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A Comparative Study of Two Techniques for Analyzing Optimal Bounds in Fully Fuzzy Transportation Problems Using Parametric Methods

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ABSTRACT This study makes two major contributions: first, it presents a new approach to solving the fuzzy transportation problem (FTP); second, it contrasts two different Parametric methods for analyzing optimal bounds. The first Mechanism is to obtain a fuzzy optimal solution by solving the FTP directly. Then, in order to attain the best crisp outcome, we apply the alpha-cut and parametric function to the fuzzy decision variables and their associated cost values. This produces a sequence of ordinary values for the objective function, and we choose the best one. In the second method, we again employ the alpha-cut and parametric function, but this time we transform all of the Fuzzy transportation problem table components, which are costs, supply, and demand, into crisp numbers. Then we get ordinary Transportation Problem (TP) and solve it to acquire the optimal solution for the objective function. For the values of alpha and beta, we will get a series of crisp solutions this enabling us to determine the best optimal crisp value. This paper offers a thorough analysis of the two approaches, stressing their benefits, drawbacks, and suitability for solving practical issues.

Keywords: Fuzzy Transportation Problem, Trapezoidal Fuzzy Number, Parametric Methods, Alpha-Cut,Optimal Bounds,Crisp Solutions

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دراسة مقارنة تقنيتين لتحليل الحدود المثلى في مسائل النقل الضبابية تماما باستخدام الأساليب البارامترية

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المستخلص:
يقدم هذا البحث مساهمتين رئيسيتين: أولاً، يُقدم طريقة جديدة لحل مشكلة النقل الضبابي (FTP)، وثانياً، يُقارن بين تقنيتين مختلفتين لتحليل الحدود المثلى باستخدام الطرق البارامترية. في الطريقة الأولى، نحل مشكلة النقل الضبابي مباشرةً للحصول على حل أمثل ضبابي، ثم نطبق دالة $\alpha - cut$ والدالة البارامترية على متغيرات القرار الضبابي وقيم التكلفة المقابلة لها للوصول إلى النتيجة المثلى، مما يُولّد $\alpha - cut$ سلسلة من الحلول لدالة الهدف بأعداد حقيقية، مما يتيح لنا ان نحدد أفضل هذه الحلول. اما في الطريقة الثانية، نستخدم مرة أخرى دالة $\alpha - cut$ والدالة البارامترية، ولكن هذه المرة نُحوّل جدول مشكلة النقل بأكمله - بما في ذلك التكاليف والعرض والطلب - إلى قيم واضحة حقيقية، ومن ثم نحل مشكلة النقل الواضحة الحقيقية الناتجة هذه لإيجاد القيمة المثلى الواضحة لدالة الهدف. لكل نطاق مُحدد بواسطة ألفا وبيتا، نشتق مشكلة نقل واضحة مع القيم المثلى المقابلة لها، مما يُمكّننا من تحديد أفضل قيمة مثالية واضحة. لذا تُقدم هذه الدراسة مقارنة شاملة بين الطريقتين، مع تسليط الضوء على مزاياهما وقيودهما وإمكانية تطبيقهما في مسائل واقعية.

Introduction

The methodical process of optimization entails picking the best options from a wide range of possibilities in order to maximize or minimize particular objectives or standards. The roots of this problem can be found in the work of mathematician Gaspard Monge in 1781, even though the classical framework for it only appeared in the early 20th century. George B. Dantzig improved on Monge's original investigation of resource distribution in 1941, and his work laid the groundwork for theories of resource allocation and logistical optimization. In order to effectively manage resources, the transportation problem mainly focuses on reducing transportation costs while concurrently meeting supply and demand constraints. In 1965, Lotfi Zadeh [1] created fuzzy set theory, which was a revolutionary addition to mathematical theory. The optimization field was drastically altered by this novel framework, which made it possible to rigorously represent ambiguous or imprecise parameters. Investigating novel optimization techniques is becoming more and more crucial since real-world situations frequently involve uncertainties pertaining to data and goals.

Due to this change, the traditional transportation problem has been expanded into what is now called the fuzzy transportation problem (FTP), which combines the ideas of fuzzy set theory and fuzzy logic. We can efficiently capture and represent ambiguous or uncertain values across crucial parameters, including cost coefficients, supply levels, demand quantities, and capacity constraints, by adopting the FTP. Fuzzy optimization methods designed to address these issues can be used to produce more resilient and flexible solutions that take into account the complexity of real-world situations. In this context, fuzzy numbers are commonly used because they enable the representation of uncertain values. Because fuzzy transportation problems have special characteristics, it is necessary to use specialized solution techniques. Over the years, researchers have proposed a number of methods to address fuzzy transportation problems (FTPs). Early approaches mostly concentrated on using ranking techniques to transform hazy parameters like costs, supply, and demand into precise values. One notable contribution came from researchers Chanas and Kuchta [2], who developed techniques for handling fuzzy numbers in transportation contexts. Additionally, Nagoor Gani and Abdul Rezak [3] presented solutions using trapezoidal fuzzy numbers for two-stage cost-minimizing fuzzy TPs, demonstrating the potential of these more advanced methods.

More sophisticated techniques, like fuzzy linear programming, were investigated as the study went on. To improve computational efficiency and offer more accurate solutions for completely fuzzy transportation problems, hybrid approaches that combine fuzzy logic with classical optimization algorithms were created. Furthermore, researchers Liu and Kao [4] expanded the applicability of these ideas to more complicated situations by proposing the extension principle for resolving fuzzy transportation problems. In conjunction, Gani and Abdul Razak provided a methodical parametric approach that enables decision-makers to minimize costs. Additionally, Pandian and Natarajan [5] used the novel fuzzy zero-point approach to tackle complex problems related to fuzzy transportation. Additionally, they presented a recently proposed method for obtaining a fuzzy optimal solution.

The difficulty of identifying the optimal solution is a significant component of this field because fuzzy transportation problems do not always have a single, distinct solution. We used a parametric approach to tackle this complexity, which enables a two-pronged analysis of the issue. This two-pronged strategy helps to develop more effective solutions and promotes a deeper comprehension of the issue. We will present a numerical example that demonstrates the real-world implementation of these optimization techniques in order to demonstrate these concepts and methodologies.

Methodology

First: Preliminaries

Definition 2.1: Let X be a nonempty universal set, A fuzzy set Ã in X with a membership Function $\mu_{\tilde{A}}: X \rightarrow [0,1]$is represented by: [6]

$$\tilde{A} = \{(x, \mu_{\tilde{A}}) \mid x \in X,\ 0 \le \mu_{\tilde{A}}(x) \le 1\} \quad (1)$$

Definition 2.2: A fuzzy number Ã is a subset of the real line with the following properties: [6] [7]
Ã is normal, that is there exists at least one x ∈ ℝ such that Ã(x) = 1.
Have a continuous membership function.
Convex Set, $\forall\ x, y \in R$ and $\lambda \in [0,1]$, $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\,(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y))$.

Definition 2.3: For real-valued parameters a, b, c, and d, we define a Trapezoidal fuzzy number (a, b, c, d) by:
The fuzzy number's upper and lower bounds are a and d, respectively.
The core's lower and upper bounds (the most likely values) are denoted by b and c, respectively.
A trapezoidal fuzzy number's piecewise linear membership function has the following definition:

$$\mu_A(x) = \begin{cases} 0 & x \leq a \text{ or } x \geq d \\ \frac{x-a}{b-a} & a \leq x \leq b \\ \frac{d-x}{d-c} & c \leq x \leq d \\ 1 & b \leq x \leq c \end{cases} \qquad (2)$$

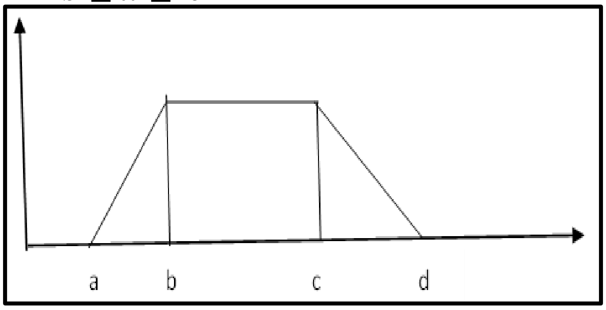

Figure 1. The membership function of a trapezoidal fuzzy number [8]

Definition 2.4: Basic arithmetic operations on trapezoidal fuzzy numbers:
Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers then: [7] [8] [5]
We define addition ⊕ , subtraction ⊖ and Multiplication ⊗ as the following:

$$(1)\ (a_1, a_2, a_3, a_4) \oplus (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$
$$(2)\ (a_1, a_2, a_3, a_4) \ominus (b_1, b_2, b_3, b_4) = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$$
$$(3)\ (a_1, a_2, a_3, a_4) \otimes (b_1, b_2, b_3, b_4) = (t_1, t_2, t_3, t_4)$$

Where $t_1 = minimum\,\{a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4\}$
$t_2 = minimum\,\{a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3\}$
$t_3 = maximum\,\{a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4\}$
$t_4 = maximum\,\{a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3\}$

Definition 2.5: If we have a trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$, the ranking value or crisp value of $\tilde{A}$ called *magnitude* of $\tilde{A}$ it can be written as the following: [9] [5]

$$Mag(\tilde{A}) = \frac{a+5b+5c+d}{12} \quad (3)$$

Definition 2.6: Let $\tilde{u}$ and $\tilde{v}$ be any two trapezoidal fuzzy numbers. Then we can say: [5] [9]
(1) $Mag\ (\tilde{u}) > Mag\ (\tilde{v})$ if and only if $\tilde{u} \succ \tilde{v}$;
(2) $\text{Mag}\ (\tilde{u}) < \text{Mag}\ (\tilde{v})$ if and only if $\tilde{u} \prec \tilde{v}$ and
(3) $\text{Mag}\ (\tilde{u}) = \text{Mag}(\tilde{v})$ if and only if $\tilde{u} \approx \tilde{v}$.

And the ordering $\succeq$ and $\preceq$ between any two trapezoidal fuzzy numbers are as the following: [5]
(1) $\tilde{u} \succeq \tilde{v}$ if and only if $\tilde{u} \succ \tilde{v}$ or $\tilde{u} \approx \tilde{v}$ and
(2) u $\preceq$ v if and only if $\tilde{u} \prec \tilde{v}$ or $\tilde{u} \approx \tilde{v}$.
(3) $\tilde{u} = (a,b,c,d) \approx \tilde{0}$ if and only if $\text{Mag}\ (\tilde{u}) = 0$;
(4) $\tilde{u} = (a,b,c,d) \succeq \tilde{0}$ if and only if $\text{Mag}\ (\tilde{u}) \geq 0$ and
(5) $\tilde{u} = (a,b,c,d) \preceq \tilde{0}$ if and only if $\text{Mag}\ (\tilde{u}) \leq 0$.

Definition 2.7: The α-cut of a fuzzy number denoted by A_α is defined by: [6]

$$A_\alpha = \{x \in X: \mu_A(x) \geq \alpha, \alpha\epsilon(0,1)\} \quad (4)$$

Where alpha cuts reduce the complexity of fuzzy numbers by converting them into intervals at different levels of confidence (alpha levels). This makes mathematical analysis and manipulation simpler.

Definition 2.8: Consider we have the following TRFN $\tilde{A}$ = (a, b, c, d), then we can convert trapezoidal fuzzy number to an interval by using α -cut definition: [10]

$$\text{Interval}\ (\tilde{A}) = [\underline{a}, \bar{a}]=[a+\alpha(b-a), d-\alpha(d-c)] \quad (5)$$

Where $\alpha \in [0,1]$

Definition 2.9: If we have a trapezoidal fuzzy number $\tilde{A} = (a,b,c,d)$, then the parametric function of TRFN is defined by:

$$Par(\tilde{A}) = Par([\underline{a}, \bar{a}]) = \underline{a} + \beta(\bar{a} - \underline{a})$$
$$=(1-\alpha-\beta+\alpha\beta)a + (\alpha-\alpha\beta)b + (\alpha\beta)c + (\beta-\alpha\beta)d \quad (6)$$

Where $\alpha, \beta\epsilon[0,1]$

Second: Fuzzy Transportation Problem Formulation

Fuzzy transportation problem (FTP) is a TP with fuzzy costs, fuzzy sources and fuzzy demands, it's a special type of fuzzy linear programming and it can be written as:

$$\begin{aligned} & Minimize\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \otimes \tilde{x}_{ij} \\ & Subject\ to: \\ & \sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i \qquad for\ i = 1,2,\dots,m \\ & \sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j \qquad for\ j = 1,2,\dots,n \\ & \tilde{x}_{ij} \geq 0 \qquad for\ i = 1,2,\dots,m\ and\ j = 1,2,\dots,n, \end{aligned} \quad (7)$$

Where:

m: the number of supply points.
n: the number of demand points.
$\tilde{x}_{ij} \approx \left({x_{ij}}^1, {x_{ij}}^2, {x_{ij}}^3, {x_{ij}}^4\right)$ is the uncertain number of units shipped from supply i to demand j.
$\tilde{c}_{ij} \approx \left({c_{ij}}^1, {c_{ij}}^2, {c_{ij}}^3, {c_{ij}}^4\right)$ is the uncertain cost of shipping one unit from supply i to the demand j.
$\tilde{a}_i \approx ({a_i}^1, {a_i}^2, {a_i}^3, {a_i}^4)$ is the uncertain supply at supply point i, and
$\tilde{b}_j \approx \left({b_j}^1, {b_j}^2, {b_j}^3, {b_j}^4\right)$ is the uncertain demand at demand point j.
we can represent the transportation problem in a tabular form:

Table 1: Tabular form of transportation problem

	1	. . .	n	Supply
1	$\tilde{c}_{11}$	. . .	$\tilde{c}_{1n}$	$\tilde{\boldsymbol{a}}_1$
. . .	. . .	. . .	. . .	. . .
m	$\tilde{c}_{m1}$	. . .	$\tilde{c}_{mn}$	$\tilde{a}_n$
Demand	$\tilde{\boldsymbol{b}}_1$		$\tilde{\boldsymbol{b}}_n$	

Third: Algorithm of New Method for Solving Fuzzy Transportations Problems:

Step 1: Check if the fuzzy transportation problem is balanced, if it is not balanced add a dummy row or column with zero fuzzy cost to make it balanced. [11]

$$\mathrm{Mag}\left(\sum \tilde{a}_\imath\right) = \mathrm{Mag}\left(\sum \tilde{b}_\imath\right)$$

Step2: Applying Algorithm of Max {Supplies, demands} method in fuzzy environment to find initial fuzzy solution as the following: [12]
Determine the maximum fuzzy supply or maximum fuzzy demand and then start by locating the minimum fuzzy cost cell in the corresponding row or column. Assign the maximum possible quantity to that cell and eliminate that row and column. We will continue on this procedure till all fuzzy demand and supply are satisfied..

Step 3: t's necessary to verify the primary solution for optimality as follows:

(i) We should be sure that the number of primary basic feasible solutions is equal to m+n-1.
(ii) We utilize the Modified Distribution Method (MODI) or UV Method in the fuzzy system, and here we mean by ($\approx$) is equivalncy in Maginitude.
In order to determine the values of $\tilde{u}_i$ and $\tilde{v}_i$we solve the set of equations $\tilde{u}_i \oplus \tilde{v}_i \approx \tilde{c}_{ij}$, for basic cells, and then compute the penalties of non-basic cells by applying this formula: $\tilde{p}_{ij} \approx \tilde{c}_{ij} \ominus \tilde{u}_i \ominus \tilde{v}_i$.
(iii) If all mag($\tilde{p}_{ij}$) ≥ 0 then the solution is optimal, then go to Step (5).
If there exist mag($\tilde{p}_{ij}$) < 0 then the solution is not optimal going to the next step.

Step 4: Modifying the primary solution by calculating the fuzzy value θ as follows: In the non-basic cells, starting from the maximum negative magnitude cell and drawing a closed loop with basic cells, and then determining the value of θ ::

$$\theta \approx min\,\{negative\ sing\ corner\ fuzzy\ value\}$$

We will obtain a new set of fuzzy solutions by adding fuzzy value θ to the loop's positive corner fuzzy value and subtracting it from its negative corner fuzzy value.
Proceed to step (3).
Step 5: compute the objective function: $Minimize\ z \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c_{ij}} \otimes \tilde{x_{ij}}$.
Stop the Algorithm.

Fourth: Optimal Bounds Analysis utilizing the Parametric Function in Two Ways:

Drive parametric function:
Consider we have a fuzzy number $\tilde{A} = $ (a, b, c, d) clearly from the membership function in equation (2), Clearly $\alpha \geq 0$, and by definition of the $\alpha - cut$, we can say $\frac{x-a}{b-a} \geq \alpha$, implies that: $x \geq a + \alpha(b-a)$ and also $\frac{d-x}{d-c} \geq \alpha$ implies that $x \leq d - \alpha(d-c)$ and then, we can say: $a + \alpha(b-a) \leq x \leq d - \alpha(d-c)$ which means the interval of the fuzzy number according to value $\alpha[\underline{a}, \bar{a}]$ =$[a + \alpha(b-a), d - \alpha(d-c)]$.

We have the parametric which converts the interval to parametric value.

$$[\underline{a}, \bar{a}] = \underline{a} + \beta(\underline{a} - \bar{a}) \quad (8)$$

$$\underline{a} = a + \alpha(b-a) \quad (9)$$

$$\bar{a} = d - \alpha(d-c) \quad (10)$$

From equations (8), (9) and (10) we conclude:

$$Z = [\underline{a}, \bar{a}] = (a - \alpha a + \alpha b) + \beta(d - a - \alpha d + \alpha c + \alpha a - \alpha b)$$

$$Z = (1 - \alpha - \beta + \alpha\beta)a + (\alpha - \alpha\beta)b + (\alpha\beta)c + (\beta - \alpha\beta)d \quad (11)$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$.

Method (I):
The first method relies on the optimal set of fuzzy decision variables and the fuzzy costs associated with them. The fuzzy decision variables and fuzzy costs are transformed into ordinary real values by employing a parametric function. Finally, the value of the objective function is determined, and then the sequence of optimal solutions is obtained according to the Alpha and Beta values.

Method (II):

Transforming the entire Fuzzy transportation table to an ordinary traditional transportation problem by applying the parametric function on fuzzy numbers, and then utilizing the usual standard approaches to obtain a series of optimal values according to the values of Alpha and Beta, we can choosing $\alpha = 0$, $\alpha = 0.1, \dots$, $\alpha = 0.9$, $\alpha = 1$, and selecting three values for $\beta = 0, \beta = 0.5$, $\beta = 1$, The final decision regarding the ideal solution might be left to the individual of the decision maker.

1. Numerical Example:

Suppose we have this Fuzzy transportation problem:

Table 2: Fuzzy transportation problem

	B1	B2	B3	B4	supply
A1	(8,9,11,12)	(8,9,13,14)	(14,15,17,18)	(12,13,15,16)	**(33,34,36,37)**
A2	(17,18,22,23)	(9,10,12,13)	(10,11,13,14)	(13,14,16,17)	**(8,9,11,12)**
A3	(10,11,13,14)	(6,7,9,10)	(17,18,22,23)	(5,6,8,9)	**(38,39,41,42)**

Demand	(13,14,16,17)	(18,19,21,22)	(34,35,37,38)	(12,13,15,16)

Using our new algorithm for solving the problem.
Step1: Since $\sum \widetilde{a}_i \approx (79, 82, 88, 91),\ mag = 85$
and $\sum \widetilde{b}_i \approx (77, 81, 89, 93)$, mag=85
Since $\text{Mag}(\widetilde{a}_i) = \text{Mag}(\widetilde{b}_i) = 85$, the given problem is a balanced.
Step2: Applying Algorithm of Max {Supplies, demands} method in fuzzy environment to find initial fuzzy solution, as the following:
Mag Supply {(33,34,36,37), (8,9,11,12), (38,39,41,42)} = {35, 10,40}
Mag Demand {(13,14,16,17), (18,19,21,22), (34,35,37,38), (12,13,15,16)} = {15,22,36,14}
Max {Supplies, demands} ≈ (38,39,41,42) because mag {(38,39,41,42)} =40
We will take Row (A3) and find minimum fuzzy cost in Row (A3).
Mag {row(A3)} = {(10,11,13,14), (6,7,9,10), (17,18,22,23), (5,6,8,9)}
= {12, 8,20,7}

So, Min {row(A3)} ≈ (5,6,8,9) because Mag {(5,6,8,9)} =7
We allocate as much as possible fuzzy amount for cell (3, 4).
Clearly for this cell we have demand ≈ (12,13,15,16), Mag=14 and Supply≈ (38,39,41,42), Mag=40, so we can assign (12,13,15,16) for Cell (3,4)
And then we cross out column B4.
Calculate reaming supply (38,39,41,42) ⊖ (12,13,15,16) ≈ (22, 24, 28, 30), Mag=26
Again, Mag Supply {(33,34,36,37), (8,9,11,12), (22, 24, 28, 30)} = {35, 10,26}
Mag Demand {(13,14,16,17), (18,19,21,22), (34,35,37,38)} = {15,22,36}
Max {Supplies, demands} ≈ (34,35,37,38), We will take column (B3) and the minimum fuzzy cost in this column is (10,11,13,14), mag=12, so we allocate (8,9,11,12) for cell (2,3), (34,35,37,38) ⊖ (8,9,11,12) ≈ (22, 24, 28, 30) mag=26 the cross-out row(A2),
By the same way we continue on the allocation till we obtain the following primary solution:

Table 3: primary feasible solution

	B1	B2	B3	B4	supply
A1	(8,9,11,12) **(13, 14, 16, 17)**	(8,9,13,14)	(14,15,17,18) **(16, 18, 22, 24)**	(12,13,15,16)	**(33,34,36,37)**
A2	(17,18,22,23)	(9,10,12,13)	(10,11,13,14) **(8, 9, 11, 12)**	(13,14,16,17)	**(8,9,11,12)**
A3	(10,11,13,14)	(6,7,9,10) **(18, 19, 21, 22)**	(17,18,22,23) **(0, 3, 9, 12)**	(5,6,8,9) **(12, 13, 15, 16)**	**(38,39,41,42)**
Demand	**(13,14,16,17)**	**(18,19,21,22)**	**(34,35,37,38)**	**(12,13,15,16)**	

Let's check the initial solution:
(13, 14, 16, 17)⊕(16, 18, 22, 24) ≈ (29,32,38,41) mag=35 and clearly for (33,34,36,37) mag=35, so they are equivalent and the supply of row (A1) are met.
(18,19,21,22) ⊕(0, 3, 9, 12) ⊕ (12,13,15,16) ≈ (30,35,45,50), mag=40 and clearly that:
Mag (38,39,41,42) =40, so (30,35,45,50)≈ (38,39,41,42) and the supply of row (A3) are met.
(16, 18, 22, 24) ⊕(8,9,11,12) ⊕ (0, 3, 9, 12) ≈ (24,30,42,48) mag=36

(34,35,37,38) mag=36, we see also (24,30,42,48) ≈(34,35,37,38)
So, Fuzzy demand and supply satisfied. Going to next step.
Step 3: verifying the optimality by applying MODI Method or UV Method in fuzzy system:
Clearly, we have m+n-1=4+3-1=6 basic feasible fuzzy solution cells
For Basic cells we find $\tilde{u}_i \oplus \tilde{v}_i \approx \tilde{c}_{ij}$, Also we consider fuzzy zero $\tilde{0} \approx (0,0,0,0)$

$$\tilde{u}_1 \oplus \tilde{v}_1 \approx (8,9,11,12)$$
$$\tilde{u}_1 \oplus \tilde{v}_3 \approx (14,15,17,18)$$
$$\tilde{u}_2 \oplus \tilde{v}_3 \approx (10,11,13,14)$$
$$\tilde{u}_3 \oplus \tilde{v}_2 \approx (6,7,9,10)$$
$$\tilde{u}_3 \oplus \tilde{v}_3 \approx (17,18,22,23)$$
$$\tilde{u}_3 \oplus \tilde{v}_4 \approx (5,6,8,9)$$
Set $\tilde{u}_3 \approx \tilde{0},\ then\ \tilde{v}_2 \approx (6,7,9,10),\ \tilde{v}_3 \approx (17,18,22,23),\ \tilde{v}_4 \approx (5,6,8,9)$

$\tilde{u}_1 \approx (14,15,17,18) \ominus (17,18,22,23) \approx (-9, -7, -1, 1)$
$\tilde{v}_1 \approx (8,9,11,12) \ominus (-9, -7, -1, 1) \approx (7, 10, 18, 21)$
$\tilde{u}_2 \approx (10,11,13,14) \ominus (17,18,22,23) \approx (-13, -11, -5, -3)$
For non-basic cells we use: $\tilde{p}_{ij} \approx \tilde{c}_{ij} \ominus \tilde{u}_i \ominus \tilde{v}_i$
$\tilde{p}_{12} \approx \tilde{c}_{12} \ominus \tilde{u}_1 \ominus \tilde{v}_2$
$\tilde{p}_{12} \approx (8,9,13,14) \ominus (-9,-7,-1,1) \ominus (6,7,9,10) \approx (-3, 1, 13, 17), mag = 7 > 0$
By the same way:
$\tilde{p}_{14} \approx (2, 6, 16, 20)$ mag=11, $\tilde{p}_{21} \approx (-1, 5, 23, 29)$ mag=14
$\tilde{p}_{22} \approx (2, 6, 16, 20)$ mag=11, $\tilde{p}_{24} \approx (7, 11, 21, 25)\ mag = 16$
$\tilde{p}_{31} \approx (-11, -7, 3, 7)$, mag=-2 <0, so $(-11, -7, 3, 7) \preceq \tilde{0}$
The solution is not optimal we should enhance the solution towards the optimal. Going to cell (3,1) creating close loop with basic feasible solution cells. From cell (3,1) we make close loop with cells (1,1), (1,3), and (3,3):

$$\theta \approx \min(\text{cell}(1,1), \text{cell}(3,3)) \approx \min\{(13,14,16,17), (0, 3, 9, 12)\}$$
$$\approx (0, 3, 9, 12), and\ \ mag = 6$$

Therefore; $(0, 3, 9, 12)$ will be added to cell (3,1) and then we subtracting
$(13, 14, 16, 17) \ominus (0, 3, 9, 12) \approx (1, 5, 13, 17)$ and also
$(16, 18, 22, 24) \oplus (0, 3, 9, 12) \approx (16,21,31,36)$
Finally, the solution modified to:

Table 4: Modified solution of fuzzy transportation problem

	B1	B2	B3	B4	supply
A1	(8,9,11,1 2) **(1, 5, 13, 17)**	(8,9,13,14)	(14,15,17,18) **(16, 21, 31, 36)**	(12,13,15,16)	**(33,34,36,37)**
A2	(17,18,22,23)	(9,10,12,13)	(10,11,13,14) **(8, 9, 11, 12)**	(13,14,16,17)	**(8,9,11,12)**
A3	(10,11,13,14) **(0, 3, 9, 12)**	(6,7,9,10) **(18, 19, 21, 22)**	(17,18,22,23)	(5,6,8,9) **(12, 13, 15, 16)**	**(38,39,41,42)**
Demand	**(13,14,16,17)**	**(18,19,21,22)**	**(34,35,37,38)**	**(12,13,15,16)**	

by the same way as before we will check optimality, we can see that $mag(\tilde{p}_{ij}) > 0$ for all non-basic cells, the value of the objective function is:

$$\tilde{z} \approx (480, 703, 1239, 1552), \text{ and } Mag\,(480, 703, 1239, 1552) = 978.5$$

Analyzing optimal bounds using the parametric function in two methods:

Method (I): For the fuzzy decision variables and associated fuzzy costs, we can utilize alpha-cut and parametric functions to observe how optimal solutions vary based on alpha and beta as follows:

Consider a fuzzy number $\tilde{A}$ = (a, b, c, d) then the Interval of $\tilde{A} = [a + \alpha(b - a), d - \alpha(d - c)]$. Also transform to parametric value by $[\underline{a}, \bar{a}] = \underline{a} + \beta(\overline{a} - \underline{a})$, therefore;

$$Z = [\underline{a}, \bar{a}] = (a - \alpha a + \alpha b) + \beta(d - a - \alpha d + \alpha c + \alpha a - \alpha b)$$
$$Z = (1 - \alpha - \beta + \alpha\beta)a + (\alpha - \alpha\beta)b + (\alpha\beta)c + (\beta - \alpha\beta)d$$

If $\alpha = 0$ and $\beta = 0$ then we will calculate the crisp value of the objective function as the following:

Table 5: Optimal solution for particular values of $\alpha = 0, \beta = 1$

Fuzzy Decision variable	Parametric conversion (ParCon1)	Fuzzy costs	Parametric conversion (ParCon2)	ParCon1 x ParCon2
(1, 5, 13, 17)	1	(8,9,11,12)	8	8
(16,21,32,36	16	(14,15,17,18)	14	224
(8,9,11,12)	8	(10,11,13,14)	10	80
(0,3,9,12)	0	(10,11,13,14)	10	0
(18,19,21,22)	18	(6,7,9,10)	6	108
(12,13,15,16)	12	(5,6,8,9)	5	60
			(Objective function value) z value=	480

By the same way calculating other z- values which is shown in Table 6.

Table 6: Optimal solution for particular values of $0 \le \alpha \le 1, 0 \le \beta \le 1$

Alpha	Beta=0 (Z value)	Beta=0.5 (Z value)	Beta=1 (Z value)
0	480	956	1552
0.1	500.95	956	1519.35
0.2	522.2	956	1487
0.3	543.75	956	1454.95
0.4	565.6	956	1423.2
0.5	587.75	956	1391.75
0.6	610.2	956	1360.6
0.7	632.95	956	1329.75
0.8	656	956	1299.2
0.9	679.35	956	1268.95
1	703	956	1239

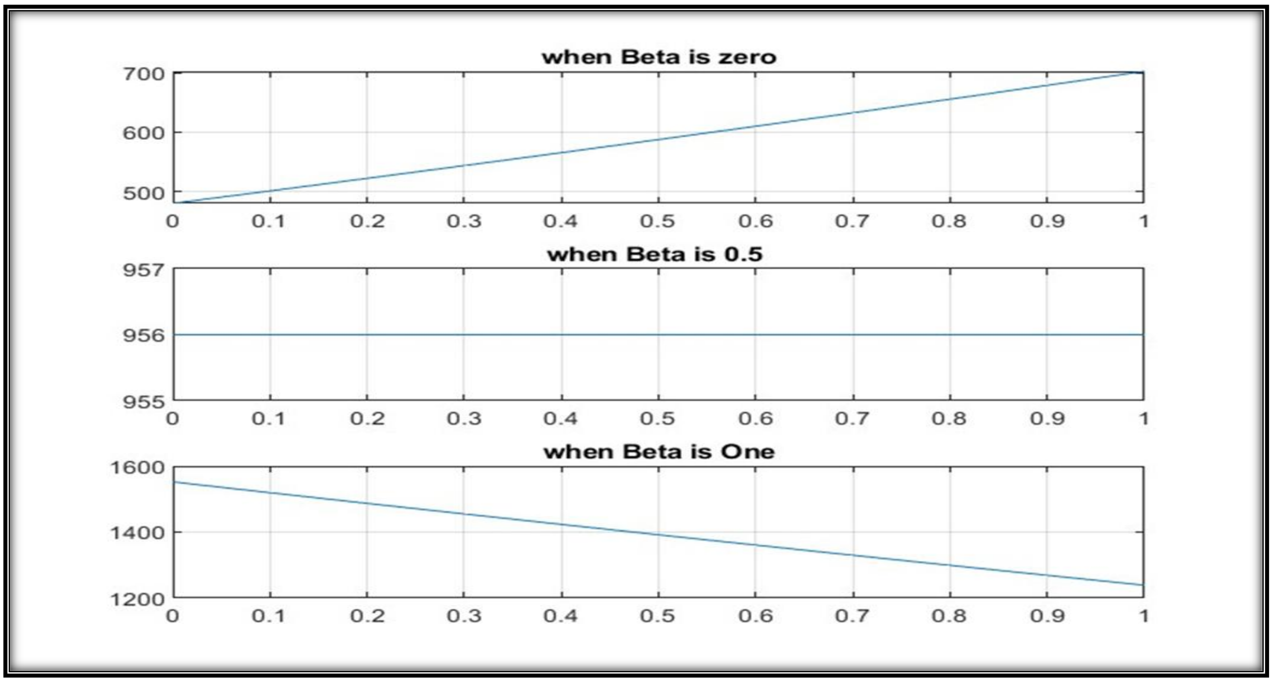

Figure 2: Optimal solutions when $0 \leq \alpha \leq 1$

Method (II):
Step 1: We can use the above parametric conversion for the costs, demand, and supply for the transportation table, convert the fuzzy transportation problem to crisp transportation problem, and then solve it using the usual methods to find the optimal solution.
Step 2: Take $\alpha = 0$, $\alpha = 0.1, \dots$, $\alpha = 0.9, \alpha = 1$, also take three values for $\beta = 0, \beta = 0.5, \beta = 1$, for each Alpha and Beta value, we will get a crisp Transportation problem and solve it find an optimal solution.

According to method (II) the whole Fuzzy transportation problem will be converted to the following usual transportation problems If $\alpha = 0$ and $\beta = 0$.

Table 7: crisp Transportation problem

	B1	B2	B3	B4	supply
A1	8	8	14	12	33
A2	17	9	10	13	8
A3	10	6	17	5	38
Demand	13	18	34	12	

Solving the transportation problems in usual ways we got:

Table 8: solution of the crisp Transportation problem

	B1	B2	B3	B4	supply

A1	8 [7]	8	14 [26]	12	33
A2	17	9	10 [8]	13	8
A3	10 [6]	6 [18]	17	5 [12]	38
Demand	13	18	34	12	

Z value = $(8 \times 7) + (26 \times 14) + (8 \times 10) + (6 \times 10) + (18 \times 6) + (12 \times 5) = 728$
By the same we will calculate Z-value for other values of alpha and Beta as shown in the Table 8 below:

Table 9. The series of optimal solutions according to the value of Alpha and Beta

Alpha	Beta=0 (Z value)	Beta=0.5 (Z value)	Beta=1 (Z value)
0	728	956	1176
0.1	738.579	956	1164.43
0.2	749.36	956	1152.92
0.3	760.1091	956	1141.47
0.4	771.0346	956	1130.08
0.5	782	956	1118.75
0.6	793.04	956	1107.48
0.7	804.16	956	1096.27
0.8	815.36	956	1085.12
0.9	827.43	956	1074.03
1	838	956	1063

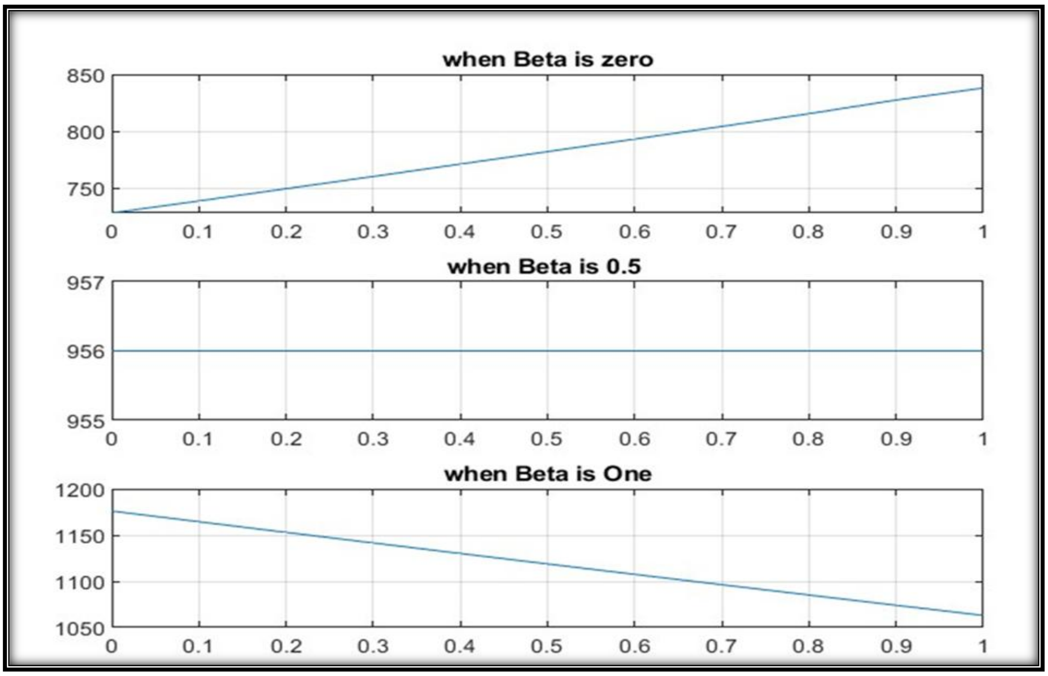

Figure 3: Optimal solutions when $0 \leq \alpha \leq 1$

From the above tables and figures, we see that alpha cuts provide a way to analyze fuzzy numbers at different levels of certainty. Decision-makers can choose an alpha level that reflects their desired confidence level, allowing for more informed and flexible decision-making, as well as the alpha cuts help in visualizing fuzzy numbers by breaking them down into crisp intervals at various alpha levels. This can make it easier to understand and interpret the fuzzy number.

Results and discussion:

In assessing the two methods for solving the fuzzy transportation problem, we observe distinct outcomes regarding their lower bound crisp values. The first method yields a minimum lower bound crisp value of 480, while the second method produces a higher minimum lower bound crisp value of 728.
This indicates that the first method is more effective for our objective of minimizing transportation costs, as it directly addresses the fuzzy transportation problem and allows us to work only on the fuzzy decision variable results and their corresponding costs. By applying this method, we can generate a series of crisp solutions based on varying values of Alpha and Beta.
In contrast, the second method requires us to transform the entire transportation problem into a crisp transportation problem for each combination of Alpha and Beta. This results in the need to solve a total of 33 crisp transportation problems in our example, given that we have eleven values for Alpha and three values for Beta. This process can become increasingly complicated, especially when the transportation problem involves larger tables with more extensive supply and demand figures, as well as additional values for Alpha and Beta.
Overall, the first method not only offers a better optimal solution in terms of cost but also simplifies the process significantly compared to the second method.

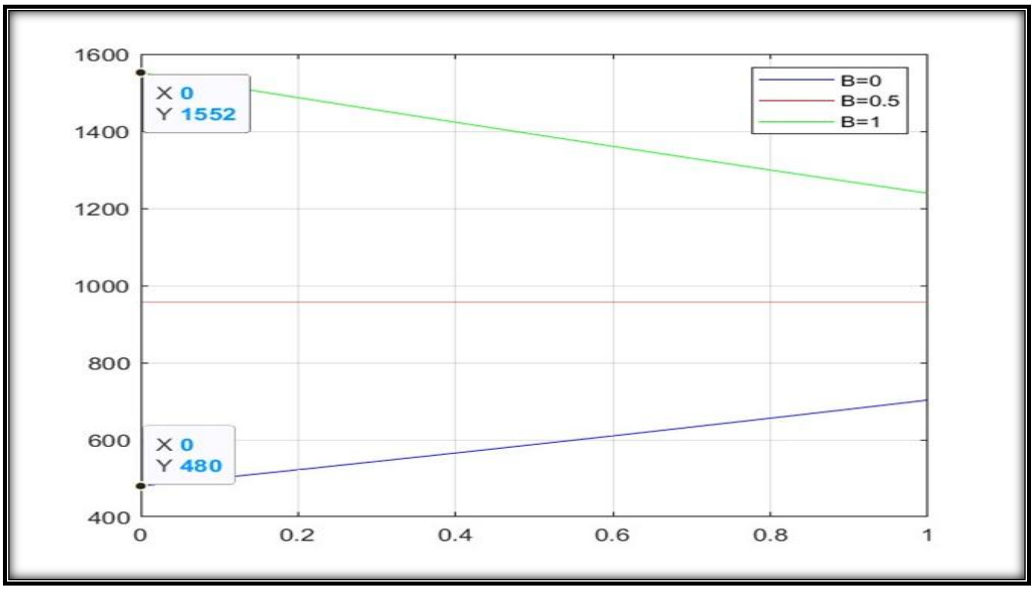

Figure 4: Method (I) Optimal solutions $0 \leq \alpha \leq 1$ and $\beta = 0, 0.5, 1$

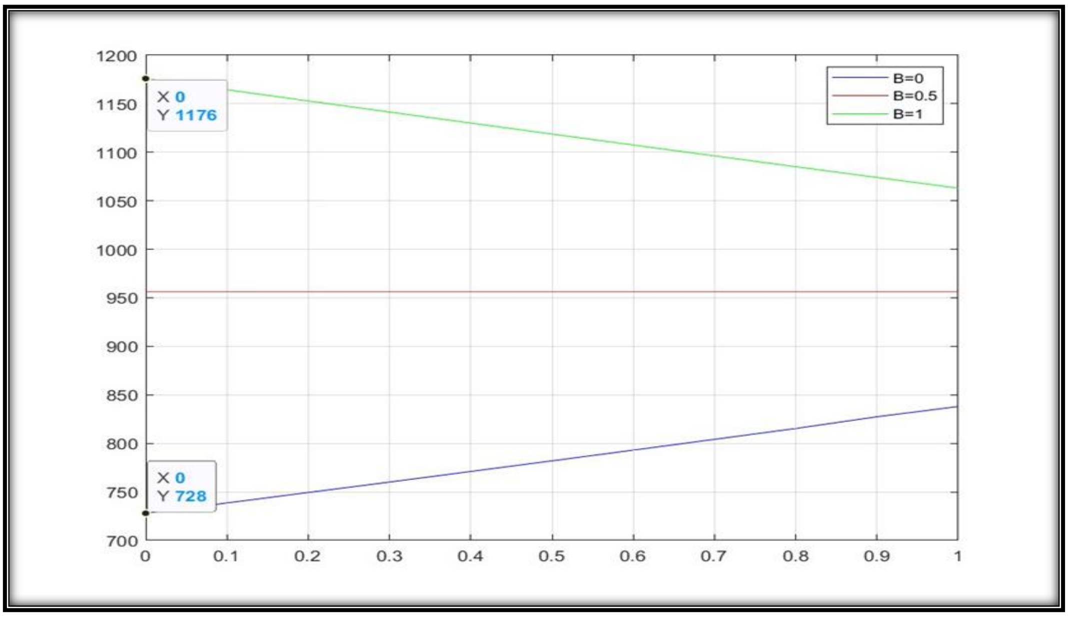

Figure 5: Method (II) Optimal solutions $0 \leq \alpha \leq 1 \; and \; \beta = 0, 0.5, 1$

2. Conclusion

A novel approach is created to address transportation issues in which supply, demand, and cost are all completely ambiguous. Compared to conventional approaches that make the assumption of clear or partially fuzzy parameters, this represents a substantial improvement. Our suggested approach makes a substantial addition to the fuzzy optimization community. It offers a practical and adaptable framework for resolving transportation issues under uncertainty by utilizing trapezoidal fuzzy numbers and a parametric approach. Additionally, alpha cuts are a potent tool in fuzzy set theory that offer a means of simplifying, analyzing, and manipulating fuzzy numbers, making them easier to work with in real-world scenarios. The method's efficacy is illustrated by the numerical example, and decision-makers can gain important insights from the analysis of uncertainty ranges.

Sources

[1] L. Zadeh, "Fuzzy sets," *Information and Control,* vol. 8, no. 3, pp. 338-353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] S. &. K. Chanas, "A concept of the optimal solution of the transportation problem with fuzzy cost coefficients," *Fuzzy Sets and Systems,* vol. 82, no. 3, pp. 299-305, 1996 . https://doi.org/10.1016/0165-0114(95)00278-2

[3] N. G. a. K. A. Razak, "Two Stage Fuzzy Transportation Problem," *Journal of Physical Sciences,* vol. 10, p. 63 – 69, 2006 . https://www.researchgate.net/publication/241902343.

[4] S.-T. L. a. C. Kao, "Solving fuzzy transportation problems based on extension principle," *European Journal of Operational Research,* vol. 153, no. 3, pp. 661-674, 2004. https://doi.org/10.1016/S0377-2217(02)00731-2

[5] P. P. a. G. Natarajan, "A New Algorithm for Finding a Fuzzy Optimal Solution for Fuzzy Transportation Problems," *Applied Mathematical Sciences,* vol. 4, no. 2, pp. 79 - 90, 2010. https://www.m-hikari.com/ams/ams-2010/ams-1-4-2010/pandianAMS1-4-2010.pdf

[6] D. H. &. P. Kumam, "A method for solving a fuzzy transportation problem via Robust ranking technique and ATM," *Cogent Mathematics,* vol. 4, pp. 1-11, 2017 . https://doi.org/10.1080/23311835.2017.1283730

[7] K. a. S. Subramanian, "An Approach for Solving Fuzzy Transportation Problem," *International Journal of Pure and Applied Mathematics,* vol. 119, no. 17, pp. 1523-1534, 2018 .

https://www.researchgate.net/publication/354776804

[8] A. A. a. S.Santhi, "A New Technique for Solving Fuzzy Transportation Problem Using Trapezoidal Fuzzy Numbers," *JOURNAL OF ALGEBRAIC STATISTICS,* vol. 13, no. 2, pp. 2216-2222, 2022 . https://publishoa.com/index.php/journal/article/download/409/374

[9] S. Vimala and S. Krishna Prabha2, "Fuzzy Transportation Problem through Monalisha's Approximation Method," *British Journal of Mathematics & Computer Science,* vol. 17, no. 2, pp. 1-11, 2016. DOI: 10.9734/BJMCS/2016/26097

[10] S. K. a. T. D. Rao, "Optimal Bounds for Fully Fuzzy Transportation Problems: A Parametric Approach," *Research Square,* pp. 1-12, 2024 . https://doi.org/10.21203/rs.3.rs-3890262/v1

[11] A. R. a. J. Singh, "Fully Fuzzy Solution of Transportation Problem in Fuzzy Environment," *International Journal of Science and Research,* vol. 11, no. 12, pp. 794-799, 2022 . https://www.ijsr.net/archive/v11i12/SR221215125834.pdf.

[12] D. A. S. K. a. M. S. Taher, "MAXIMUM {SUPPLIES, DEMANDS} METHOD TO FIND THE INITIAL TRANSPORTATION PROBLEM," *Journal of University of Zakho,* vol. 1, no. 2, pp. 849-853, 2013 . https://sjuoz.uoz.edu.krd/index.php/sjuoz/article/download/430/259/655