

**EXISTENCE SOLUTION OF  
NONLINEAR INTEGRABLE  
EQUATION WITH N-TH TIMES  
INTEGRAL**

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## الملخص

في هذا البحث, قمنا باعطاء بعض التعاريف حول التقريب للدالة, ثم اخذنا معادلة لدالة غير خطية قابلة للتكامل نون من المرات وقمنا بدراسة الاستمرارية ووجود الحل لها حيث هذه المعادلة هي:

$$f(x_1, x_2, \dots, x_n) = f(x_{01}, x_{02}, \dots, x_{0n}) + \int_a^b \int_a^b \dots \int_a^b k(s_1, s_2, \dots, s_n, x_1, x_2, \dots, x_n, f(s_1, s_2, \dots, s_n)) ds_1 ds_2 \dots ds_n \quad \dots(1)$$

where  $k \in \left[ \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \right] \rightarrow \mathbb{R}$ ,  $f \in \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ .

## Abstract

In this paper, we given some definitions about convergence of function, then we taken an equation of nonlinear n-th integrable function and studied the continuity and existence of solution of this equation while the equation has the form:

$$f(x_1, x_2, \dots, x_n) = f(x_{01}, x_{02}, \dots, x_{0n}) + \int_a^b \int_a^b \dots \int_a^b k(s_1, s_2, \dots, s_n, x_1, x_2, \dots, x_n, f(s_1, s_2, \dots, s_n)) ds_1 ds_2 \dots ds_n \quad \dots(1)$$

where  $k \in \left[ \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \right] \rightarrow \mathbb{R}$ ,  $f \in \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ .

## 1.Introduction

we know that every continues function is integrable, but in the case of integrable function there is some conditions to be continues. We want to have these conditions to the important of the integral subject and continuity.

The object of integration is to find lengths, areas, and volumes of irregular shapes,[1], [2].

In [6] Maria Dobritoiu and Ana-Maria Dobritoiu found the existence and uniqueness to n-th order nonlinear integral equation In [4] Akbar H. Borzabadi, and Omid S. Fard found the existence of solution to first kind single integral equation. In this paper we'll find the existence to equation (1).

**2.Continuity:**

Definition: [3]

The sequence  $f_n$  converges uniformly to function  $f$  if for every  $\varepsilon > 0$  there is an integer  $N$  such that  $n \geq N$  implies

$$|f_n(t) - f(t)| \leq \varepsilon$$

Where  $t \in R$ .

By using the above definition, we'll give the following new theorem:

**Theorem 1:**

Let  $f_n$  be a sequence of bounded, continues, real-valued integrable functions on a metric space  $(x,d)$ . If  $f_n \rightarrow f$  uniformly, then  $f$  is uniformly continues where  $f$  is the function (1).

**Proof:**

By the triangle inequality:

$$\left| f(x_1, x_2, \dots, x_n) - f_n(y_1, y_2, \dots, y_n) \right| \leq \left| f(x_1, x_2, \dots, x_n) - f_n(x_1, x_2, \dots, x_n) \right| + \left| f_n(x_1, x_2, \dots, x_n) - f_n(y_1, y_2, \dots, y_n) \right| + \left| f_n(y_1, y_2, \dots, y_n) - f(y_1, y_2, \dots, y_n) \right| .$$

Since  $f_n \rightarrow f$  uniformly, there is an  $n$  such that:

$$\left| f(x_1, x_2, \dots, x_n) - f_n(x_1, x_2, \dots, x_n) \right| < \varepsilon/3, \quad \left| f_n(y_1, y_2, \dots, y_n) - f(y_1, y_2, \dots, y_n) \right| < \varepsilon/3 \quad \text{for } x_i, y_i \in R^n.$$

Since  $f_n$  from above is continues, there is a  $\delta > 0$  such that  $d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) < \delta$  this implies

$$\left| f_n(x_1, x_2, \dots, x_n) - f_n(y_1, y_2, \dots, y_n) \right| < \varepsilon/3.$$

It follows that:

$d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) < \delta$  implies  $\left| f(x_1, x_2, \dots, x_n) - f(y_1, y_2, \dots, y_n) \right| < \varepsilon$  this means  $f$  is continues at  $(x_1, x_2, \dots, x_n)$ .

**3.existence of solution:**

We'll give the following new theorem to prove the existence of solution of (1).

**Theorem 2:**

Suppose at  $k \in C [R^n \times R^n \times R^n]$  and  $\partial k$  (partial derivative of  $k$ ) exist and it is bounded vector function on  $R^n \times R^n$ . then if  $\{f_n(s_1, s_2, \dots, s_n)\}$  be uniformly convergence sequence of solutions to a function  $f^*(s_1, s_2, \dots, s_n)$  then a function solution to (1).

**Proof:**

Suppose  $\{f_n(s_1, s_2, \dots, s_n)\}$  be a uniformly convergent sequence which is obtained from solution of  $f^*(s_1, s_2, \dots, s_n)$  with every refined partition. Define

$Y_n(s_1, s_2, \dots, s_n) = k(s_1, s_2, \dots, s_n, X_n, f_n(s_1, s_2, \dots, s_n))$ ,  $\forall s_1, s_2, \dots, s_n \in R^n$  and  $X_n$  is the sequence of  $x_1, x_2, \dots, x_n$ .

Trivially  $Y_n(s) \in C [R^n \times R^n \times R^n]$ . For given  $\varepsilon > 0$  and for

each  $s_i \in \mathbb{R}^n$  we have

$$| Y_m(s_1, s_2, \dots, s_n) - Y_n(s_1, s_2, \dots, s_n) | = | k(s_1, s_2, \dots, s_n, X_n, f_m(s_1, s_2, \dots, s_n)) - k(s_1, s_2, \dots, s_n, X_n, f_n(s_1, s_2, \dots, s_n)) |, X_n = x_1, x_2, \dots, x_n.$$

$$\leq | k(s_1, s_2, \dots, s_n, X_n, f_m(s_1, s_2, \dots, s_n)) - k(s_1, s_2, \dots, s_n, X_n, f^*(s_1, s_2, \dots, s_n)) | + | k(s_1, s_2, \dots, s_n, X_n, f^*(s_1, s_2, \dots, s_n)) - k(s_1, s_2, \dots, s_n, X_n, f_n(s_1, s_2, \dots, s_n)) |$$

By [5]

$$\begin{aligned} & | k(s_1, s_2, \dots, s_n, X_n, f_m(s_1, s_2, \dots, s_n)) - k(s_1, s_2, \dots, s_n, X_n, f^*(s_1, s_2, \dots, s_n)) | \\ & + | k(s_1, s_2, \dots, s_n, X_n, f^*(s_1, s_2, \dots, s_n)) - k(s_1, s_2, \dots, s_n, X_n, f_n(s_1, s_2, \dots, s_n)) | \\ & \leq | [\partial k(s_1, s_2, \dots, s_n, x_1, x_2, \dots, x_n, f_m(s_1, s_2, \dots, s_n)) / \partial X_n] (X - X_m) | \\ & + | [\partial k(s_1, s_2, \dots, s_n, X_n, f_n(s_1, s_2, \dots, s_n)) / \partial f(s_1, s_2, \dots, s_n)] (f_m(s_1, s_2, \dots, s_n) - f^*(s_1, s_2, \dots, s_n)) | + \\ & | [\partial k(s_1, s_2, \dots, s_n, x_1, x_2, \dots, x_n, f_m(s_1, s_2, \dots, s_n)) / \partial X_n] (X - X_n) | \\ & + | [\partial k(s_1, s_2, \dots, s_n, X_n, f_n(s_1, s_2, \dots, s_n)) / \partial f(s_1, s_2, \dots, s_n)] (f_n(s_1, s_2, \dots, s_n) - f^*(s_1, s_2, \dots, s_n)) | \\ & \leq M(|X_n - X| + |f_n(s_1, s_2, \dots, s_n) - f^*(s_1, s_2, \dots, s_n)|) \\ & + (|X_m - X| + |f_m(s_1, s_2, \dots, s_n) - f^*(s_1, s_2, \dots, s_n)|) \end{aligned}$$

where  $M$  is a large positive number such that

$$| [\partial k(s_1, s_2, \dots, s_n, X_n, f_n(s_1, s_2, \dots, s_n)) / \partial X_n] | \leq M.$$

Now for  $\varepsilon > 0$ , there is a positive integer number  $N$  such that for each  $l > N$

$$| X_j - X | < \varepsilon' \quad \text{and} \quad | f_j(s_1, s_2, \dots, s_n) - f^*(s_1, s_2, \dots, s_n) | < \varepsilon'.$$

Thus for given  $\varepsilon' > 0$  and the above  $N$ , for each

$m, n \geq N$  we have

$$| Y_m(s_1, s_2, \dots, s_n) - Y_n(s_1, s_2, \dots, s_n) | < 4M \varepsilon'.$$

and thus by choosing  $\varepsilon' = \varepsilon / 4M$

we have

$$| Y_m(s_1, s_2, \dots, s_n) - Y_n(s_1, s_2, \dots, s_n) | < \varepsilon,$$

that is the sequence  $\{Y_n\}$  is uniformly convergence,

, and thus  $Y_n \rightarrow Y$  where  $Y(s_1, s_2, \dots, s_n) = k(s_1, s_2, \dots, s_n, X, f^*(s_1, s_2, \dots, s_n))$ ,

Now we can result:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_a^b \int_a^b \dots \int_a^b Y_n(s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n = \\ & \int_a^b \int_a^b \dots \int_a^b \lim_{n \rightarrow \infty} Y_n(s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n = \int_a^b \int_a^b \dots \int_a^b Y(s_1, s_2, \dots, s_n) ds_1 ds_2 \dots ds_n \\ & = \int_a^b \int_a^b \dots \int_a^b k(s_1, s_2, \dots, s_n, x_1, x_2, \dots, x_n, f^*(s_1, s_2, \dots, s_n)) ds_1 ds_2 \dots ds_n \end{aligned}$$

and it means that  $\{f^*(s_1, s_2, \dots, s_n)\}$  is a solution of (1).

The following example is applying to the above theorem by taking  $n=3$ :

**Example (1)**

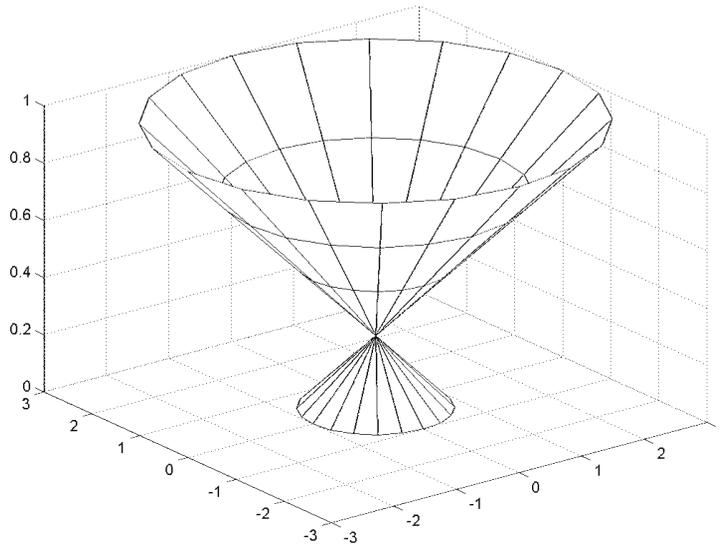
The nonlinear n-th times integrable equation is

$$f(x_1, x_2, x_3) = 1/4[\exp(8 x_1 x_2 x_3)-1] [\exp(-11 x_1 x_2 x_3)-\exp(-3 x_1 x_2 x_3)-\exp(-7 x_1 x_2 x_3)+\exp(x_1 x_2 x_3)]/ x_1^3 x_2^3 x_3^3, \text{ where}$$

$$f(x_1, x_2, x_3) = 0 + \int_{-1}^3 \int_{-1}^3 \int_{-1}^3 s_1 s_2 \exp((f(s_1, s_2, s_3)(x_1 x_2 x_3)) ds_1 ds_2 ds_3$$

The exact solution of this equation is  $f(s_1, s_2, s_3) = s_1^2 - s_2^2 - s_3$ .

The following figure shows by using matlab program the exact solution  $f(s_1, s_2, s_3)$ :

**References:**

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- [5] P. DARANIA, A. EBADIAN, and A. V. OSKOI," Linearization Method For Solving Nonlinear Integral Equations" Vol. 2006, Article ID 73714, Pages 1–10, 23 October 2005.
- [6] Maria Dobritoiu and Ana-Maria Dobritoiu," A Generalization Of Some Integral Equations" vol.1,2009.