - d_j - d_k - 2r]$
$\Delta_{ijk}=\delta_{ijk} - \delta_{ikj}$
$= -2p_{ik} + 2p_{ij} \leq 0$
Since $p_{ij} < p_{ik}$, then $j \rightarrow k$

Case 2: If $d_j \leq \Phi + p_{ij}$, $\Phi + p_{ik} \leq d_k \leq \Phi + p_{ij} + p_{ik}$, job (j) be constantly tardy and job (k) is tardy assuming that not scheduled at first.

Proof: since job j is tardy
$thus\ E_{ij} = 0$,job (k) is tardy if not scheduled first $T_{ik} = 0\ and\ \in V_{ik}\{c_{ik} - d_{ik}, 0\}$
(see job (k) that early or partial early)

(a) When $V_{ik} = 0$
$\delta_{ijk} = [(\Phi + p_{ij}) - r + (\Phi + p_{ij} - d_j) + 0 + p_{ij} + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik}) - d_k) + o + p_{ik})]$
$= [4\Phi + 3p_{ik} + 5p_{ij} - d_j - d_k - 2r]$
$\delta_{ikj} = [(\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij}]$
$= [2\Phi + 3p_{ij} + 2p_{ik} - d_j + d_k - 2r]$
$\Delta_{jki}=\delta_{jki} - \delta_{kji}$
$= 2\Phi + 2p_{ij} + p_{ik} - 2d_k > 0$
$then\ job\ k\ \ preced\ job\ j.$

(b) When $V_{ik} = c_{ik} - d_{ik} = \Phi + p_{ij} + p_{ik} - d_{ik}$
$\delta_{ijk} = [(\Phi + p_{ij}) - r + (\Phi + p_{ij} - d_{ij}) + 0 + p_{ij} + (\Phi + p_{ij} + p_{ik}) - r + ((\Phi + p_{ij} + p_{ik}) - d_{ik}) + o + p_{ik})]$
$= [4\Phi + 3p_{ik} + 5p_{ij} - d_{ij} - d_{ik} - 2r]$
$\delta_{ikj} = [(\Phi + p_{ik}) - r + 0 + (d_{ik} - \Phi - p_{ik}) + (\Phi + p_{ik} - d_{ik}) + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_{ij}) + 0 + p_{ij}]$
$= [3\Phi + 3p_{ij} + 3p_{ik} - d_{ij} - 2r]$
$\Delta_{ijk}=\delta_{ijk} - \delta_{ikj}$

$= \Phi + 2p_{ij} - d_{ik} > 0$
then job k preced job j.

Case 3: If $d_j \leq \Phi + p_{ij}$, $\Phi + p_{ij} + p_{ik} \leq d_k$, then(j) be constantly tardy and the (k) be constantly early.

Proof: Since(j) be constantly tardy and(k)be constantly early then $E_{ij} = 0$, $T_{ik} = 0$ *and* $V_{ik} \in \{c_{ij} - d_j, 0\}$

(a) When $V_{ik} = 0$ (i.e k is early)

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + (\Phi + p_{ij} - d_j) + 0 + p_{ij} + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d_k - \Phi - p_{ij} - p_{ik}) + 0]$$

$$= [2\Phi + 3p_{ij} - d_j + d_k - 2r]$$

$$\delta_{ikj} = [(\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij}]$$

$$= [2\Phi + 3p_{ij} + 2p_{ik} - d_j + d_k - 2r]$$

$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$

$$= -2p_{ik} \leq 0,$$

Since job j preced job k.

(b) When $V_{ik} = c_{ik} - d_k = \Phi + p_{ij} + p_{ik} - d_k$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + (\Phi + p_{ij} - d_j) + 0 + p_{ij} + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d_k - \Phi - p_{ij} - p_{ik}) + (\Phi + p_{ij} + p_{ik} - d_k)]$$

$$= [3\Phi + 4p_{ij} + p_{ik} - d_j - 2r]$$

$$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + (\Phi + p_{ik} - d_k) + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$$

$$= [3\Phi + 3p_{ij} + 3p_{ik} - d_j - 2r]$$

$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$

$$= p_{ij} - 2p_{ik} \leq 0 \rightarrow p_{ij} < p_{ik}$$

Since *job j precede job k* .

Case 4: If $\Phi + p_{ij} \leq d_j \leq d_k \leq \Phi + p_{ik}$, (see, the job (k) be constantly tardy and (j) is tardy assuming that not scheduled firstly.

Proof: Since $E_{ik} = 0$ if job j schedule first

Then $T_{ij} = 0$ and $V_{ij} \in \{c_{ij} - d_j, 0\}$

(a) When $V_{ij} = 0$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + 0 + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik} - d_k) + 0 + p_{ik}]$$

$$= [2\Phi + 3p_{ik} + 2p_{ij} + d_j - d_k - 2r]$$

$$\delta_{ikj} = [(\Phi + p_{ik}) - r + 0 + (\Phi + p_{ik} - d_k) + p_{ik} + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$$
$$= [4\Phi + 3p_{ij} + 5p_{ik} - d_j - d_k - 2r]$$
$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$
$$= (-2\Phi - p_{ij} - 2p_{ik} - 2d_j)$$

Since $job\ j\ precede\ job\ k$

(b) When $V_{ij} = c_{ij} - d_k$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + (\Phi + p_{ij} - d_j) + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik} - d_k) + 0 + p_{ik}]$$
$$= [3\Phi + 3p_{ik} + 3p_{ij} - d_k - 2r]$$
$$\delta_{ikj} = [(\Phi + p_{ik}) - r + (\Phi + p_{ik} - d_k) + 0 + p_{ik} + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$$
$$= [4\Phi + 3p_{ij} + 5p_{ik} - d_j - d_k - 2r]$$
$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$
$$= -\Phi - 2p_{ik} + d_j \leq 0, then\ job\ j\ preced\ job\ k.$$

Case 5: If $\Phi + p_{ij} \leq d_j$, $\Phi + p_{ik} \leq d_k \leq \Phi + p_{ij} + p_{ik}$, (see, both of the two jobs (j) and (k) are tardy assuming that they not scheduled first).

Proof:

(a) When $V_{ij} = 0, V_{ik} = 0,$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + 0 + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik} - d_k) + 0 + p_{ik}]$$
$$= [2\Phi + 2p_{ij} + 3p_{ik} + d_j - d_k - 2r]$$
$$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$$
$$= [2\Phi + 3p_{ij} + 2p_{ik} - d_j + d_k - 2r]$$
$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$
$$= p_{ik} - p_{ij} + 2d_j - 2d_k$$

$Since\ \Phi + p_{ij} \leq d_j\ and\ \Phi + p_{ik} \leq d_k \rightarrow p_{ik} - p_{ij} + 2d_j - 2d_k \leq 0$
$then\ job\ k\ preced\ job\ j.$

(b) When $V_{ij} = c_{ij} - d_j, V_{ik} = c_{ik} - d_k$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + (\Phi + p_{ij} - d_j) + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik} - d_k) + 0 + p_{ik}]$$
$$= [3\Phi + 3p_{ij} + 3p_{ik} - d_k - 2r]$$

$$\delta_{kji} = [((\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + (\Phi + p_{ik} - d_k) + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$$
$$= [3\Phi + 3p_{ij} + 3p_{ik} - d_j + 2r]$$
$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$
$$= -d_j - d_k \leq 0, then\ job\ k\ preced\ job\ j$$

(c) When $V_{ij} = 0, V_{ik} = c_{ik} - d_k$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + 0 + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik} - d_k) + 0 + p_{ik}]$$
$$= [2\Phi + 2p_{ij} + 3p_{ik} + d_j - d_k - 2r]$$
$$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + (\Phi + p_{ik} - d_k) + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$$
$$= [3\Phi + 3p_{ij} + 3p_{ik} - d_j - 2r]$$
$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$
$$= -\Phi - p_{ij} + 2d_j - d_k \leq 0, then\ job\ k\ preced\ job\ j.$$

(d) When $V_{ij} = c_{ij} - d_{ij}, V_{ik} = 0$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + (\Phi + p_{ij} - d_j) + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik} - d_k) + 0 + p_{ik}]$$
$$= [3\Phi + 3p_{ij} + 3p_{ik} - d_k - 2r]$$
$$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$$
$$= [2\Phi + 3p_{ij} + 2p_{ik} - d_j + d_k - 2r]$$
$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$
$$= \Phi + p_{ik} + d_j - 2d_k, Since\ \Phi + p_{ik} \leq d_k$$
$$= \Phi + p_{ik} + d_j - 2d_k \leq 0, then\ jobj\ preced\ job\ k.$$

Case 6: If $\Phi + p_{ij} \leq d_j \leq \Phi + p_{ij} + p_{ik} \leq d_k$ (see, job (j) is tardy assuming that not scheduled first, the job (k) be constantly early)

Proof:

(a) $when\ V_{ij} = 0, V_{ik} = 0$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + 0 + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d_k - \Phi - p_{ij} - p_{ik}) + 0]$$

$$= [d_j + d_k - 2r]$$

$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$

$= [2\Phi + 3p_{ij} + 2p_{ik} - d_j + d_k - 2r]$

$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$

$= -2\Phi - 3p_{ij} - 2p_{ik} + 2d_j \leq 0,$

then jobj preced job k.

$(\boldsymbol{b})$ *when* $V_{ij} = \Phi + p_{ij} - d_j,$

$$V_{ik} = \begin{cases} \Phi + p_{ik} - d_k & if\ (k) is\ 1^{st} \\ \Phi + p_{ij} + p_{ik} - d_k & if\ (k) is\ 2^{nd} \end{cases}$$

$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + (\Phi + p_{ij} - d_j) + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d_k - \Phi - p_{ij} - p_{ik}) + (\Phi + p_{ij} + p_{ik} - d_k)]$

$= [2\Phi + 2p_{ij} + p_{ik} - 2r]$

$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + (\Phi + p_{ik} - d_k) + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d_j) + 0 + p_{ij})]$

$= [3\Phi + 3p_{ij} + 3p_{ik} - d_j - 2r]$

$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$

$= -\Phi - p_{ij} - 2p_{ik} + d_j \leq 0,$

then jobj preced job k.

Case 7: If $\Phi + p_{ij} + p_{ik} \leq d_j \leq d_k$,(see, both (j) and (k) are early)

Proof:

(a) *when* $V_{ij} = 0, V_{ik} = 0$

$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + 0 + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d_k - \Phi - p_{ij} - p_{ik}) + 0]$

$= [d_j + d_k - 2r]$

$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d_k - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij} - p_{ik}) + 0)$

$= [d_j + d_k - 2r]$

$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$

$\Delta_{ijk} = [d_j + d_k - 2r] - [d_j + d_k - 2r] = 0,$

thenjob j or k first is irrelevant

$$(\boldsymbol{b}) when\ V_{ij} = \begin{cases} \Phi + p_{ij} - d_j & if\ (j) is\ 1^{st} \\ \Phi + p_{ij} + p_{ik} - d_j & if\ (j) is\ 2^{nd}, \end{cases}$$

$$V_{ik} = \begin{cases} \Phi + p_{ik} - d_k & if\ (k) is\ 1^{st} \\ \Phi + p_{ij} + p_{ik} - d_k & if\ (k) is\ 2^{nd} \end{cases}$$

$$\delta_{ijk} = [(\Phi + p_{ij}) - \mathrm{r} + 0 + (d_j - \Phi - p_{ij}) + (\Phi + p_{ij} - d_j) + (\Phi + p_{ij} + p_{ik}) - \mathrm{r} + 0 + (d_k - \Phi - p_{ij} - p_{ik}) + (\Phi + p_{ij} + p_{ik} - d_k)]$$

$$= [2\Phi + 2p_{ij} + p_{ik} - 2\mathrm{r}]$$

$$\delta_{ikj} = [((\Phi + p_{ik}) - \mathrm{r} + 0 + (d_k - \Phi - p_{ik}) + (\Phi + p_{ik} - d_k) + (\Phi + p_{ik} + p_{ij}) - \mathrm{r} + 0 + (d_j - \Phi - p_{ik} - p_{ij}) + (\Phi + p_{ij} + p_{ik} - d_j)]$$

$$= [2\Phi + p_{ij} + 2p_{ik} - 2\mathrm{r}]$$

$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$

$= p_{ij} - p_{ik} \leq 0$, then job j comes befor j.

$$(\boldsymbol{c})\ when\ V_{ij} = \begin{cases} \Phi + p_{ij} - d_j & if\ (j) is\ 1^{st} \\ \Phi + p_{ij} + p_{ik} - d_j & if\ (j) is\ 2^{nd} \end{cases},$$

$$V_{ik} = 0$$

$$\delta_{ijk} = [(\Phi + p_{ij}) - \mathrm{r} + 0 + (d_j - \Phi - p_{ij}) + (\Phi + p_{ij} - d_j) + (\Phi + p_{ij} + p_{ik}) - \mathrm{r} + 0 + (d_k - \Phi - p_{ij} - p_{ik}) + 0]$$

$$= [\Phi + p_{ij} + d_k - 2\mathrm{r}]$$

$$\delta_{ikj} = [((\Phi + p_{ik}) - \mathrm{r} + 0 + (d_k - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - \mathrm{r} + 0 + (d_j - \Phi - p_{ik} - p_{ij}) + (\Phi + p_{ik} + p_{ij} - d_j)]$$

$$= [\Phi + p_{ij} + p_{ik} + d_k - 2\mathrm{r}]$$

$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$

$= -p_{ik} \leq 0$, then job j comes before j.

Second: Assuming that $r > \Phi$,then we demonstrato that the theorem holds within alike technique as we close that earlier. (Integral)

Theorem (4.2): The jobs(j), and (k) are adjacent jobs in the problem $\mathcal{P}$ if $\mathrm{r_j} = \mathrm{r}$ and$\mathrm{d_j} = \mathrm{d}$ for all $(\mathrm{i} \in \mathrm{N})$ and if $\mathrm{p_{ij}} \leq \mathrm{p_{ik}}$ and $\mathrm{d_j} = \mathrm{d}$, then then job (j) should precedes job (k) for at least one sequences with the optimum value for problem p

Proof: Through resing the identical regulation as previously we can demonstrate the validity of this theorem in the following situations:

First $r \leq \Phi$

Case 1: If $d \leq \Phi + p_{ij} \leq \Phi + p_{ik}$ **(**see, both of the jobs (j), in addition to (k) are repeatedly tardy)

Proof: once the jobs (j), in addition to (k) are both tardy then $E_{ji} = E_{ki} = 0$ and $v_{ij} = p_{ij}$, $v_{ik} = p_{ik}$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + (\Phi + p_{ij} - d) + 0 + p_{ij} + (\Phi + p_{ij} + p_{ik}) - r + (\Phi + p_{ij} + p_{ik} - d) + 0 + p_{ik}]$$
$$= [4\Phi + 5p_{ij} + 3p_{ik} - 2d - 2r]$$
$$\delta_{ikj} = [(\Phi + p_{ik}) - r + (\Phi + p_{ik} - d) + 0 + p_{ik} + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d) + 0 + p_{ij})]$$
$$= [4\Phi + 5p_{ik} + 3p_{ij} - 2d - r)]$$
$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$
$$= 2p_{ij} - 2p_{ik} \leq 0, \text{then job j comes before k.}$$

Case 2: If $d \leq \Phi + p_{ij}, \Phi + p_{ij} + p_{ik} \leq d$ **,** the job (j) is tardy and the job (k) is early repeatedly)

Proof: As the job (j) is tardy repeatedly, then $E_{ij} = 0$, and the job (k) is repeatedly on time, then $T_{ik} = 0, V_{ik} = \{c_{ik} - d, 0\}$

(a) When $V_{ik} = 0$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + (\Phi + p_{ij} - d) + 0 + p_{ij} + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d - \Phi - p_{ij} - p_{ik}) + 0]$$
$$= [2\Phi + 3p_{ij} - 2r]$$
$$\delta_{ikj} = [(\Phi + p_{ik}) - r + 0 + (d - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d) + 0 + p_{ij}]$$
$$= [2\Phi + 2p_{ik} + 3p_{ij} - 2r]$$
$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$
$$= -2p_{ik} \text{ then job j precede job k.}$$

(b) When $V_{ik} = c_{ik} - d$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + (\Phi + p_{ij} - d) + 0 + p_{ij} + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d - \Phi - p_{ij} - p_{ik}) + (\Phi + p_{ij} + p_{ik} - d)]$$
$$= [3\Phi + 4p_{ij} + p_{ik} - d - 2r]$$
$$\delta_{ikj} = [(\Phi + p_{ik}) - r + 0 + (d - \Phi - p_{ik}) + (\Phi + p_{ik} - d) + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d) + 0 + p_{ij}]$$

$= [3\Phi + 3p_{ik} + 3p_{ij} - d - 2r]$

$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$

$= p_{ij} - 2p_{ik} \leq 0$, then job j precede job k.

Case 3: If $\Phi + p_{ij} \leq \Phi + p_{ij} + p_{ik} \leq d$(see, job (j) is tardy if not scheduled first and job(k)is repeatedly early)

Proof:

(a) when $V_{ij} = 0, V_{ik} = 0$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d - \Phi - p_{ij}) + 0 + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d - \Phi - p_{ij} - p_{ik}) + 0]$$

$= [2d - 2r]$

$$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d) + 0 + p_{ij})]$$

$= [2\Phi + 2p_{ik} + 3p_{ij} - 2r]$

$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$

$= [2d - 2r] - [2\Phi + 2p_{ik} + 3p_{ij} - 2r] = 0$

$= -2\Phi - 2p_{ik} - 3p_{ij} + 2d \leq 0$

(**b**)when $V_{ij} = \Phi + p_{ij} - d$,

$$V_{ik} = \begin{cases} \Phi + p_{ik} - d & \text{if (k)is } 1^{st} \\ \Phi + p_{ij} + p_{ik} - d & \text{if (k)is } 2^{nd} \end{cases}$$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d - \Phi - p_{ij}) + (\Phi + p_{ij} - d) + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d - \Phi - p_{ij} - p_{ik}) + (\Phi + p_{ij} + p_{ik} - d)$$

$= [2\Phi + 2p_{ij} + p_{ik} - 2r] =$

$$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d - \Phi - p_{ik}) + (\Phi + p_{ik} - d) + (\Phi + p_{ik} + p_{ij}) - r + (\Phi + p_{ik} + p_{ij} - d) + 0 + p_{ij})]$$

$= [3\Phi + 3p_{ij} + 3p_{ik} - d - 2r]$

$\Delta_{jki} = \delta_{jki} - \delta_{kji}$

$= -\Phi - p_{ij} + 2p_{ik} + d \leq 0$

Case 4: If $\Phi + p_{ij} + p_{ki} \leq d$ (jobs (j)in addition to (k) are early)

Proof:

(a) when $V_{ij} = 0, V_{ik} = 0$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d - \Phi - p_{ij}) + 0 + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d - \Phi - p_{ij} - p_{ik} + 0)]$$

$$= [2d - 2r]$$

$$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d - \Phi - p_{ik}) + 0 + (\Phi + p_{ik} + p_{ij}) - r + 0 + (d - \Phi - p_{ik} - p_{ij}) + 0]$$

$$= [2d - 2r]$$

$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$

$$= [2d - 2r] - [2d - 2r] = 0$$

then scheduling jor k is irrelevant

(**b**)when $V_{ij} = \begin{cases} \Phi + p_{ij} - d & \text{if } (j) \text{is } 1^{st} \\ \Phi + p_{ij} + p_{ik} - d & \text{if } (k) \text{is } 2^{nd'} \end{cases}$

$$V_{ik} = \begin{cases} \Phi + p_{ik} - d & \text{if } (k) \text{is } 1^{st} \\ \Phi + p_{ij} + p_{ik} - d & \text{if } (k) \text{is } 2^{nd} \end{cases}$$

$$\delta_{ijk} = [(\Phi + p_{ij}) - r + 0 + (d_j - \Phi - p_{ij}) + (\Phi + p_{ij} - d) + (\Phi + p_{ij} + p_{ik}) - r + 0 + (d - \Phi - p_{ij} - p_{ik}) + (\Phi + p_{ij} + p_{ik} - d)$$

$$= [2\Phi + 2p_{ij} + p_{ik} - 2r]$$

$$\delta_{ikj} = [((\Phi + p_{ik}) - r + 0 + (d - \Phi - p_{ik}) + (\Phi + p_{ik} - d) + (\Phi + p_{ik} + p_{ij}) - r + 0 + (d - \Phi - p_{ik} - p_{ij}) + (\Phi + p_{ij} + p_{ik} - d)]$$

$$= [2\Phi + p_{ij} + 2p_{ik} - 2r]$$

$$\Delta_{ijk} = \delta_{ijk} - \delta_{ikj}$$

$= p_{ij} - p_{ik} \leq 0$, then job j come before job k

Conclusion:

We focus on unrelated parallel-machine scheduling problems with release date. In the course of this study several dominant were developed, the aims of minimizing the sum of the experiments' whole flow time, whole tardiness, whole earliness, and whole late work to find the best solution to the hunger of the goal function. .the questions of its schedule unrelated parallel machines have been addressed on business. In the present study, our focus is on the NP-hard problem for scheduling n jobs on unrelated parallel machines Because this is a difficult issue, and in this search a set of dominance rules has been derived, which are considered to be an important focus, which is used in optimal and unsurpassable resolution methods because they reduce the time in the search for a contract that leads to the best solution. These cases were taken with conditions for business. The naturally inquire that be raised in the future work is whether our returns can be utilized to issues involving release dates that are undoubtedly pertinent to express contexts (like computer strategy). This significant issue is left for further study.

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نهج نظري لحل مسألة الجدولة على آلات موازية غير متجانسة ذات تواريخ إتاحة بغرض تقليل أربعة معايير

هادي حسن هادي (1)
قسم الرياضيات، كلية العلوم،
الجامعة المستنصرية، بغداد، العراق
07710639151

مروة بنيان كاظم (2)
قسم الرياضيات، كلية العلوم،
الجامعة المستنصرية، بغداد، العراق

حنان علي جاجان (3)
قسم الرياضيات، كلية العلوم،
الجامعة المستنصرية، بغداد، العراق

hadi.h.hadi@uomustansiriyah.edu.iq
Marwa.B.Kadhim@uomustansiriyah.edu.iq
hanan_altaai@yahoo.com

مستخلص البحث:
يتناول هذا البحث تحدي جدولة متعددة الأهداف على آلات موازية غير متجانسة، حيث يُجدول مجموعة مكونة من n مهمة على m آلة موازية غير متجانسة. في دراسة حالة تطبيقية في ورشة قطع، هدفت الدراسة إلى تقليل مجموع زمن التدفق الكلي، والتأخر الكلي، والمباكرة الكلية، وإجمالي حجم العمل المتأخر. يُؤخذ في الاعتبار تاريخ إتاحة كل مهمة، وقد تم اقتراح الخصائص الرئيسية ومعايير التقييم على عدد كبير من الحالات التي تم توليدها عشوائيًا.
الكلمات المفتاحية :الآلات الموازية غير المتجانسة، تاريخ الإتاحة، دالة الهدف متعددة المعايير، خصائص الهيمنة.
ملاحظة : هل البحث مستل من رسالة ماجستير او اطروحة دكتوراه ؟ كلا