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"A Simulation Study on Parameter Estimation for the Lomax-Pareto Mixture Distribution"

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Abstract

This paper proposed a new distribution called the Lomax-Pareto distribution using the theory of mean generalized distributions applied to the Lomax distribution. The new distribution (Lomax-Pareto distribution) had four parameters (α, β, θ,ʎ) , and some crucial functions were developed for this distribution, such as the joint function, the probability function, and the reliability function. Two estimation procedures for the parameters of this new distribution were proposed, namely, the maximum likelihood estimation and the least squares estimation methods. Monte Carlo simulation was then used to compare methods, where samples of sizes 10, 25, 50, and 100 were generated, and criteria such as MSE and Bias were used. The results showed that the LSE method was better than the MLE method, as it yielded some good and acceptable results.

Keywords

Odd Lomax-Pareto distribution, Lomax distribution, Pareto distribution, MLE, LSE.

- **Introduction**

The search for new statistical distributions forms one of the major undertakings towards understanding complicated phenomena seen in real life [1]. Next, this work introduces a new statistical distribution, named a new statistical distribution (Lomax Pareto), by mixing the

Lomax distribution with the Pareto distribution [2]. The new distribution preserves unique properties of both distributions and offers flexibility to model a large set of data [3]. Furthermore, estimation methods such as (maximum likelihood, least squares) were used to estimate the parameters of the Lomax-Pareto distribution [4] [5], The researcher Mintode and others identified the Lomax distribution and the Lomax family, its main purpose, and derived its mathematical properties (moments, entropy function, etc.) The maximum likelihood method was used to estimate the parameters[17] , The researcher Jamilu and others postulated a class of distributions with two parameters called the Lomax-G class, and derived the properties (moments, Renny entropy, and other properties). and these were used to estimate the parameters of simulation[18]. For example, maximum probability and at least classes used projection methods, and simulation was used to verify the results. However, the LSE method was better than the MLE method because of its good results

2- Pareto distribution

The general form of Pareto distribution as below [10], [11]:

$$G(x) = 1 - \left(\frac{\theta}{x}\right)^{\lambda} \qquad x > 0; \quad \alpha = \theta$$

$$g(x) = \lambda \theta^{\lambda} x^{-\lambda-1} \qquad \beta = \lambda$$

By the substitution the previous PDF and CDF in equation (1) and (2), the result will be getting the CDF and PDF to Lomax-Pareto Distribution:

$$F(x) = 1 - \beta^{\alpha}\left[\beta + \frac{1}{\left(\frac{\theta}{x}\right)^{\lambda}} - 1\right]^{-\alpha} \qquad \ldots(1)$$

$$f(x) = \left[\frac{\alpha\beta^{\alpha}\lambda x^{\lambda-1}}{\theta^{\lambda}}\right]\left[(\beta - 1) + \frac{x^{\lambda}}{\theta^{\lambda}}\right]^{-(\alpha+1)} \quad \ldots(2)$$

3-Lomax Distribution

Lomax delivery (also known as Pareto Type II distribution) is used, and it is used for a well-known science, business failure, as well as medical and biological science, income and wealth inequality, and engineering science; moreover, it is applied in dataset for life and reliability [6].

In fact, Lomax distribution (also known as Pareto Type II distribution) is used, and it is used for modeling a well-known science, business failure, medical and biological science, income and wealth, and engineering science; therefore, it supports lifetime and credibility dataset. Furthermore, Lomax distribution has been used for modeling some real activities, such as income and wealth data [7]. For example, for a fixed size data [8], and for reliability and life testing [4], as well as for the receiver; operating characteristic (ROC) curve analysis [9], and so on. Thus, the cumulative distribution function for Lomax generalized will be :

$$F(x) = 1 - \beta^{\alpha} \left[\beta + \frac{G(x)}{1-G(x)}\right]^{-\alpha} \qquad \ldots(3)$$

The cumulative density function (CDF) for Lomax generalized will be:

$$\mathrm{f}(x) = \alpha\beta^{\alpha} \frac{g(x)}{[1-G(x)]^2} \left[\beta + \frac{G(x)}{1-G(x)}\right]^{-(\alpha+1)} \qquad \ldots(4)$$

Where g(x) و G(x) represented the PDF and CDF for each other continuous distribution.

3- Reliability Function

effectiveness and longevity of items and systems. The likelihood that a system will successfully carry out its intended function within a given time frame and operational conditions is represented by this. Reliability research is closely linked to failure time probability distributions, which are used to characterize the lifespan of parts or systems. Dependability is an essential tool for comprehending the temporal behavior of failure and forecasting the lifespan of the system in the context of probability distributions like the Lomax and Pareto distributions.The system's capacity to run flawlessly for extended periods of time is indicated by a higher reliability function. Reliability analysis is practically crucial in many domains, such as engineering system design, electronics, aviation, nuclear, and biostatistical and medical applications.Because it incorporates the characteristics of the Pareto and Lomax distributions, the suggested distribution (Odd Lomax-Pareto) offers a more adaptable framework for researching reliability. When working with practical and real-world datasets, this combination produces solutions that are more precise and flexible. As a result, using this novel distribution to analyze reliability creates more opportunities for future study and real-world applications.

$$R(x) = 1 - F(x)$$

$$\boldsymbol{R(x) = 1 - \left[1 - \beta^{\alpha}\left[\beta + \frac{1}{\left(\frac{\theta}{x}\right)^{\lambda}} - 1\right]^{-\alpha}\right]}$$

$$R(x) = \beta^{\alpha}\left[\beta + \frac{1}{\left(\frac{\theta}{x}\right)^{\lambda}} - 1\right]^{-\alpha} \qquad \ldots(5)$$

4- Estimation methods

There are many estimation methods used to estimate the parameters of the Lomax Pareto distribution From her:

(1-4) Maximum Likelihood Method estimation (MLE)

It is one of the statistical methods used to estimate the parameters of a distribution, and it is named as such because it maximizes the possibility function, and when the sample size is large, its values have less variance[12],[13].

$$L = \prod_{i=1}^{n} f(x_i)$$

By substitution the PDF of the Lomax-Pareto Distribution such as in equation (4), it possible to get the below (ML) function:

$$L = \prod_{i=1}^{n}\left[\frac{\alpha\beta^{\alpha}\lambda x^{\lambda-1}}{\theta^{\lambda}}\right]\left[(\beta-1) + \frac{x^{\lambda}}{\theta^{\lambda}}\right]^{-(\alpha+1)}$$

$$L = \left[\frac{\alpha\beta^{\alpha}\lambda}{\theta^{\lambda}}\right]^{n} \prod_{i=1}^{n} x^{\lambda-1} \prod_{i=1}^{n}\left[(\beta-1) + \frac{x^{\lambda}}{\theta^{\lambda}}\right]^{-(\alpha+1)} \qquad \ldots(6)$$

$$l(\alpha,\theta,\lambda) = nlog(\alpha) + n\alpha log\beta + nlog(\lambda) - n\lambda log\theta + (\lambda-1)\sum_{i=1}^{n}\log(\mathrm{x}) - (\alpha + 1)\sum_{i=1}^{n}\log\left[(\beta-1) + \frac{x^{\lambda}}{\theta^{\lambda}}\right] \qquad \ldots(7)$$

$\hat{\theta}$=min($\hat{y_i}$)=y_1

To get the estimators of (MLE) for the parameters of the distribution, it is possible to utilize the first partial derivation for the previous function and for each parameter from the parameters of the distribution as below:

$$\frac{\partial l(\alpha,\beta,\lambda)}{\partial \alpha} = \frac{n}{\alpha} + nlog(\beta) - \sum_{i=1}^{n} \log\left[(\beta-1) + \frac{x^{\lambda}}{{y_1}^{\lambda}}\right] \quad \ldots(8)$$

$$\frac{\partial l(\alpha,\beta,\lambda)}{\partial \beta} = \frac{n\,\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n} \frac{1}{\left[(\beta-1)+\frac{x^{\lambda}}{{y_1}^{\lambda}}\right]}(1) \quad \ldots(9)$$

$$\frac{\partial l(\alpha,\beta,\lambda)}{\partial \lambda} =$$

$$\frac{n}{\lambda} - nlog(y_1) + \sum_{i=1}^{n} \frac{1}{x^{\lambda-1}}\ log(x)x^{\lambda-1} - (\alpha+1)\sum_{i=1}^{n} \frac{1}{\left[(\beta-1)+\frac{x^{\lambda}}{{y_1}^{\lambda}}\right]} \frac{{y_1}^{\lambda}x^{\lambda}[log(x)-log(y_1)]}{{y_1}^{2\lambda}} \quad \ldots(10)$$

And by equaling the previous derivations to the zero as below:

$$0 = \frac{n}{\alpha} + nlog(\beta) - \sum_{i=1}^{n} \log\left[(\beta-1) + \frac{x^{\lambda}}{{y_1}^{\lambda}}\right] \quad \ldots(11)$$

$$0 = \frac{n\,\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n} \frac{1}{\left[(\beta-1)+\frac{x^{\lambda}}{{y_1}^{\lambda}}\right]} \quad \ldots(12)$$

$$0 = \frac{n}{\lambda} - nlog(y_1) + \sum_{i=1}^{n} \frac{1}{x^{\lambda-1}} log(x)x^{\lambda-1}$$

$$-(\alpha+1)\sum_{i=1}^{n} \frac{1}{\left[(\beta-1)+\frac{x^{\lambda}}{{y_1}^{\lambda}}\right]} \frac{{y_1}^{\lambda}x^{\lambda}[log(x)-log(y_1)]}{{y_1}^{2\lambda}} \ldots(13)$$

The previous derivations were unsolvable by utilized the elimination and substitution, but it is possible to utilize the Iterative numerical methods with a successive substitution for reaching out to the MLE (Like as these methods is Newton–Raphson method) as below:

(2-4) Least Square Error

It is possible to get the (LSE) function by finding [14], [15], [16]:

$$F(x) = 1 - \beta^{\alpha}\left[\beta + \frac{1}{\left(\frac{\theta}{x}\right)^{\lambda}} - 1\right]^{-\alpha} \quad \ldots(14)$$

$$Q = \sum_{i=1}^{n} [F(x_i) - F(x)]^2 \qquad \dots(15)$$

$$\hat{F}(x_i) = p_i \qquad \frac{i-0.5}{n} \qquad i = 1,2,3 \dots n$$

$$v = \ln(1 - \hat{p}_i)$$

$$\ln\left(1 - F(x_i)\right) = ln\left[1 - \left[1 - \beta^{\alpha}\left[\beta + \frac{1}{\left(\frac{\theta}{x}\right)^{\lambda}} - 1\right]^{-\alpha}\right]\right] \qquad \dots(16)$$

$$\ln\left(1 - F(x_i)\right) = \alpha\ln(\beta) - \alpha\ln\left[\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1\right] \qquad \dots(17)$$

$$Q = \sum_{i=1}^{n}\left[v - \alpha\ln(\beta) + \alpha\, ln\left[\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1\right]\right]^2 \qquad \dots(18)$$

$$\frac{\partial Q}{\partial \alpha} = 2\sum_{i=1}^{n}\left[v - \alpha\ln(\beta) + \alpha\, ln\left[\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1\right]\right]\left[-\ln(\beta) + \ln\left[\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1\right]\right] \qquad \dots(19)$$

$$\frac{\partial Q}{\partial \beta} = 2\sum_{i=1}^{n}\left[v - \alpha\ln(\beta) + \alpha\, ln\left[\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1\right]\right]\left[\frac{-\alpha}{\beta} + \frac{\alpha(1)}{\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1}\right] \qquad \dots(20)$$

$$\frac{\partial Q}{\partial \lambda} = 2\sum_{i=1}^{n}\left[v - \alpha\ln(\beta) + \alpha\, ln\left[\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1\right]\right]\left[\frac{-\alpha ln\left(\frac{\theta}{x}\right)\left(\frac{\theta}{x}\right)^{-\lambda}}{\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1}\right] \qquad \dots(21)$$

$$\frac{\partial Q}{\partial \theta} = 2\sum_{i=1}^{n}\left[v - \alpha\ln(\beta) + \alpha\, ln\left[\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1\right]\right]\left[\frac{\frac{-\alpha\lambda\theta^{-(\alpha+1)}}{x^{-\lambda}}}{\beta + \left(\frac{\theta}{x}\right)^{-\lambda} - 1}\right] \qquad \dots(22)$$

5- Simulation

Simulation can be defined as the process of imitating or representing reality using specific models.

Simulation is of great importance as it is one of the solutions to overcome the problems researchers face when they need a large number of samples. This requires significant time, effort, and money; therefore, simulation is used, and these experiments involve repeating the process a sufficient number of times (1000) times A random variable was generated with a Pareto Lomax distribution for the parameters (θ،λ،β), and values were assumed for them, and sample sizes (25, 50, 100) were assumed using the program (R).

Table 1. Predictive values (**α=2 , β=0.9, λ=1.5, θ=4**) by utilizing the MLE and LSE methods. to the parameters

method		MLE	LSE	MLE	LSE	MLE	LSE	MLE	LSE
n		10		25		50		100	
Estimator	**α**	2.628788	1.632054	2.1285	1.726199	2.032365	1.854429	2.008161	1.92584
	β	1.295708	0.727026	0.987632	0.708756	0.942921	0.809504	0.91424	0.856875
	λ	1.628354	0.898237	1.517924	1.166058	1.512819	1.341355	1.503886	1.421868
	θ	4.115809	3.153249	4.047816	3.669482	4.025793	3.838056	4.012582	3.919278
Bias	**α**	0.628788	-0.36795	0.1285	-0.273801	0.032365	-0.145571	0.008161	-0.07416
	β	0.395708	-0.17297	0.087632	-0.191244	0.042921	-0.090496	0.01424	-0.04313
	λ	0.128354	-0.60176	0.017924	-0.333942	0.012819	-0.158645	0.003886	-0.07813
	θ	0.115809	-0.84675	0.047816	-0.330518	0.025793	-0.161944	0.012582	-0.08072
MSE	**α**	0.869755	0.807233	0.084418	0.097522	0.01914	0.022222	0.004629	0.005603
	β	0.557326	0.46741	0.0899	0.059307	0.021726	0.010596	0.005431	0.002064
	λ	0.374241	0.671586	0.079341	0.136794	0.019593	0.027841	0.005086	0.006356

	θ	0.027825	0.811237	0.004495	0.113367	0.001309	0.026606	0.000321	0.006563

Table 1-showed that the LSE method was better than the MLE method, as it yielded some good and acceptable results

Figure 1. The changing of t**he** $\boldsymbol{\alpha} = 2$**, β=0.9, λ=1.5, θ=4** by utilizing the MLE and LSE methods

MSE values when

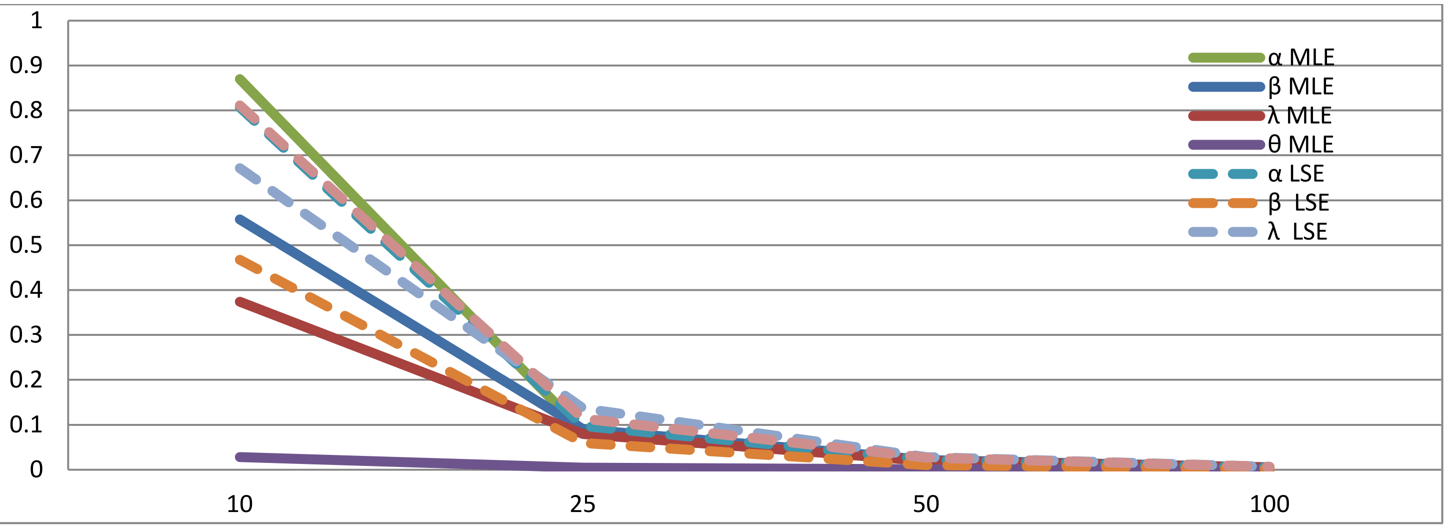
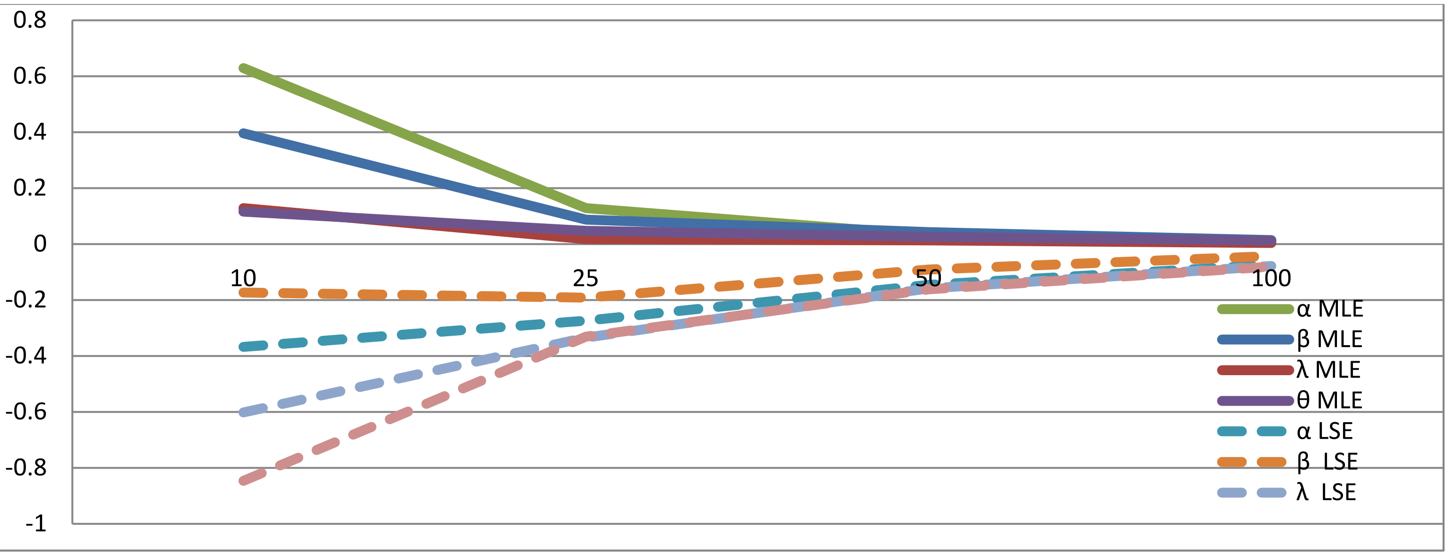


Figure 2. The changing of the(**α=2, β=0.9, λ=1.5, θ=4)** by utilizing the MLE and LSE methods

Bias values when



Conclusion and discussion:

In this study was provided a new distribution is well known "odd Lomax-Pareto distribution". Also it was focused comprehensively and extensively on the properties of odd Lomax-Pareto distribution. The Maximum Likelihood Method estimation (MLE) and least square error were used by using the simulation. The results showed that the least square error (LSE) was the best method because it was achieved good results and it was better than the maximum likelihood method estimation (MLE). As a future works, it is possible to apply the current distribution in the real life applications or it can possible to improve the performance of the current distribution by mixing with other statistical distributions and so on.

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