مجلة كلية التربية الاساسية
كلية التربية الاساسية – الجامعة المستنصرية
Journal of the College of Basic Education Vol.31 (NO. 133) 2025, pp. 10-18

Numerical solution of Stochastic Variable-Order Fractional Differential Equations Using Euler Method

Douaa Khudheyer Abid
Directorate Of Edition of First Karkh, The Secondary School for Distinguished Boys in Harithiyah, Baghdad, Iraq
Douaa.Abid.da@gmail.com
07723554084

Abstract

The stochastic variable order fractional differential equations have so many difficulties in their analytic solution, therefore, numerical methods may be the most cases are the suitable methods of finding the solution. This research is presenting a numerical method for solving stochastic variable order fractional differential equations. The approach is based on the Eulers method, the calculations are written using the mathematical software MATLAB R2025a.

Key Words: Variable-Order Fractional Differential Equations , white noise , Euler Method.

1.Introduction:

Recently, much attention has been focused on fractional differential equations(FDEs) because they can be used to model computational processes with memory and inherited properties1 which are not exactly matched by the conventional integer models. These come from a very wide field of applications that include engineering, environmental, financial biological and physics among others[1] [2]. To further enhance the flexibility of modeling, variable order fractional derivatives have been recently introduced wherein the order can vary with respect to time or space hence able to capture more details in systems whose dynamics change non-uniformly[3] [4].

Stochastic variable-order fractional differential equations (SVFDEs) have recently become a powerful mathematical apparatus in modeling complex dynamical systems with randomness and memory effects. Indeed, they turn out to be deeply suitable for describing engineering, physical, environmental, biological, and financial phenomena when the system is based on the history of the phenomenon. Therefore, an accurate solution is a major challenge for this type of equation because of its complex dependence on randomness and variable order derivative.

However, most real phenomena are random and noisy. This has inspired the formulation of stochastic fractional differential equations(SFDEs) which inherit non-locality from fractional calculus and randomness from real phenomena [5]. This led to the integration of variable-order derivatives to the emergence of a new type of models known as stochastic variable-order fractional differential equations (SVFDEs).

The analytical solution of SVFDEs becomes cumbersome due to the randomness and the time-varying memory effect. Therefore, to suffice this more effective numerical method that approximates the solution, in this paper we use and improve Euler's classic method for approximating such solutions [6][7].

The aim of the research is to find an approximate solution to solve stochastic differential equations of variable order using Euler's method and calculating the approximate relative error to ensure the accuracy of the solution with the analytical solution.

2. Preliminaries

In this section, we briefly review key definitions and concepts related to fractional calculus, variable-order derivatives, and stochastic processes that will be used throughout the paper.

2.1 Definition (Caputo Fractional Derivative):

The Caputo fractional derivative of order $\alpha \in (0,1)$ of a function $f(t)$, is defined as[1]:

$$D_t^{\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau, \qquad 2.1$$

where Γ (.) is the Gamma function.

2.2 Definition (Variable-Order Caputo Derivative):

The variable-order Caputo derivative is defined as [2]:

let $\alpha(t) \in (0,1)$ be a continuous function the value of α depended on time t

$$D_t^{\alpha(t)} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha(t)} f'(\tau) d\tau, \qquad 2.2$$

2.3 Definition (Wiener Process):

A stochastic process $\{W(t), t \geq 0\}$is called a standard Wiener process (or Brownian motion) if [8]:-*

1. $W(0) = 0$ almost surely,
2. $W(t)$ has independent and normally distributed increments,
3. $W(t) - W(s) \sim N(0, t-s)$,
4. Paths of $W(t)$ are continuous almost surely.

2.4 Stochastic Fractional Differential Equation (SFDE):

A basic form of a stochastic fractional differential equation with variable order is given by:

$$D_t^{\alpha(t)} X(t) = a(t, X(t)) + b(t, X(t)) \dot{W}(t), \qquad 2.3$$

where $D_t^{\alpha(t)}X(t)$ is the variable-order fractional derivative, $a(t,X)$ and $b(t,X)$ are deterministic functions, and $\dot{W}(t)$ represents white noise (the formal derivative of a Wiener process).

2.4 Relative Error

Relative error puts absolute error into perspective because it compares the size of absolute error to the size of the true value. Note that the units drop off in this calculation, so relative error is dimensionless (unitless).[12]

$$Relative\ Error = E_i = \frac{|X_{i+1} - X_i|}{X_{i+1}} * 100$$

3. Mathematical model Problem Statement

We consider the following stochastic variable-order fractional differential equation (SVFDE) of the form:

$$D_t^{\alpha(t)}X(t) = a\big(t,X(t)\big) + b\big(t,X(t)\big)\dot{W}(t), \qquad t \in [0,T], \qquad 3.1$$

with the initial condition:

$$X(0) = X_0, \qquad 3.2$$

where:

• $D_t^{\alpha(t)}$ denotes the Caputo fractional derivative of variable order $\alpha(t) \in (0,1)$,

• $X(t)$ is the unknown stochastic process,

• $a(t,X)$ and $b(t,X)$ are known continuous functions,

• $W(t)$ is a standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathrm{P}, \{\mathcal{F}t\} \geq 0)$,

• X_0 is a given random variable independent of $W(t)$.

This type of SVFDE arises in modeling real-world systems exhibiting both memory effects and randomness, such as anomalous diffusion in heterogeneous media [9];[3]. The variable-order derivative captures time-varying memory effects, which makes it more flexible than constant-order models [10].

The aim of this paper is to construct a stable and convergent numerical method to approximate the solution of (3.1).

4. Fractional Euler Scheme Method to Solve SVFDE

To find an approximate solution to the stochastic variable-order fractional differential equation (SVFDE) given in (3.1), we use a modified version of the modified Euler-Maruyama method for Kabuto's variable-order derivatives. Let the time interval $[0,T]$ be divided into N uniform steps of size $h = \frac{T}{N}$, and let $t_n = \text{nh}$, for $n = 0,1,\ldots,N$. We denote the numerical approximation of $X(t_n)$ by X_n. The variable-order Caputo derivative at $t = t_n$ is approximated by the Grünwald–Letnikov-type discretization:

$$D_t^{\alpha(t_n)} X(t_n) \approx \frac{1}{h^{\alpha(t_n)}} \sum_{j=0}^{n} \omega_j^{(\alpha(t_n))} X_{n-j}, \qquad 4.1$$

where the weights are given by:

$$\omega_j^{\alpha} = (-1)^j \binom{\alpha}{j}, \text{ where} \binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} \qquad 4.2$$

Using this, the fractional Euler scheme for the SVFDE (3.1) becomes:

$$X_{n+1} = X_n + h^{\alpha(t_n)} \sum_{j=0}^{n} \omega_j^{(\alpha(t_n))} a(t_n, X_n) + b(t_n, X_n)\Delta W_n, \qquad 4.3$$

where

- X_{n+1} the approximate solution at time t_{n+1}.
- X_n the approximate solution at time t_n.
- $h^{\alpha(t_n)}$ the time step raised to the variable order $\alpha(t_n)$.
- $\omega_j^{(\alpha(t_n))}$ the weights associated with the fractional order $\alpha(t_n)$.
- $a(t_n, X_n)$ the deterministic part of the equation (usually represents a deterministic equation).
- $b(t_n, X_n)$ The stochastic part of the equation (interaction with the Wiener process).
- ΔW_n The Wiener increment, where $W(t)$ is the Wiener process, $\Delta W_n = W(t_{n+1}) - W(t_n) \sim N(0,T)$.

This method provides a first-order approximation in the weak sense, and its convergence can be established under Lipschitz continuity and linear growth conditions on $a(t,X)$ and $b(t,X)$ [3]; [11].

5. Numerical Experiment

In this part of the research, we will give mathematical examples to prove the variable_order Euler method for solving the stochastic variable_order fractional differential equation (SVFDE).

5.1 Example

Consider the following stochastic fractional differential equation with variable order:

$$D_t^{\alpha(t)} X(t) = -X(t) + 0.1\frac{dW(t)}{dt}, \qquad 0 < \alpha < 1 \qquad 5.1$$

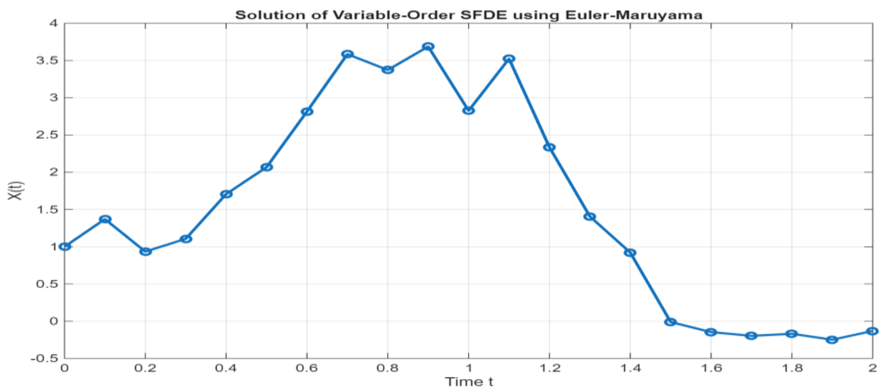
with $X_0 = 1$ and $\alpha(t) = 0.8 + 0.1\sin t$.

According to suggested methodology, the numerical solution to (5.1) in the interval $[0,2]$ with fixed step $h = 0.1$ and number of steps $n = 20$, the numerical approximation is given by:

$$X_{n+1} = X_n + h^{\alpha(t_n)} \sum_{j=0}^{n} \omega_j^{(\alpha(t_n))}(-1)X_j + 0.1\,\Delta W_n$$

Table1:Numerical Solution Table (Euler Method with Variable Order)

Steps n	Time t_n	X_{n+1}	E_n
0	0	1	0
1	0.1	1.3715	0.8715
2	0.2	0.9343	0.24855
3	0.3	1.1063	0.63915
4	0.4	1.71	0.115685
5	0.5	2.068	0.1213
6	0.6	2.82	0.1786
7	0.7	3.59	0.218
8	0.8	3.373	0.1578
9	0.9	3.69	0.20035
10	1	2.83	0.985
11	1.1	3.5243	0.21093
12	1.2	2.34	0.57785
13	1.3	1.41	0.24
14	1.4	0.923	0.218
15	1.5	-0.011032	0.47253
16	1.6	-0.146	0.14048
17	1.7	-0.197	0.124
18	1.8	-0.17	0.0715
19	1.9	-0.25	0.165
20	2	-0.131	0.006



5.2 Example

Consider the following stochastic fractional differential equation with variable order:

$$D_t^{\alpha(t)} X(t) = -0.8X(t) + 0.3\frac{dW(t)}{dt}, \qquad 0 < \alpha < 1 \qquad 5.2$$

with $X_0 = 1$ and $\alpha(t) = 0.9 - 0.05t$.

According to suggested methodology, the numerical solution to (5.2) in the interval $[0,1]$ with fixed step $h = 0.1$ and number of steps $n = 10$, the numerical approximation is given by:

$$X_{n+1} = X_n + h^{\alpha(t_n)} \sum_{j=0}^{n} \omega_j^{(\alpha(t_n))} (-0.8) X_j + 0.3\Delta W_n$$

Table2:Numerical Solution Table (Euler Method with Variable Order)

Steps n	**Time t_n**	X_{n+1}	E_n
0	0	1	0
1	0.1	1.34	0.84
2	0.2	0.884	0.214
3	0.3	1.04	0.598
4	0.4	1.596	0.1076
5	0.5	1.9	0.10102
6	0.6	2.61	0.166
7	0.7	3.314	0.1009
8	0.8	3.12	0.1463
9	0.9	3.41	0.185
10	1	2.62	0.1125

When there is difficulty in finding the analytical solution, the approximate relative error is employed to verify the accuracy and convergence of the numerical solution. This is considered one of the commonly used methods.
As we know, when the error close to the zero, it helps us to determine the stability of the method used to solve the stochastic variable_order fractional differential equation (SVFDE).
Using Mathlab software in both examples, we get (table1, Fig.1, table2, Fig.2) for Eulr Maruyama method.

6.Conclusions

In this paper, we derive a numerical method that is the simplest method, Euler's method, for solving fractional-order differential equations of a physical nature. The randomness of the equation and the change in the time-dependent order between zero and one made it difficult to find the solution analytically, so we resorted to numerical methods to find the solution in an approximate way to solve this type of equations, and the methods can be generalized to a larger number of variable orders From the results we obtained for the variable $X(t)$, we can conclude the behavior of the equation and how randomness and damping affect the variable $X(t)$. If this equation represents any phenomenon on the ground (physics, engineering, finance, environment, biology), then the results can provide conclusions, theories, and scientific facts based on these results.

For future work the following problems could be recommended:

1- Solve the problem it was given by equation (3.1) by using Runge-Kutta(RK) method
2- Use the method to solve SVFEDs with multi_order or with non-linear diffusion coefficients
3- Derives other methods that highest order.

Through these points, we can apply numerical methods to models of mathematics finance, biology and physics.

Reference

[1] Podlubny, I. (1999). Fractional Differential Equations. Academic Press.

[2] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and Applications of Fractional Differential Equations. Elsevier.

[3] Li, C., & Zeng, F. (2015). Numerical Methods for Fractional Calculus. CRC Press.

[4] Wang, Z., & Li, Y. (2020). “Numerical methods for variable-order fractional differential equations: A review”, Computers & Mathematics with Applications.

[5] Zhang, Y., & Sun, Z. (2012). “Finite difference methods for stochastic fractional differential equations”, Applied Mathematics and Computation.

[6] Atangana, A., & Blaenau, D. (2016). “New fractional derivatives with non-local and non-singular kernel”, Chaos, Solitons & Fractals.

[7] Chen, M., & Qian, Y. (2021). “Hybrid numerical algorithms for solving stochastic fractional differential equations”, Mathematics and Computers in Simulation.

[8] Øksendal, B. (2003). Stochastic Differential Equations: An Introduction with Applications. Springer.

[9] Sun, H., Zhang, Y., Blaenau, D., Chen, W., & Chen, Y. (2018). A new collection of real-world applications of fractional calculus in science and engineering. Commun. Nonlinear Sci. Number. Simul., 64, 213–231.

[10] Sun, H., & Chen, W. (2009). Variable-order fractional differential operators in anomalous diffusion modeling. Physical A, 388(21), 4586–4592.

[11] Garrappa, R. (2018). Numerical methods for fractional differential equations: A survey and a software tutorial. Mathematics, 6(2), 16.

[12] Hanewinckel, Michiel, ed. (2001). “Theory of Errors.” *Encyclopedia of Mathematics*. Springer Science Business Media B.V. / Kluwer Academic Publishers. ISBN 978-1-55608-010-4.

الحل العددي للمعادلات التفاضلية الكسرية العشوائية ذات الرتبة المتغيرة باستخدام طريقة أويلر

مستخلص البحث:
يواجه حل المعادلات التفاضلية الكسرية العشوائية ذات الرتبة المتغيرة صعوباتٍ تحليلية عديدة، لذا تُعدّ الطرق العددية هي الأكثر ملاءمة لإيجاد الحل. لذلك، يُقدّم هذا البحث طريقة عددية لحل المعادلات التفاضلية الكسرية العشوائية ذات الرتبة المتغيرة. تعتمد هذه الطريقة على طريقة أويلر، وتُكتب الحسابات باستخدام برنامج الرياضيات.MATLAB R2025a

الكلمات المفتاحية: المعادلات التفاضلية الكسرية ذات الرتبة المتغيرة، الضوضاء البيضاء، طريقة أويلر.

ملاحظة : هل البحث مستل من رسالة ماجستير او اطروحة دكتوراه ؟ كلا