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Direct Sum of Intuitionistic Fuzzy Quasi Injective S-Act

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Abstract

In this paper, we develop concept of direct sum of $s-$act that mean L be intuitionistic fuzzy $s-$act on A. With $\mu_L(0) = \mu_{L_i}(0)$ and $\lambda_L(0) = \lambda_{L_i}(0)$ $\forall i$, then L is called intuitionistic fuzzy direct sum of L_i (for short IF $-$direct sum-act) and denoted by $L = \oplus_{i=1}^{n} L_i$. We gave example of intuitionistic fuzzy direct sum act and theorem that show direct sum $s-$act of intuitionistic fuzzy quasi-injective $s-$act such that if $L = (\mu_L, \lambda_L) = L_i \oplus L_j$ be IFQI $-$ act, then L_i is L_j $-$injective and its theorem converse is true, see (5). proposition and corollaries have been introduced.

Keywords: Fuzzy Injective s-act, Fuzzy quasi injective s-act, Direct sum s-act, Intuitionistic fuzzy s-act.

Introduction

Injective and quasi-injective $s-$act was studied by [1, 2] after that [3] initiated the direct sum $s-$act as a generalization of direct sum module. In application point of view, fuzzy $s-$act was widely used in many engineering applications such as model fuzzy discrete system. This has been giving the motivation in recent years to investigate the fuzziness of different kinds of $s-$acts.

In 2021, fuzzy injective, fuzzy quasi-injective s-acts and fuzzy direct sum were defined and studied by many others (see [4, 5]).

Intuitionistic fuzzy injective (IFI-act) and Intuitionistic fuzzy quasi-injective s-acts (IFQI-act) have been introduced by [6].

In this research, we investigated the direct sum of Intuitionistic fuzzy quasi-injective s-acts. First, we gave example of the Intuitionistic fuzzy direct sum s-act and $L = L_i \oplus L_j$ is Intuitionistic fuzzy quasi-injective s-act, then L_i is $L_j -$injective, see Theorem (4) and the converse theorem is true, see Theorem (5) such that if L_i is $L_j -$injective, then L is Intuitionistic fuzzy quasi-injective $s -$act.

Preliminary

In this paper, we will introduce some notation and state a few well-known results that will be used in the sequel, and will recall some definitions, theorems, lemmas, corollaries, results and notations.

Definition 1 [2]: An $s -$act A is called quasi-injective $s -$act (QI $-$act) if it is A $-$injective, such that if any subact B of A and any $s -$ homomorphism $f: B \to A$, there exists $s -$endomorphism $g: A \to A$, that mean g is an extension of f, that is, $goi = f$, where i is the inclusion mapping of B in to A.

Definition 2 [3]: The disjoint union $A \cup B$ or $(A \oplus B)$, is the direct sum of two $s -$acts A and B, in the case of being B is subact of A, then B is called direct summand of A and if there exist any subact C of A, then $B \cap C = 0$ and $B \cup C = A$.

Definition 3 [4]: Suppose that A is B $-$injective and let $\mu_A$and μ_B be two fuzzy $s -$acts, that mean μ_A is $\mu_B -$ injective if for each fuzzy subact μ_C of μ_B and for each fuzzy $s -$homomorphism $f: \mu_C \to \mu_B$ be extended to a fuzzy $s -$homomorphism$g: \mu_A \to \mu_B$.

Definition 4 [5]: Let A be quasi injective $s -$act (QI $-$act) and μ_A is a fuzzy $s -$act, such that μ_A is called fuzzy quasi injective $s -$act (FQI $-$act) if for each fuzzy subact μ_B of π_A and for each fuzzy $s -$homomorphism $f: \mu_B \to \mu_A$ of there exists a fuzzy $s -$homomorphism $g: \mu_A \to \mu_A$ that mean g is an extension of f.

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Lemma 5 [5]: Let A be quasi-injective, then A is A_j −injective s −act, such that $A = A_i \oplus A_j$ for $j \in \{1,2\}$.

Definition 6 [5]: Let $A = \oplus_{i=1}^{n} A_i$ and let μ_A be the fuzzy s −act on A with $\mu_A(o) = \mu_{Ai}(o)$ for all i, then μ_A is called the direct sum of the fuzzy s −act μ_{Ai}, such that is denoted by $\mu_A = \oplus_{i=1}^{n} \mu_{Ai}$.

Definition 7 [6]: Let A be B −injective s −acts if $L = (\mu_L, \lambda_L)$ and $U = (\mu_U, \lambda_U)$ be IF −act on A and B respectively, then L is U −injective if

$\mu_L\left(f(b)\right) \geq \mu_U(b)$ and $\lambda_L\left(f(b)\right) \leq \lambda_U(b)$ $\forall f \in \mathrm{Hom}\,(B, A)$ so L is said to be intuitionistic fuzzy injective s −act (IFI −ac).

Definition 8 [6]: Let A be QI −act and $L = (\mu_L, \lambda_L)$ is any IF −act on A, then L is said to be intuitionistic fuzzy quasi-injective s −act (IFQI −act) if

$\mu_L\left(f(a)\right) \geq \mu_L(a)$ and $\lambda_L\left(f(a)\right) \leq \lambda_L(a)$ for each $f \in \mathrm{End}\,(A)$ and $a \in A$.

Definition 9 [7]: Let A and B be two s − acts, then the function $f: A \to B$ is called s −homomorphis, if $f(as) = f(a)s$ $\forall a \in A$ and $s \in S$.

Definition 10 [7]: Let A and B be two s −acts, then A is B −injective if for all s −subact C of B and any homomorphism $f: C \to A$ can be extended to a homomorphismg: $B \to A$.

Proposition 11 [7]: Let $A = \pi {Ai}_{i \in I}$ be injective if and only if A_i is A_j −injective.

Definition 12 [7]: Let S be a monoid and A be anon-empty set, this set is said to be s −actA, if we have $f: S \times A \to A, s - t\,(s, a) \to sa = f\,(s, a)$ such that $S(ta) = (st)a$ $\forall a \in A$ and $s, t \in S$. If S has an identity element $1(1.a = a, \forall a \in A)$ then we called B a unitary right s −act and if a semigroup S with zero and the following hold: $0 - a = 0$ $\forall a \in A$, then we called A is aright s −act with zero element.

Definition 13 [8]: Let (A, μ_A) and (B, μ_B) be two s −acts, then an S −homomorphism $f: A \to B$ is called a fuzzy s −homomorphism from (A, μ_A) to (B, μ_B) if, $\mu_B(f(a)) \geq \mu_A(a)$ $\forall a \in A$.

Definition 14 [9]: Let A be an s −act and $\mu: A \to [0, 1]$ be a fuzzy set on A then (A, μ) is called fuzzy s −act it is denoted by (F −act) if $\mu(as) \geq \mu(a)$ for every $a \in A$, $s \in S$.

Definition 15 [10]: Let S be a monoid with two sided zero and A be a rights-act with zero element θ. Then IFs $-$ act $L = (\mu_{(L)}, \lambda_L)$ in A is said to be an intuitionistic fuzzy s $-$ act (IF $-$act) if

1. $\mu_L(as) \geq \mu_L(a)\ \forall a \in A$ and $s \in S$, 2. $\lambda_L(as) \leq \lambda_L(a)\ \forall a \in A$ and $s \in S$.

Definition 16 [11]: A fuzzy set on A is the set of order pairs $(x, \mu(x))\ \forall x \in A$ (universal set) $B = \{(x, \mu(x)): \forall x \in A\}$ the fuzzy power set $(F(B))$; $\mu: A \to [0, 1]$ is called membership function μ (x): The grade of membership.

Definition 17 [12]: L in a nonempty set, A is an object having the shape $L = \{\mu_L(a), \lambda_L(a), \forall a \in A\}$. When the functions $\mu_L: A \to [0,1]$ denoted the degree of membership and $\lambda_L: A \to [0,1]$ denoted the degree of non-membership so that $0 \leq \mu_L(a) + \lambda_L(a) \leq 1$ then L is called Intuition fuzzy set (IFS) we can write L in form $L = (\mu_L, \lambda_L)$.

Direct sum of IF QI $-$ Act

In this section, we will define and investigate Direct sum of IFQI $-$act.

Proposition 1: Let $A = \oplus A_i$ be an s $-$act where A_i are subact of A for each i. If $L_i = (\mu_{L_i}, \lambda_{L_i})$ $(1 \leq i \leq n)$ are IF $-$ act on A_i, then $L = (\mu_L, \lambda_L)$ where $\mu_L(a) = \min\{\mu_{Li}(ai) | i = 1,2, \dots, n\}$ and $\lambda_L(a = \max\{\lambda_{Li}(ai) | i = 1,2, \dots, n\}$ $\forall a \in A$, $ai \in A_i$ is an IF $-$act on A.

Proof: Since each L_i is IF $-$act on A for every $s \in S$ and $a = ai \in A$,

Then $\mu_{(L)}(sa) = \wedge\{\mu_{(Li)}(sai) | i = 1,2, \dots \dots, n\}$

$= \mu_{Li}(sai)$, for some i

$\geq \mu_{Li}(a_i)$ (Li is IF $-$ act)

$\geq \mu_L(a)$

and $\lambda_L(sa) = \vee\{\lambda_{L_i}(sai) | i = 1,2, \dots \dots, n\}$

$= \lambda_{Li}(sai)$, for some i

$\leq \lambda_{Li}(a_i)$ (Li is IF $-$ act)

$\leq \lambda_L(a)$

There for L is an IF $-$act on A. □

The above proposition gives as the motivation to define the following:

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Definition 2: Let L be IF $-$act on A in proposition (3.1) with $\mu_L(o) = \mu_{Li}(o)$ and $\lambda_L(o) = \lambda_{Li}(o)$ $\forall i$, then L is called Intuitionistic fuzzy direct sum of L_i (IF $-$direct sum-act) and denoted by $L = \oplus_{i=1}^{n} L_i$.

Example 3: Let $L = L_1 \oplus L_2$ where L_1 be IF $-$act on $S_1 = \{0, a, b\}$ where S_1 is S $-$act over itself under operation $a^2 = ab = a$ and $b^2 = ba = b$ and L_2 be IF $-$act on $S_2 = \{x, y, z\}$ where S_2 is S $-$act over itself under operation $y^2 = yz = zy = y$ and $z^2 = z$. It is easy to check that IF $-$act, where:

$$\mu_{L1}(s_1) = \begin{cases} \frac{1}{5} & \text{if } s_1 = a, b \\ 1 & \text{if } s_1 = 0 \end{cases}, \quad \mu_{L2}(s_2) = \begin{cases} \frac{1}{4} & \text{if } s_2 = x, y \\ \frac{1}{5} & \text{if } s_2 = z \end{cases} \text{ and}$$

$$\lambda_{L1}(s_1) = \begin{cases} \frac{4}{5} & \text{if } s_1 = a, b \\ 0 & \text{if } s_1 = 0 \end{cases}, \quad \lambda_{L2}(s_2) = \begin{cases} \frac{3}{4} & \text{if } s_2 = x, y \\ \frac{4}{5} & \text{if } s_2 = z \end{cases}$$

Then $S = S_1 \oplus S_2 = \{(s_1, s_2) | s_1 \in S_1 \text{ and } s_2 \in S_2\}$ such that $S_1 \oplus S_2 = \{(0,x), (0,y), (0,z), (a,x), (a,y), (a,z), (b,x), (b,y), (b,z)\}$, $L = (\mu_L, \lambda_L)$ can be find as follows:

$\forall\, s = (s_1, s_2) \in S$, $\mu_L(s) = \mu_{L1}(s_1) \wedge \mu_{L2}(s_2)$ and $\lambda_L(s) = \lambda_{L1}(s_1) \vee \lambda_{L2}(s_2)$

If $s = (0, x)$, then:

$$\mu_L(s) = \mu_{L1}(0) \wedge \mu_{L2}(x) = 1 \wedge \frac{1}{4} = \frac{1}{4}$$

And

$$\lambda_L(s) = \lambda_{L1}(0) \vee \lambda_{L2}(x) = 0 \vee \frac{3}{4} = \frac{3}{4}$$

If $s = (a, y)$, then:

$$\mu_L(s) = \mu_{L1}(a) \wedge \mu_{L2}(y) = \frac{1}{5} \wedge \frac{1}{4}$$

$= \frac{1}{5}$

And

$\lambda_L(s) = \lambda_{L1}(a) \vee \lambda_{L2}(y)$

$= \frac{4}{5} \vee \frac{3}{4}$

$= \frac{4}{5}$

In the same way we find μ_L, λ_L for each $s \in S = \oplus_{i=1}^{2} S_i$.

So, we have:

$$\mu_L(s) = \begin{cases} \frac{1}{4} & \text{if} \quad s = \{(0,x),(0,y) \\ \frac{1}{5} & \text{if} \quad s = \{(a,y),(b,z),(0,z),(a,x),(a,z),(b,x),(b,y) \end{cases}$$

And

$$\lambda_L(s) = \begin{cases} \frac{3}{4} & \text{if} \quad s = \{(0,x),(0,y) \\ \frac{4}{5} & \text{if} \quad s = \{(a,y),(b,z),(0,z),(a,x),(a,z),(b,x),(b,y) \end{cases}$$

$\Rightarrow$ L is IF −direct sum-act. □

Theorem 4: Let $L = (\mu_L, \lambda_L) = L_i \oplus L_j$ be IFQI −act on the s −act $A = A_i \oplus A_j$, then L_i is L_j−injective-act for $i, j \in \{1,2\}$.

Proof: $L = (\mu_L, \lambda_L)$ is IFQI −act, then: i- $A = A_i \oplus A_j$ is QI −act, by lemma (2.5), A_i is A_j −injective. ii- $\mu_L(\psi(a)) \geq \mu_L(a)$ and $\lambda_L(\psi(a)) \leq \lambda_L(a)$ $\forall \psi \in \text{Hom}(A, A)$.

To prove L_i is L_j −injective-act for each $\alpha \in \text{Hom}(A_j, A_i)$, consider inclusion homomorphism $i: Ai \to A = A_i \oplus A_j$ then $\varphi = i o \alpha: A_j \to A$ (see Figure 1).

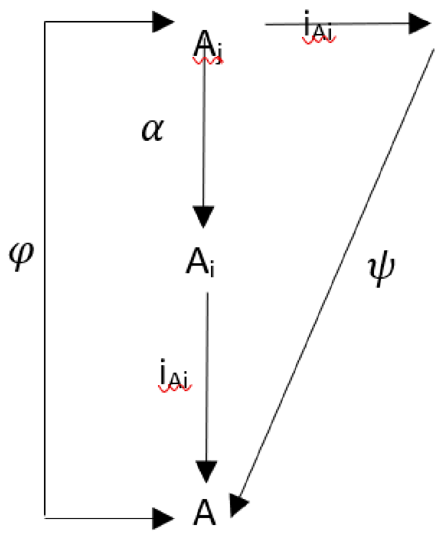

Figure 1: IFQI s-act

But A is QI −act, then there exist ψ ∈ End (A) which be extension of φ such that $\psi|A_j =$ ……... (1)

We have $\mu_L(\psi(a)) \geq \mu_L(a)\ \forall a \in A$ ……. (2)

Since $A = Ai \oplus A_j$ if $a_j \in A_j$, then $a_j = (0, a_j) \in A$

From (2), we get:

$\mu_{(L)}\left(\psi(a_j)\right) \geq \mu_{(L)}(a_j)\ \forall a_j \in A_j$ ……. (3)

Also $\mu_L(a_j) = \mu_L(0, a_j) = \mu_{Li}(0) \wedge \mu_{Lj}(a_j)$

$= \mu_{Lj}(a_j)$ ……… (4)

From (1) we get: $\psi(a_j) = \varphi(a_j) = i(\alpha(a_j)) = \alpha(a_j)$

Therefor $\mu_L(\psi(a_j)) = \mu_L(\alpha(a_j)) = \mu_L(\alpha(a_j), 0) = \mu_{Li}(\alpha(a_j)) \wedge \mu_{Lj}(0)$

$= \mu_{Li}(\alpha(a_j))$ ….... (5)

From (3), (4) and (5) we get:

$\mu_{Li}(\alpha(a_j)) \geq \mu_{Lj}(a_j)\ \forall \alpha \in Hom(A_j, A_i)$ and $a_j \in A_j$ and we have

$\lambda_L(\psi(a)) \leq \lambda_L(a)\ \forall a \in A$ ……… (6)

Since $A = A_i \oplus A_j$ if $a_j \in A_j$, then: $a_j = (0, a_j) \in A$, from (6) we get:

$\lambda_L(\psi(a_j)) \leq \lambda_L(a_j)\ \forall a_j \in A_j j$ ……. (7)

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Also $\lambda_L(a_j) = \lambda_L(0, a_j) = \lambda_{Li}(0) \vee \lambda_{Lj}(a_j)$

$= \lambda_{Lj}(a_j)$ ……… (8)

From (1) we get: $\psi(a_j) = \varphi(a_j) = i(\alpha(a_j)) = \alpha(a_j)$

Therefor $\lambda_L(\psi(a_j)) \leq \lambda_L(a_j) = \lambda_L(\alpha(a_j), 0) = \lambda_{Li}(\alpha(a_j)) \vee \lambda_{Lj}(0)$

$= \lambda_{Li}(\alpha(a_j))$………… (9)

From (7), (8) and (9) we get:

$\lambda_{Li}(\alpha(a_j)) \leq \lambda_{Lj}(a_j)$ $\forall \alpha \in \mathrm{Hom}(A_j, A_i)$ and $a_j \in A_j$

$\Rightarrow L_i$ is L_j −injective s −act. □

The following theorem gives the convers of theorem (3.4).

Theorem (5) convers of the theorem 4: Let L_i be two IF −acts on $A_i (i = 1,2)$ such that $L = L_i \oplus L_j$ on s − act $(A = A_i \oplus A_j)$ and if L_i is L_j − injective for $i, j \in \{1,2\}$, then L is IFQI −act.

Proof: Suppose L_i is L_j −injective, then A_i is A_j −injective so by (2.11) ⇒A is QI −act.

Since L_i is L_j −injective for $i, j \in \{1,2\}$

$\mu_{Li}(\alpha_j(a_j)) \geq \mu_{Lj}(a_j)$ ………… (1)

$\lambda_{Li}(\alpha_j(a_j)) \leq \lambda_{Lj}(a_j)$ ………… (2)

$\forall \alpha_j \in \mathrm{Hom}(A_j, A_i)$

Also L_j is L_i –injective.

$\mu_{Lj}(\alpha_i(a_i)) \geq \mu_{Li}(a_i)$ ………… (3)

$\lambda_{Lj}(\alpha_i(a_i)) \leq \lambda_{Li}(a_i)$ …………... (4)

From (1) and (3) we get:

$\mu_{Li}(\alpha_j(a_j)) \wedge \mu_{Lj}(\alpha_i(a_i)) \geq \mu_{Lj}(a_j) \wedge \mu_{Li}(a_i)$

$\mu_L(\alpha_j(a_j), \alpha_i(a_i)) \geq \mu_L(a_i, a_j)$

$\mu_L(\alpha(a)) \geq \mu_L(a)$

From (2) and (4) we get:

$\lambda_{Li}(\alpha_j(a_j)) \vee \lambda_{Lj}(\alpha_i(a_i)) \leq \lambda_{Lj}(a_j) \vee \lambda_{Li}(a_i)$

$\lambda_L(\alpha_j(a_j), \alpha_i(a_i)) \geq \lambda_L(a_i, a_j)$

$\lambda_L(\alpha(a)) \geq \lambda_L(a)$

$\forall \alpha \in Hom(A, A)$ and $a \in A$.

There for L is L −injective-act

⇒ L is IFQI-act. □

Corollary 6

Let $A = \oplus_{i=1}^{n} A_i$, where A_i are the S −subacts of A. Let Li = $(\mu_i\ \lambda_i)$ are IF −acts of Ai such that $L = \oplus_{i=1}^{n} L_i$. If L is IFQ I −act then L_i is L_j −injective for all i, j $(1 \leq i, j \leq n)$.

Corollary 7

Let $A = \oplus_{i=1}^{n} A_i$, where A_i are the S −subacts of A. Let $L_i = (\mu i\ \lambda i)$ are IF −acts of Ai such that $L = \oplus_{i=1}^{n} L_i$. If L_i is Lj −injective for all i, j $(1 \leq i, j \leq n)$ then L is IFQ I −act.

Conclusion

In this paper, we have introduced the notion of direct sum of intuitionistic fuzzy s −act, this notion is very useful in studying some properties of intuitionistic fuzzy s −act. We discussed the direct sum of intuitionistic fuzzy quasi-injective s −act and showed that if $L = L_i \oplus L_j$ is IFQI −act, then L_i is L_j −injective and the converse is true. The present work can be extended of the study of properties of the fuzziness and the intuitionistic fuzziness of s-act.

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