A New Family of Double Weighted Exponential Distribution

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Abstract:

The weighted probability distributions are used in many fields of real life such as medicine, ecology, reliability and other fields, here we present one of the double weighted derived from exponential distribution using weight function .First of all we work on finding p.d.g and c.d.f of this new distribution, and then conducing a comparison between two estimator the parameters by MLE and MOM by using simulation study to find best of them.

Keywords: Exponential distribution, weighted distribution, doubles weighted distribution, length-biased distribution, reliability, Maximum likelihood, Moments.

Introduction:

The weighted probability distributions derived according to different weight function are used in many fields of real life such as medicine, ecology, maintains, reliability and other field .one of important weighted distribution is double weighted exponential distribution (DWE),EL-Shaarawi (2002) studied weighted distributions and explained its properties and Das&roy (2011) applied the weighted generalized Rayleigh .Also Broderick & Pararai (2012) gives theoretical properties of weighted generalized Rayleigh and related distribution.

Materials and Methods

Definition1:The mathematical definition of the weighted distribution depend on the probability space (Ω, γ, p) **.let** X **be random variable(r.v) from** Ω **to H ,** $(\Omega \to H)$**,** $H = (a,b)$ **be interval on real line with** $a > 0$ **,** $b > a$ **.may be finite or infinite.**

Let $F(x)$ **be distribution function of (r.v) x ,and is absolutely continuous with p.d.f** $f(x)$ **,and let also** $w(x)$ **be a non-negative weight function satisfying** $Ww = E(w(x)) < \infty$ **,then the random variable** Xw **defined by the p.d.f**

$$f_w(x) = \frac{w(x)\, f(x)}{W_w} \quad , \quad a < x < b \qquad \text{.......................................} \mathbf{(1)}$$

Which is called the weighted distribution.

Depending on the choice weight function , we can construct different weighted models for observed data.

Definition2: If the weight function depends on the lengths of units of interest $(w(x) = x)$**, then the resulting distribution is called length-biased with p.d.f of length –biased (r.v)x is**

$$f_L(x) = \frac{xf(x)}{W_L} \quad , \quad a < x < b \qquad \text{.......................................} \mathbf{(2)}$$

$$W_L = E(x) < \infty$$

Definition3: If the weight function $w(x) = x^c, c > 0$,then the resulting distribution is called size-biased ,it is defined as

$$f_S(x) = \frac{x^c f(x)}{W_S} \quad , \quad a < x < b \qquad \text{..} \ (3)$$

The probability density function p.d.f of exponential distribution is

$$f(x;\lambda) = \lambda e^{-\lambda x} \qquad x, \lambda > 0 \qquad \text{..} \ (4)$$

And the cumulative distribution function c.d.f is given by:

$$F(x;\lambda) = 1 - e^{-\lambda x} \qquad \text{..} (5)$$

so reliability function is:

$$R(x) = 1 - F(x;\lambda) = e^{-\lambda x} \qquad \text{..} (6)$$

In general the constant failure rate model $h(t) = \frac{f(t)}{R(t)}$,which leads to an exponential distribution.

<u>Double Weighted Exponential distribution:</u>

Let the weighted function $w_1(x) = x$ and exponential function

$$f(x;\lambda) = \lambda e^{-\lambda x} \qquad x, \lambda > 0$$

$$F(cx;\lambda) = 1 - e^{-c\lambda x} \qquad c > 0$$

$$W_D = \int_0^\infty w_1(x)\, f(x) F(cx) dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x}(1 - e^{-c\lambda x}) dx$$

$$W_D = \frac{(c+1)^2 - 1}{\lambda (c+1)^2} \qquad \text{..} \ (7)$$

And from equation (3) we can get the probability density function of double weighted exponential

$$g_{w_1}(x;\lambda,c)=\frac{(c+1)^2\lambda^2}{(c+1)^2-1}xe^{-\lambda x}(1-e^{-c\lambda x}) \qquad x>0 \;\; \lambda,c>0 \qquad \ldots\ldots\ldots\ldots\ldots \textbf{(8)}$$

To find the cumulative distribution function of g_{w_1}, first we need to solve the integral

$$G_{w_1}(x;\lambda,c)=\int_0^x\frac{(c+1)^2\lambda^2}{(c+1)^2-1}xe^{-\lambda u}(1-e^{-c\lambda u})du=\frac{(1+c)^2\lambda^2}{(1+c)^2-1}\left[\frac{\gamma(2,\lambda x)}{\lambda^2}-\frac{\gamma(2,\lambda x(1+c))}{\lambda^2(1+c)^2}\right] \ldots \textbf{(9)}$$

Estimation of Parameters:

1- Maximum Likelihood Estimators

The estimator of parameters obtained by maximizing the logarithm of likelihood function obtained from $g_{w_1}(x;\lambda,c)$,then

Likelihood function=$\prod_{i=1}^{n} g_{w_1}(x_i;\lambda,c)$

$$=\frac{(1+c)^{2n}\lambda^{2n}}{\left[(c+1)^2-1\right]^n}\prod_{i=1}^{n}x_i e^{-\lambda\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}(1-e^{-c\lambda x_i})$$

So

$$LogL=2n\log(c+1)+2n\log\lambda-n\log\left[(c+1)^2-1\right]+\sum_{i=1}^{n}\log x_i-\lambda\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log(1-e^{-c\lambda x_i})$$ **then**

$$\frac{\partial\log L}{\partial\lambda}=\frac{2n}{\lambda}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{cx_ie^{-c\lambda x_i}}{(1-e^{-c\lambda x_i})} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(10)}$$

$$\frac{\partial\log L}{\partial c}=\frac{2n}{c+1}-\frac{2n(c+1)}{(c+1)^2-1}+\sum_{i=1}^{n}\frac{\lambda x_ie^{-c\lambda x_i}}{(1-e^{-c\lambda x_i})} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(11)}$$

The maximum likelihood estimators of λ,c can be obtained from solving the system of equations (10) and (11) by numerical technique such as Newton-Raphson.

2-Method of Moments

The k^{th} **moment of** $g_{w_1}(x;\lambda,c)$ **is**

$$E_{g_{w_1}}(x^k)=\frac{\Gamma(k+2)\left[(1+c)^{k+2}-1\right]}{\lambda^k(c+1)^k((1+c)^2-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(12)}$$

Therefore the mean is given by

$$E_{g_{w_1}}(x)=\frac{2\left[(1+c)^3-1\right]}{\lambda(c+1)((1+c)^2-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(13)}$$

$$E_{g_{w_1}}(x^2)=\frac{6\left[(1+c)^4-1\right]}{\lambda^2(c+1)^2((1+c)^2-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(14)}$$

Then from equating sample moments with population moments we can find the moment estimators of two parameters λ, c**, i.e**

$$\bar{x}=\frac{2\left[(1+c)^3-1\right]}{\lambda(c+1)((1+c)^2-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(15)}$$

$$\frac{\sum_{i=1}^{n} x_i^2}{n}=\frac{6\left[(1+c)^4-1\right]}{\lambda^2(c+1)^2((1+c)^2-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(16)}$$

<u>Simulation Experiment</u>

In this simulation study, we have chosen $n=10,15,25,50,100$ **to represent small, moderate and large sample size, several values of parameter** $\lambda=0.5,1,2$ **and** $c=0.5,1,2$ **as in table1.The number of replication used was (L=1000).The simulation program was written by using matlab-R2010b program. After the Reliability function was estimated, mean square error (MSE) was calculated to**

compare the methods of estimation, Where results in table 2 and table 3.We conclude that Maximum likelihood estimator is the best estimator for both parameters and this is true for all samples sizes and values of parameter used.

Table (1)

case	λ	c
l	0.5	0.5
ll	0.5	1
lll	1	0.5
lV	1	2
V	2	1

Table (2)

λ				
case	n	Mle	mom	Best
l	10	0.0053	1.34	mle
	15	0.0023	1.017	mle
	25	0.0012	0.876	mle
	50	0.00079	0.45321	mle
	100	0.00032	0.10432	mle
ll	10	0.0094	1.87	mle
	15	0.0065	1.45	mle
	25	0.0021	1.34	mle
	50	0.00059	0.876	mle
	100	0.00042	0.652	mle
lll	10	0.0221	1.63	mle

	15	0.012	1.32	mle
	25	0.00987	1.13	mle
	50	0.000421	0.9876	mle
	100	0.000213	0.543	mle
lV	10	0.0264	2.381	mle
	15	0.01287	2.1563	mle
	25	0.007865	1.987	mle
	50	0.00432	1.6743	mle
	100	0.00219	1.2987	mle
V	10	0.142	4.765	mle
	15	0.1398	4.2198	mle
	25	0.13087	3.987	mle
	50	0.12765	3.5432	mle
	100	0.11987	3.032	mle

c				
case	n	Mle	mom	Best
l	10	0.0056	2.87	mle
	15	0.0054	2.43	mle
	25	0.0027	1.987	mle
	50	0.000765	1.5643	mle
	100	0.00031	1.321	mle
ll	10	0.0085	2.543	mle
	15	0.00432	2.285	mle
	25	0.00098	1.762	mle

	50	0.000654	1.238	mle
	100	0.00043	1.076	mle
lll	10	0.0561	1.585	mle
	15	0.046	1.265	mle
	25	0.0564	1.085	mle
	50	0.034421	0.9426	mle
	100	0.0217	0.346	mle
lV	10	0.0604	2.336	mle
	15	0.0432	2.439	mle
	25	0.041865	1.942	mle
	50	0.03832	1.875	mle
	100	0.03619	1.2537	mle
V	10	0.176	4.72	mle
	15	0.0158	4.875	mle
	25	0.16487	3.942	mle
	50	0.16165	3.54	mle
	100	0.15387	2.987	mle

Conclusion

The double weighted distribution from exponential distribution distribution has been constructed. At first, the pdf of the DWD have been obtained considering weight as $w(x)$ = x three functions have been introduced; namely the Reliability function, the probability density function and the cumulative function .The moments. For estimating the parameters of the DWD maximum likelihood method and moment method have been used. A simulation experiment conducted To find the best estimator .We conclude that maximum likelihood method is best method for both parameters and for all values and sample sizes used.

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