*Al-Mustansiriyah Journal of Science*
*ISSN: 1814-635X (print) 2521-3520 (electronic)*

*ORIGINAL ARTICLE* *OPEN ACCESS*

# A New Fractional Predator–Prey Model for Studying the Dynamic Relationship between Financial Corruption and Society-Simulation Study

Dilbar M. Sharif Abdullah [a] and Faraj Y. Ishak [a]

[a] *Department of Statistics, College of Administration & Economics, University of Duhok, Iraq*

**CORRESPONDENCE**
*Dilbar M. Sharif Abdullah*
*dilbar.abdullah88m@gmail.com*

**ARTICLE INFO**
*Received: Aug. 01, 2025*
*Revised: Oct. 04, 2025*
*Accepted: Oct. 29, 2025*
*Published: Dec. 30, 2025*



**ABSTRACT:** ***Background:*** *This study investigates the existence and uniqueness of solutions for a class of nonlinear fractional differential systems governed by Caputo derivatives. Such systems are effective in modeling dynamic processes with hereditary behaviors and anomalous responses, which are commonly observed in socio-economic, biological, and engineering contexts. Traditional integer-order differential models fail to capture these features, motivating the use of fractional-order formulations.* ***Objective:*** *The primary goal is to establish rigorous mathematical conditions ensuring the well-posedness of nonlinear fractional systems involving coupled variables. By proving existence and uniqueness, the study provides a solid theoretical foundation for analysis, simulation, and application of fractional-order models to complex systems with hereditary dynamics.* ***Methods:*** *The fractional differential system is transformed into an equivalent integral formulation using Riemann–Liouville fractional integral operators. Within a Banach space, two operators are defined: one is continuous and compact, while the other is contractive. The existence of solutions is established via Krasnoselskii's fixed point theorem, and the boundedness and Lipschitz continuity of the nonlinear terms ensure uniqueness through Banach's contraction principle. The system admits a unique solution confined to a closed, invariant subset of the function space, providing a rigorous framework for analytical and numerical investigations.* ***Results:*** *Analytical results confirm that the proposed fractional-order system satisfies conditions for existence and uniqueness of solutions. The integral representation demonstrates the well-posedness of the model and validates its ability to capture hereditary dynamics accurately. Operator-based analysis ensures stability and supports convergence of numerical methods, making the model practically applicable for simulations of socio-economic and biological processes.* ***Conclusions:*** *This work establishes a rigorous theoretical foundation for nonlinear fractional differential systems with Caputo derivatives. By addressing existence, uniqueness, and stability, it provides a reliable framework for modeling systems where past states influence current behavior. The findings facilitate both analytical and numerical studies of fractional-order dynamics, expanding the applicability of fractional calculus in diverse fields, including economics, population dynamics, control theory, and material science.*

**KEYWORDS:** *Boundary value problems; Fractional differential equations; Fixed-point theorem; Krasnoselskii's theorem; Predator–Prey model*

## INTRODUCTION

The earliest mathematical formulation of predator–prey interactions dates back to 1925, when American biophysicist Alfred Lotka aimed to model oscillatory behaviors in chemical reactions. A year later, Vito Volterra expanded on this by analyzing fluctuations in fish populations in the Adriatic Sea during World War I. His study revealed that the suspension of commercial fishing, caused by the conflict between Austria and Italy, led to an increase in predator fish and a corresponding decline in prey species—an observation that supported the theoretical framework now known as the

Lotka–Volterra model [1]–[5]. Alfred Lotka and Vito Volterra made key contributions to understanding interactions between predators and prey, and the model describing these relationships is named in their honor. Their groundbreaking work established the foundation of population biology and opened the door to further research on how biological populations change over time [6]–[8]. The dynamic Lotka–Volterra predator-prey model captures the complex, time-dependent interactions within ecosystems by providing a mathematical framework for analyzing the relationships among ecological competitors [9], [10]. The Lotka–Volterra model, initially introduced to represent the interaction between two biological species—typically a predator and its prey, has gradually developed into a versatile tool for examining complex ecological relationships. At its core, the model employs differential equations to depict how various forms of interspecies interactions, including competition, predation, and symbiosis, influence population dynamics over time [11]–[13]. By allowing derivatives of non-integer order, fractional differential equations (FDEs) provide a more generalized version of classical differential equations, broadening the definition of differentiation beyond integer values [14]. Systems with dynamics dependent on past states, hereditary behaviors, and anomalous diffusion can be modeled using this extension. These traits are often seen in domains including control theory, economics, physics, engineering, and biology [15].

The theoretical foundations of fractional calculus can be found in the early works of mathematicians such as Leibniz, Liouville, and Riemann. Caputo made further contributions, and his formulations of fractional derivatives serve as the foundation for many modern applications [16]–[18]. Formulations such as the Riemann-Liouville and Caputo derivatives appear in publications quite often. Because it agrees with the initial conditions given by classical (integer-order) derivatives, the Caputo derivative is very useful in physical models [18]–[20]. Recent developments in fractional differential equations have focused on analyzing the existence, uniqueness, stability, and long-term behavior of solutions. These advances have strengthened the theoretical foundation of fractional calculus through the use of tools like special functions (e.g., the Mittag-Leffler function), integral transforms, and fixed-point theorems [15], [21], [22]. In our earlier work [23], we introduced an integer-order Lotka–Volterra model to examine the dynamic relationship between financial corruption and society, focusing on the existence, uniqueness, stability, and Ulam–Hyers as well as Ulam–Hyers–Rassias stability of the system. The present manuscript extends this work by formulating a fractional-order Lotka–Volterra model using Caputo derivatives, providing a broader framework for analyzing the dynamics of financial corruption and society. We establish the existence, uniqueness, and stability of the proposed system using fixed-point theorems and eigenvalue analysis, while also employing analytical tools such as boundary value problems, fractional differential equations, the fixed-point theorem, Krasnoselskii's theorem, and the predator–prey model. To clarify the novelty of the present study, we note that although several fractional predator–prey models have been proposed, including [24], [25], these works primarily focus on biological systems and phenomena such as harvesting or the Allee effect. In contrast, our research extends the previous integer-order model by formulating a fractional-order Lotka–Volterra framework specifically for socio-economic dynamics. This approach allows us to investigate existence, uniqueness, and Ulam–Hyers–Rassias stability using Caputo derivatives, while employing analytical tools such as boundary value problems, fixed-point theorems, and Krasnoselskii's theorem.

In this study, we use numerical simulations with assumed parameter values to demonstrate the qualitative dynamics and stability of the proposed fractional-order model, thereby providing a solid theoretical foundation for future empirical applications. This approach is motivated by the fact that reliable and structured datasets on financial corruption are rarely available, as corruption is often hidden, underreported, or indirectly measured, which makes direct calibration of the model with real data highly challenging. At present, our work is theoretical and based on numerical simulations. However, the model could be applied to real-world contexts by calibrating its parameters with proxy datasets such as Transparency International's Corruption Perceptions Index (CPI), World Bank governance indicators, or survey-based measures of institutional trust and social behavior. Furthermore, our numerical simulations reveal dynamic features unique to socio-economic systems, which are not addressed in the aforementioned biological models. Therefore, this work extends the fractional predator–prey modeling framework to a novel context, demonstrating the originality and significance of our contribution. In order to examine the dynamic relationship between the proliferation of financial corruption and population behavior, we develop a nonlinear fractional-order Lotka-Volterra model. The hereditary characteristics inherent to socio-economic systems are accounted for by the model through the use of tools from fractional calculus, specifically Caputo derivatives and the Riemann–Liouville fractional integral. To guarantee the well-posedness of the corresponding boundary value problems, we establish the existence and uniqueness of solutions under standard conditions of boundedness and Lipschitz continuity. The analysis is grounded in fixed-point theory within the context of Banach spaces. Integral operators involved in the formulation are shown to possess continuity and compactness, supported by the application of the Arzelà–Ascoli theorem to establish equicontinuity and relative compactness of the corresponding solution set. The solution is represented in an explicit integral form using the Riemann–Liouville fractional operator, with an additional correction term incorporated to satisfy the boundary conditions. Uniqueness is rigorously justified by employing the contraction mapping principle, thereby providing a solid theoretical basis for the formulation of robust numerical algorithms capable of capturing the system's asymptotic dynamics.

In this paper, we present a nonlinear Lotka-Volterra model to describe the interaction between population and corruption levels. The system is governed by the following differential equations:

$$\begin{aligned} &{}^{c}D^{\alpha}\Phi(\tau) = \xi_1\Phi(\tau) - \xi_2\Phi(\tau)\Psi(\tau) - \alpha\Phi^2(\tau), \\ &{}^{c}D^{\alpha}\Psi(\tau)\Phi(\tau) = -\rho_1\Psi(\tau) + \rho_2\Psi(\tau) - \beta ln\left(\Psi(\tau)\right). \\ &\Phi(0) = \Phi_0, \quad \Psi(0) = \Psi_0, \quad \Phi(T) = a, \quad \Psi(T) = b \quad , \quad \tau \in [0, T] \end{aligned} \tag{1}$$

In this model, $\Phi(\tau)$ signifies a social or demographic variable that evolves over time, while $\Psi(\tau)$ captures the corresponding corruption level. The parameters $\xi_1,\ \xi_2,\ \alpha,\ \rho_1,\ \rho_2$, and $\beta$ are constants that define the system's internal behavior and the strength of interactions between variables, $\Phi_0,\ \Psi_0$,a and b are the boundary conditions, $1 < \alpha \leq 2$ is the Caputo fractional order. The nonlinear damping term $-\alpha\Phi^2(\tau)$ represents inherent constraints within the population, such as limited resources or saturation phenomena. Meanwhile, the logarithmic expression $-\beta\, ln\left(\Psi(\tau)\right)$ accounts for the decreasing marginal effect of rising corruption levels on the system's dynamics.

The quadratic term $-\alpha\Phi^2(\tau)$ is employed to represent self-limiting growth or competition, capturing the idea that expansion slows as the population $\Phi(\tau)$ becomes large due to resource scarcity or overcrowding. The logarithmic function is chosen because it reflects diminishing returns at higher values of $\Psi(\tau)$, while also capturing instability thresholds when $\Psi(\tau)$ approaches zero, since $\ln\Psi \to -\infty$. These functions were selected for their ability to model realistic nonlinear effects while preserving analytical tractability. If alternative nonlinear terms (e.g., cubic, exponential, or rational functions) were used, the qualitative behavior of the system could differ significantly: for instance, cubic terms might produce sharper thresholds, while exponential terms could amplify instability, altering the stability structure and long-term dynamics.

Beyond the theoretical proofs, the model provides dynamic insights into the relationship between corruption and institutional strength. It demonstrates the presence of threshold effects, where corruption can spiral out of control if institutional capacity $\Psi(\tau)$, falls below a critical level, underlining the importance of early reinforcement of institutions. The model also reveals nonlinear dynamics, showing that uniform policies are not always effective: corruption may self-limit at high levels, while small interventions in weak societies may fail unless thresholds are crossed. Additionally, the coupled $(\Phi,\ \Psi)$, terms highlight the interdependence of corruption and institutions, implying that effective strategies must address both simultaneously. Finally, the model offers predictive capability through simulations, which can identify tipping points, forecast corruption spikes, and guide policymakers to allocate resources efficiently and implement timed, sequenced interventions rather than relying on brute-force measures. This study is organized as follows: Section 1 (Introduction) presents the problem, motivation, and literature gaps. Section 2 (Model Formulation) introduces the fractional-order Lotka-Volterra system with assumptions and variables. Section 3 (Existence and Uniqueness) proves solution existence and uniqueness using fixed-point theorems. Section 4(Stability Analysis) studies equilibrium points and local stability with Ulam-Hyers concepts. Section 5 (Numerical Simulations) illustrates system dynamics under different parameters. Section 6 (Discussion and Conclusion) interprets results, discusses implications, and suggests future research directions.

## MATERIALS AND METHODS

### Preliminaries

This section presents essential terminology and conclusions from fractional calculus that will be employed in the following analysis.

**Definition 1 [26].** Given a real number $q > 0$, the fractional integral of order $q$ applied to a function $f(t)$, and represented by $I^q f(t)$, is defined by (2)

$$\mathrm{I}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \tag{2}$$

provided the integral exists.

**Definition 2** [26]**.** Let $q > 0$ and $n \in N$ be the integer satisfying $n-1 < q \leq n$. The fractional derivative of order q of a function $f(t)$, and represented by $D^q f(t)$, is defined by (3).

$$D^q f(t) = \frac{1}{\Gamma(n-q)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-q-1} f(s) ds, \tag{3}$$

provided the right-hand side is pointwise defined for all $t \in (0, +\infty)$.

**Lemma 1** [27]**.** Let $\alpha, \beta > 0$. Then the following properties hold:
- If $\beta > n$, then the fractional derivative of the power function satisfies:

$$D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha-1}. \tag{4}$$

- In the case of the Caputo fractional derivative, we have:
$^c D^\alpha t^k = 0$, for $k = 0, 1, ......, n-1$.

**Lemma 2** [27]**.** Let $\alpha > 0$, and let $n \in N$ be such that $n-1 < \alpha \leq n$. Then the fractional differential equation

$$^c D^\alpha_{0+} x(t) = 0 \tag{5}$$

has the unique general solution

$$x(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \quad c_i \in R, \quad i = 1, 2, ..., n. \tag{6}$$

In view of Lemma 2, it follows that:

$$I^q \, {}^c D^q x(t) = x(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \tag{7}$$
$$c_i \in R, i = 1, 2, ..., n.$$

**Theorem 1** [28] **(Krasnoselskii's Fixed Point Theorem)**. Let $M$ be a nonempty, closed, and convex subset of a Banach space $X$. Suppose that $A, B{:}M \to X$ are two operators satisfying the following conditions: (i) for all $x, y \in M$, the element $Ax + By$ belongs to $M$; (ii) the operator $A$ is continuous and compact; and (iii) The mapping $B$ is a contraction on $M$. Then there exists a point $z \in M$ such that: $z = Az + Bz$.

**Theorem 2 (Arzela -Ascoli theorem).** Let $\Omega$ be a compact Hausdorff metric space. Then $M \subset C(\Omega)$ is relatively compact $\Leftrightarrow$ $M$ is uniformly bounded and uniformly equicontinuous.

It's useful to rewrite our system as a vector equation:

$$^c D^\alpha \mathrm{X}(\tau) = \mathrm{Y}\big(\tau, \Phi(\tau), \Psi(\tau)\big), \tag{8}$$

$$\mathrm{X}(0) = \check{0}, \quad \mathrm{X}(\mathrm{T}) = \chi_T, \quad 0 \leq \tau \leq \mathrm{T}, \tag{9}$$

where

$$\mathrm{X}(\tau) = \begin{bmatrix} \Phi(\tau) \\ \Psi(\tau) \end{bmatrix}, \tag{10}$$

$$\mathrm{Y}\big(\tau, \Phi(\tau), \Psi(\tau)\big) = \begin{bmatrix} \mathcal{F}\big(\tau, \Phi(\tau), \Psi(\tau)\big) \\ g\big(\tau, \Phi(\tau), \Psi(\tau)\big) \end{bmatrix} = \begin{bmatrix} \xi_1 \Phi(\tau) - \xi_2 \Phi(\tau)\Psi(\tau) - \alpha \Phi^2(\tau) \\ -\rho_1 \Psi(\tau) + \rho_2 \Psi(\tau) - \beta l\, n\big(\Psi(\tau)\big) \end{bmatrix}, \tag{11}$$

$$\check{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathcal{X}_T = \begin{bmatrix} \mathfrak{a} \\ \mathfrak{b} \end{bmatrix}. \tag{12}$$

**Lemma 3.** Suppose Y$\in C(0,\mathrm{T}) \cap \mathcal{L}(0,\mathrm{T})$. Then the unique solution to the fractional differential system (2.1) is given by the integral representation:

$$\mathrm{X}(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \mathrm{Y}\Big(s, \Phi(s), \Psi(s)\Big) ds - \frac{\tau}{\mathrm{T}\Gamma(\alpha)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{\alpha-1} \mathrm{Y}\Big(s, \Phi(s), \Psi(s)\Big) ds + \frac{\tau\, \chi_T}{T}. \tag{13}$$

**Proof:**
The general solution of a Caputo-type fractional differential equation can be expressed as:

$$\mathrm{X}(\tau) = \mathrm{I}^{\alpha}\mathrm{Y}(\tau) + C_0 + C_1\tau, \tag{14}$$

where $\mathrm{I}^{\alpha}$ represents the Riemann-Liouville fractional integral of order $\alpha$, and $C_0, C_1 \in R^2$ are constant vectors to be determined using the boundary conditions.

From the initial condition $X(0) = 0$, it follows immediately that:

$$C_0 = 0. \tag{15}$$

Evaluating the solution at $\tau =$T, we have:

$$\mathrm{X}(\mathrm{T}) = \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{\alpha-1}\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)ds + C_1\mathrm{T} = \mathcal{X}_T. \tag{16}$$

Solving for $C_1$, we obtain:

$$C_1\mathrm{T} = \mathcal{X}_T - \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{\alpha-1}\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)ds \tag{17}$$

$$C_1 = \frac{x_T}{\mathrm{T}} - \frac{1}{\mathrm{T}\Gamma(\alpha)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{\alpha-1}\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)ds. \tag{18}$$

Substituting $C_0$ and $C_1$ back into the expression for $\mathrm{X}(\tau)$, we find:

$$\begin{aligned}\mathrm{X}(\tau) =& \frac{1}{\Gamma(\alpha)}\int_0^{\tau} (\tau-s)^{\alpha-1}\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)ds + \\ & \Big(\frac{x_T}{\mathrm{T}} - \frac{1}{\mathrm{T}\Gamma(\alpha)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{\alpha-1}\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)ds\Big)\tau.\end{aligned} \tag{19}$$

Simplifying yields, the final form:

$$\begin{aligned}\mathrm{X}(\tau) =& \frac{1}{\Gamma(\alpha)}\int_0^{\tau} (\tau-s)^{\alpha-1}\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)ds - \\ & \frac{\tau}{\mathrm{T}\Gamma(\alpha)}\int_0^{\mathrm{T}} (\mathrm{T}-s)^{\alpha-1}\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)ds + \frac{\tau x_T}{\mathrm{T}}.\end{aligned} \tag{20}$$

## RESULTS AND DISCUSSION

In this section, we analyze the existence, uniqueness, and local stability of solutions for the proposed nonlinear fractional-order system. We first establish conditions under which the system admits at least one solution and a unique solution by applying fixed-point theorems. Subsequently, we identify the equilibrium points of the system and examine their local stability using the Jacobian matrix and eigenvalue analysis. Numerical calculations are provided to illustrate the behavior of the system for specific parameter values, highlighting both stable and unstable equilibria.

**Theorem 3.** Suppose that $Y : [0, T] \times R \times R \to R$ is a continuous function satisfying a Lipschitz condition and $\|\mathrm{Y}(\tau, \Phi, \Psi)\| \leq \mathcal{L}$, for all $\tau, \Phi, \Psi \in$ [0,T] $\times R \times R, \mathcal{L} \in R^+$.

If the condition:

$$\frac{\mathcal{M}\mathrm{T}^{\alpha}}{\Gamma(\alpha+1)} < 1 \tag{21}$$

holds, then (1) admits at least one solution.

**Proof:**
Define the closed subset $\Lambda = \{X \in C : \|X\| \leq r\}$ where $r$ is chosen such that:

$$\mathcal{X}_T + \frac{2\,\mathrm{T}^\alpha \mathcal{L}}{\Gamma(\alpha+1)} \leq r. \tag{22}$$

We define two operators $\mathcal{F}, g$ on $\Lambda$ as follows:

$$(\mathrm{AX})(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \mathrm{Y}\Big(s, \Phi(s), \Psi(s)\Big) ds, \tag{23}$$

$$(\mathrm{BX})(\tau) = \quad \frac{\tau\,\mathcal{X}_T}{T} - \frac{\tau}{\mathrm{T}\Gamma(\alpha)} \int_0^\mathrm{T} (\mathrm{T} - s)^{\alpha-1} \mathrm{Y}\Big(s, \Phi(s), \Psi(s)\Big) ds. \tag{24}$$

We aim to estimate the norm:

$$\|(\mathrm{AX})(\tau) + (\mathrm{BX})(\tau)\| \tag{25}$$

For $X \in \Lambda$ we obtain:

$$\begin{aligned} \|(\mathrm{AX})(\tau) + \ (\mathrm{BX})(\tau)\| \leq & \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \left\|\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)\right\| ds \\ & + \left|\frac{\tau\,\mathcal{X}_T}{T}\right| - \left|\frac{\tau}{\mathrm{T}\Gamma(\alpha)}\right| \int_0^T (\mathrm{T} - s)^{\alpha-1} \left\|\mathrm{Y}\big(s, \Phi(s), \Psi(s)\big)\right\| ds \end{aligned} \tag{26}$$

Since $\|\mathrm{Y}(\tau, \Phi, \Psi)\| \leq \mathcal{L}$,we get:

$$\leq \frac{\tau^\alpha\,\mathcal{L}}{\Gamma(\alpha+1)} + \frac{\tau\,\mathrm{T}^\alpha\,\mathcal{L}}{\mathrm{T}\Gamma(\alpha+1)} + \frac{\tau\,\mathcal{X}_T}{T}. \tag{27}$$

Maximizing over $\tau \in [0, T]$,we obtain:

$$\leq \ \frac{\mathrm{T}^\alpha\,\mathcal{L}}{\Gamma(\alpha+1)} + \ \frac{\mathrm{T}^\alpha\,\mathcal{L}}{\Gamma(\alpha+1)} + \ \chi_T \tag{28}$$

$$\leq \ \frac{2\,\mathrm{T}^\alpha\,\mathcal{L}}{\Gamma(\alpha+1)} + \ \mathcal{X}_T \ \leq \ r. \tag{29}$$

Thus, $AX + BX \in \Lambda$.

Given that A is continuous due to the continuity of Y, our next step is to establish the compactness of A. To this end, we first show that the set $(AX)(\tau)$ is uniformly bounded on $\Lambda_r$ as follows:

$$\|(\mathrm{AX})(\tau)\| \leq \frac{\mathrm{T}^\alpha\,\mathcal{L}}{\Gamma(\alpha+1)}. \tag{30}$$

Since $Y$ is bounded on $[0, T] \times \Lambda r \times \Lambda r$ let:

$$\mathrm{Y}_{max} = \sup_{(\tau,\Phi,\Psi)\in[0,\mathrm{T}]\times\Lambda_r\times\Lambda_r} \|\mathrm{Y}(\tau, \Phi, \Psi)\|. \tag{31}$$

Then, for any $\tau_1, \tau_2 \in [0, \mathrm{T}]$,

$$\begin{aligned} \|(\mathrm{AX})(\tau_2) + (\mathrm{AX})(\tau_1)\| \leq & \frac{1}{\Gamma(\alpha)} \int_0^{\tau_2} \|(\tau_2 - s)^{\alpha-1}\,\mathrm{Y}(s, \Phi(s), \Psi(s))\| ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \|(\tau_1 - s)^{\alpha-1}\,\mathrm{Y}(s, \Phi(s), \Psi(s))\| ds\,, \end{aligned} \tag{32}$$

$$\leq \frac{Y_{max}}{\Gamma(\alpha)} \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1}\,ds - \frac{Y_{max}}{\Gamma(\alpha)} \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1}\,ds, \tag{33}$$

$$\leq \frac{Y_{max}}{\Gamma(\alpha+1)}\|\tau_2^{\alpha}-\tau_1^{\alpha}\|. \tag{34}$$

Since this bound does not depend on $\tau$,it follows that the set $(AX)(\tau)$ is relatively compact on $\Lambda_r$. Therefore, by applying the Arzelà-Ascoli theorem, we conclude that $(AX)(\tau)$ is compact in $\Lambda_r$.

Let $X_1$, $X_2 \in \Lambda_r$ and $\tau \in [0,T]$,we have:

$$\|(\mathrm{BX}_1)(\tau)-(\mathrm{BX}_2)(\tau)\| \leq \left|\frac{\tau}{\mathrm{T}\Gamma(\alpha)}\right|\int_0^T (T-s)^{\alpha-1}\|\mathrm{Y}(\mathrm{s},\phi_1(\mathrm{s}),\psi_1(\mathrm{s}))\|\,ds- \int_0^T (T-s)^{\alpha-1}\|\mathrm{Y}(\mathrm{s},\phi_2(\mathrm{s}),\psi_2(\mathrm{s}))\|\,ds, \tag{35}$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{\alpha-1}\left\|\mathrm{Y}\big(s,\Phi_1(s),\Psi_1(s)\big)-\mathrm{Y}\big(s,\Phi_2(s),\Psi_2(s)\big)\right\|ds. \tag{36}$$

Since Y satisfies a Lipschitz condition with constant $\mathcal{M}$,we get:

$$\|\mathrm{Y}\,(s,\Phi_1(s),\Psi_1(s))-\mathrm{Y}\,(s,\Phi_2(s),\Psi_2(s))\| \leq \mathcal{M}(|\,\Phi_1-\Phi_2\,|+|\,\Psi_1-\Psi_2\,|). \tag{37}$$

Hence,

$$\begin{aligned}\|(\mathrm{BX}_1)(\tau)-(\mathrm{BX}_2)(\tau)\| &\leq \frac{\mathcal{M}}{\Gamma(\alpha)}\int_0^{\mathrm{T}}(\Gamma-s)^{\alpha-1}(|\Phi_1-\Phi_2|+|\Psi_1-\Psi_2|)ds\\ &\leq \frac{\mathcal{M}\,\mathrm{T}^{\alpha}}{\Gamma(\alpha+1)}(|\,\Phi_1-\,\Phi_2\,|+|\,\Psi_1-\Psi_2\,|).\end{aligned} \tag{38}$$

It follows that $(BX)(\tau)$ is contraction, this completes the proof.

**Theorem 4.** Suppose that $Y:[0,T]\times R\times R{\to}R$ is a continuous function satisfying a Lipschitz condition with respect to the last two variables. If

$$\Omega\ = \frac{\mathcal{L}(\tau^{\alpha}+T^{\alpha})}{\Gamma(\alpha+1)}\ <1, \tag{39}$$

then the integral equation associated with the fractional differential system admits a unique solution on the interval [0,T].

**Proof:**

Define the operator $Y: C\to C$ as:

$$\mathrm{X}(\tau)=\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\mathrm{Y}\Big(s,\Phi(s),\Psi(s)\Big)ds-\frac{\tau}{\mathrm{T}\Gamma(\alpha)}\int_0^{\mathrm{T}}(\mathrm{T}-s)^{\alpha-1}\mathrm{Y}\Big(s,\Phi(s),\Psi(s)\Big)ds + \frac{\tau\,\mathcal{X}_T}{T},\quad \tau\in[0,\mathrm{T}], \tag{40}$$

$C$ represents the Banach space consisting of all continuous real-valued functions defined on the interval $[0,T]$, endowed with the supremum norm.

Assume

$$sup_{\tau\in[0,T]}|Y(\tau,0,0)|=K. \tag{41}$$

Define the closed ball

We choose the radius $r$ of a closed ball $\Lambda\subset C$ defined by

$$\Lambda=\{X\in C:\|X\|\leq r\} \tag{42}$$

and choose $r$ such that $Y(\Lambda)\subset\Lambda$. To ensure this, take

$$r\geq\frac{\tau(\mathcal{L}_1(\rho_1+\rho_2)+K)(\tau^{\alpha-1}-T^{\alpha-1})}{\Gamma(\alpha+1)}+\mathcal{X}_T. \tag{43}$$

Now, we verify that $Y(\Lambda) \subset \Lambda$. For any $X \in \Lambda$, we estimate:

$$
\begin{aligned}
\|X(\tau)\| =& \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \|Y(s, \Phi(s), \Psi(s))\| ds + \\
& \left|\frac{\tau}{T\Gamma(\alpha)}\right| \int_0^T (T - s)^{\alpha-1} \|Y(s, \Phi(s), \Psi(s))\| ds + \left|\frac{\tau \mathcal{X}_T}{T}\right| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \left(\|Y(s, \Phi(s), \Psi(s)) - Y(\tau, 0, 0)\| + \|Y(\tau, 0, 0)\|\right) ds ds - \\
& \frac{\tau}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} \left(\|Y(s, \Phi(s), \Psi(s)) - Y(\tau, 0, 0)\| + \|Y(\tau, 0, 0)\|\right) ds ds + \left|\frac{\tau x_T}{T}\right|. \quad (44)
\end{aligned}
$$

Applying the Lipschitz condition and the bound $\|Y(s, \Phi(s), \Psi(s))\| \leq \mathcal{L}_1(\|\Phi\| + \|\Psi\|) + K \leq \mathcal{L}_1(\rho_1 + \rho_2) + K$, we obtain:

$$
\leq \frac{\mathcal{L}_1(\rho_1 + \rho_2) + K}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} ds - \frac{\tau(\mathcal{L}_1(\rho_1 + \rho_2) + K)}{T\Gamma(\alpha)} \int_0^{\mathrm{T}} (T - s)^{\alpha-1} ds + \frac{\tau \mathcal{X}_T}{T}. \quad (45)
$$

$$
\leq \frac{\tau^\alpha(\mathcal{L}_1(\rho_1 + \rho_2) + K)}{\Gamma(\alpha + 1)} - \frac{\tau T^\alpha(\mathcal{L}_1(\rho_1 + \rho_2) + K)}{T\Gamma(\alpha + 1)} + \frac{\tau \mathcal{X}_T}{T}. \quad (46)
$$

$$
\leq \frac{\tau^\alpha(\mathcal{L}_1(\rho_1 + \rho_2) + K)}{\Gamma(\alpha + 1)} - \frac{\tau T^{\alpha-1}(\mathcal{L}_1(\rho_1 + \rho_2) + K)}{\Gamma(\alpha + 1)} + \frac{\tau \mathcal{X}_T}{T}. \quad (47)
$$

This simplifies to:

$$
\|\mathrm{X}(\tau)\| = \frac{\tau(\mathcal{L}_1(\rho_1 + \rho_2) + K)(\tau^{\alpha-1} - T^{\alpha-1})}{\Gamma(\alpha + 1)} + \mathcal{X}_T \leq r. \quad (48)
$$

Next, we aim to show that $X(\tau)$ defines a contraction mapping. Let $\mathcal{X}_1, \mathcal{X}_2 \in \Lambda$. Then for all $\tau \in [0, \mathcal{T}]$, the following inequality holds:

$$
\begin{aligned}
\|X_1(\tau) - X_2(\tau)\| \leq & \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \left(\|\mathrm{Y}(s, \Phi_1(s), \Psi_1(s))\|\right) ds - \\
& \frac{\tau}{\mathrm{T}\Gamma(\alpha)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{\alpha-1} \left(\|\mathrm{Y}(s, \Phi_1(s), \Psi_1(s))\| ds + \frac{\tau X_T}{T} - \right. \\
& \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \left(\|\mathrm{Y}(s, \Phi_2(s), \Psi_2(s))\|\right) ds + \\
& \frac{\tau}{\mathrm{T}\Gamma(\alpha)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{\alpha-1} \left(\|\mathrm{Y}(s, \Phi_2(s), \Psi_2(s))\|\right) ds - \frac{\tau X_T}{T}. \quad (49)
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \left(\|\mathrm{Y}(s, \Phi_1(s), \Psi_1(s)) - \mathrm{Y}(s, \Phi_2(s), \Psi_2(s))\|\right) ds - \\
& \frac{\tau}{\mathrm{T}\Gamma(\alpha)} \int_0^{\mathrm{T}} (\mathrm{T} - s)^{\alpha-1} \left(\|\mathrm{Y}(s, \Phi_1(s), \Psi_1(s)) - \mathrm{Y}(s, \Phi_2(s), \Psi_2(s))\|\right) ds. \quad (50)
\end{aligned}
$$

Using the Lipschitz condition again, we get:

$$
\leq \frac{\tau^\alpha \mathcal{L}|\Phi_1 - \Phi_2| + |\Psi_1 - \Psi_2|}{\Gamma(\alpha + 1)} - \frac{T^\alpha \mathcal{L}|\Phi_1 - \Phi_2| + |\Psi_1 - \Psi_2|}{\Gamma(\alpha + 1)} \quad (51)
$$

$$
\leq \frac{\mathcal{L}(\tau^\alpha - T^\alpha)|\Phi_1 - \Phi_2| + |\Psi_1 - \Psi_2|}{\Gamma(\alpha + 1)} \leq \Omega(|\Phi_1 - \Phi_2| + |\Psi_1 - \Psi_2|). \quad (52)
$$

Since $\Omega < 1$, the operator is a contraction on the complete metric space $\Lambda \subset C$. Therefore, by applying Banach's fixed-point theorem, we conclude that $\mathrm{X}(\tau)$ has a unique fixed point in $\Lambda$, which represents the unique solution to the corresponding integral equation.

## Model Stability

Consider the nonlinear fractional-order system governed by the Caputo derivatives:

$$^{c}D^{\alpha}\Phi(\tau) = \xi_1\Phi(\tau) - \xi_2\Phi(\tau)\Psi(\tau) - \xi_3\Phi^2(\tau), \tag{53}$$
$$^{c}D^{\alpha}\Psi(\tau) = -\rho_1\Psi(\tau) + \rho_2\Phi(\tau)\Psi(\tau) - \rho_3 ln(\Psi)(\tau),$$

where $\alpha \in (1, 2]$,and $\Phi(\tau), \Psi(\tau)$ denote the state variables. The system is subjected to the boundary conditions:

$$\Phi(\tau_0) = \Phi_0, \quad \Psi(\tau_0) = \Psi_0, \quad \Phi(\mathrm{T}) = \mathfrak{a}, \quad \Psi(\mathrm{T}) = \mathfrak{b}. \tag{54}$$

To identify the equilibrium points of the system set:

$$^{c}D^{\alpha}\Phi(\tau) = 0, \ ^{c}D^{\alpha}\Psi(\tau) = 0 \tag{55}$$

$$\xi_1\Phi - \xi_2\Phi\Psi - \xi_3\Phi^2 = 0 \text{ Equation (1)}, \tag{56}$$

$$-\rho_1\Psi + \rho_2\Phi\Psi - \rho_3 ln(\Psi) = 0 \text{ Equation (2)}, \tag{57}$$

From (1):

$$\Phi(\xi_1 - \xi_2\Psi - \xi_3\Phi) = 0. \tag{58}$$

$$\Phi = 0 \Rightarrow -\rho_1\Psi - \rho_3 ln\Psi = 0 \Rightarrow \rho_1\Psi + \rho_3 ln(\Psi) = 0. \tag{59}$$

$$\text{for } \Psi = \check{\Psi} \Rightarrow \rho_1\check{\Psi} + \rho_3 ln(\check{\Psi}) = 0. \tag{60}$$

$$E_1 = (0, \check{\Psi}). \tag{61}$$

$$\xi_1 - \xi_2\Psi - \xi_3\Phi = 0 \Rightarrow \Phi = \frac{\xi_1 - \xi_2\Psi}{\xi_3} \tag{62}$$

$$0 = -\rho_1\Psi + \rho_2(\frac{\xi_1 - \xi_2\Psi}{\xi_3})\,\Psi - \rho_3 ln(\Psi). \tag{63}$$

This is a nonlinear transcendental equation in $\Psi$, typically solved numerically or graphically depending on parameter values. Let $\Psi = \Psi^*$ be such a solution. Then, the corresponding $\Phi$ value is then given by:

$$\Phi^* = \left(\frac{\xi_1 - \xi_2\Psi^*}{\xi_3}\right) \Rightarrow E_2 = \left(\frac{\xi_1 - \xi_2\Psi^*}{\xi_3}, \Psi^*\right). \tag{64}$$

To study the local stability of the equilibrium points, we compute the Jacobian matrix of the nonlinear system and analyze its eigenvalues.

$$\Lambda(\Phi, \Psi) = \begin{bmatrix} \xi_1\Phi - \xi_2\Phi\Psi - \xi_3\Phi^2 \\ -\rho_1\Psi + \rho_2\Phi\Psi - \rho_3 ln(\Psi) \end{bmatrix}. \tag{65}$$

$$\Lambda_1(\Phi, \Psi) = \xi_1\Phi - \xi_2\Phi\Psi - \xi_3\Phi^2, \tag{66}$$

$$\frac{\partial\Lambda_1}{\partial\Phi} = \xi_1 - \xi_2\Psi - 2\xi_3\Phi, \quad \frac{\partial\Lambda_1}{\partial\Psi} = -\xi_2\Phi, \tag{67}$$

$$\Lambda_2(\Phi, \Psi) = -\rho_1\Psi + \rho_2\Phi\Psi - \rho_3 ln(\Psi), \tag{68}$$

$$\frac{\partial\Lambda_2}{\partial\Phi} = \rho_2\Psi, \quad \frac{\partial\Lambda_2}{\partial\Psi} = -\rho_1 + \rho_2\Phi - \frac{\rho_3}{\Psi}. \tag{69}$$

$$J(\Phi, \Psi) = \begin{bmatrix} \xi_1 - \xi_2\Psi - 2\xi_3\Phi & -\xi_2\Phi \\ \rho_2\Psi & -\rho_1 + \rho_2\Phi - \frac{\rho_3}{\Psi} \end{bmatrix}. \tag{70}$$

Evaluating the Jacobian matrix at $E_1$, we obtain:

$$\mathcal{J}(0, \Psi_0) = \begin{bmatrix} \xi_1 - \xi_2\check{\Psi} & 0 \\ \rho_2\check{\Psi} & -\rho_1 - \frac{\rho_3}{\check{\Psi}} \end{bmatrix}. \tag{71}$$

$$\lambda_1 = \xi_1 - \xi_2\check{\Psi}, \tag{72}$$

$$\lambda_2 = -\rho_1 - \frac{\rho_3}{\check{\Psi}}. \tag{73}$$

If $\check{\Psi} > \frac{\xi_1}{\xi_2}$, then $\lambda_1 < 0$, implying stability in that direction. Conversely, if $\check{\Psi} < \frac{\xi_1}{\xi_2}$, then $\lambda_1 > 0$, implying instability. The second eigenvalue $\lambda_2$ is always negative, i.e. $\lambda_2 < 0$, which implies stability in the corresponding direction. For the second equilibrium point $(\Phi^*, \Psi^*)$, where

$$\Phi^* = \frac{\xi_1 - \xi_2\Psi^*}{\xi_3}, \tag{74}$$

the Jacobian matrix is

$$\mathcal{J}(\Phi^*, \Psi^*) = \begin{bmatrix} \xi_1 - \xi_2\Psi^* - 2\xi_3\Phi^* & -\xi_2\Phi^* \\ \rho_2\Psi^* & -\rho_1 + \rho_2\Phi^* - \frac{\rho_3}{\Psi^*} \end{bmatrix}. \tag{75}$$

If both eigenvalues possess negative real parts, the equilibrium point is locally asymptotically stable. On the other hand, if at least one eigenvalue has a real part greater than zero, the equilibrium becomes unstable.

Given the parameters:

$$\xi_1 = 3, \xi_2 = 0.5, \xi_3 = 1, \rho_1 = 2, \rho_2 = 0.4, \rho_3 = 0.1, \tag{76}$$

$${}^cD^\alpha\Phi = 3\Phi - 0.5\Phi\,\Psi - \Phi^2, \tag{77}$$

$${}^cD^\alpha\Psi = -2\Psi + 0.4\Phi\,\Psi - 0.1ln(\Psi). \tag{78}$$

$$\Phi(3 - 0.5\Psi - \Phi) = 0, \tag{79}$$

$$\text{(i) When } \Phi = 0: \tag{80}$$

Substituting this into the second equilibrium equation ${}^cD^\alpha\Psi = 0$ leads to:

$$0 = -2\Psi - 0.1ln(\Psi), \tag{81}$$

or equivalently,

$$2\Psi + 0.1ln(\Psi) = 0. \tag{82}$$

This transcendental equation has a solution expressed in terms of the Lambert $W$ function, with approximate numerical root:

$$\Psi \approx 0.2558. \tag{83}$$

Thus, the first equilibrium point is:

$$E_1 = (0, 0.2558). \tag{84}$$

(ii) When $\Phi = 3 - 0.5\Psi$

Substituting into the second equilibrium condition yields

$$\begin{aligned} 0 = -2\Psi + (3 - 0.5\Psi)0.4\,\Psi - 0.1ln(\Psi) &= -2\Psi + 1.2\Psi - 0.2\Psi^2 - 0.1ln(\Psi), \\ &= -0.2\Psi^2 - 0.8\Psi - 0.1ln(\Psi), \end{aligned} \tag{85}$$

which can be rewritten as:

$$0.2\Psi^2 + 0.8\Psi + 0.1ln(\Psi) = 0. \tag{86}$$

Numerically solving this transcendental equation yields:

$$\Psi \approx 0.1950, \tag{87}$$

and substituting back, the corresponding value of $\Phi$ is:

$$\Phi = 3 - 0.5\Psi = 3 - 0.5(0.1950) = 2.9025 \tag{88}$$

Hence, the second equilibrium point is:

$$E_2 = (2.9025, 0.1950). \tag{89}$$

Recall that the Jacobian matrix of the nonlinear system is given by:

$$\mathcal{J}(\Phi, \Psi) = \begin{bmatrix} \xi_1 - \xi_2\Psi - 2\xi_3\Phi & -\xi_2\Phi \\ \rho_2\Psi & -\rho_1 + \rho_2\Phi - \dfrac{\rho_3}{\Psi} \end{bmatrix}. \tag{90}$$

Assuming all parameters are normalized to unity:

$$\xi_1 = 3, \xi_2 = 0.5, \xi_3 = 1, \rho_1 = 2, \rho_2 = 0.4, \rho_3 = 0.1, \tag{91}$$

and evaluating the Jacobian at the equilibrium point $E_1 = (0, 0.2558)$, we obtain:

$$\begin{aligned} \mathcal{J}(E_1) = \begin{bmatrix} 3 - 0.5\Psi - 2\Phi & -0.5\Phi \\ 0.4\Psi & -2 + 0.4\Phi - \frac{0.1}{\Psi} \end{bmatrix} &= \begin{bmatrix} 3 - 0.1279 - 0 & 0 \\ 0.1023 & -2 + 0 - \frac{0.1}{0.2558} \end{bmatrix} \\ &= \begin{bmatrix} 2.8721 & 0 \\ 0.1023 & -2.3909 \end{bmatrix}. \end{aligned} \tag{92}$$

To assess the local stability of the system, we evaluate the eigenvalues of the Jacobian matrix by solving its characteristic equation:

$$\det(\mathcal{J} - \lambda I) = 0 \tag{93}$$

Here, $I$ denotes the identity matrix and $\lambda$ represents the eigenvalues. We proceed by computing the matrix $(\mathcal{J} - \lambda I)$ as follows:

$$\mathcal{J} - \lambda I = \begin{bmatrix} 2.8721 & 0 \\ 0.1023 & -2.3909 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2.8721 - \lambda & 0 \\ 0.1023 & -2.3909 - \lambda \end{bmatrix}. \tag{94}$$

$$\det(\mathcal{J} - \lambda I\ ) = (2.8721 - \lambda)(-2.3909 - \lambda) - (0)(0.1023). \tag{95}$$

Calculating each term:

$$(2.8721 - \lambda)(-2.3909 - \lambda) = -6.8669 - 0.4812\lambda + \lambda^2 \tag{96}$$

$$(0)(0.1023) = 0 \tag{97}$$

Expanding:

$$\lambda^2 - 0.4812\lambda - 6.8669 = 0. \tag{98}$$

The eigenvalues of the Jacobian matrix at $E_1 = (0, 0.2557)$ are approximately:

$$\lambda_1 \approx 2.8721, \quad \lambda_2 \approx -2.3909. \tag{99}$$

Since the eigenvalues have opposite signs, the equilibrium point $E_1$ is a saddle point and hence locally unstable.

We evaluate the Jacobian matrix $\mathcal{J}(\Phi, \Psi)$ at the point $E_2 = (2.9025, 0.1950)$. The general form of the Jacobian is:

$$\mathcal{J}(\Phi, \Psi) = \begin{bmatrix} 3 - 0.5\Psi - 2\Phi & -0.5\Phi \\ 0.4\Psi & -2 + 0.4\Phi - \dfrac{0.1}{\Psi} \end{bmatrix}. \tag{100}$$

Substituting $\Phi = 2.9025$ and $\Psi = 0.1950$,we get:

$$\mathcal{J}(E_2) = \begin{bmatrix} 3 - 0.0975 - 5.805 & -1.4513 \\ 0.078 & -2 + 1.161 - \dfrac{0.1}{0.1950} \end{bmatrix} == \begin{bmatrix} -2.9025 & -1.4513 \\ 0.078 & -1.3518 \end{bmatrix}. \tag{101}$$

Solve the characteristic equation:

$$\det(\mathcal{J} - \lambda I) = 0 \tag{102}$$

$$\Rightarrow \det \begin{bmatrix} -2.9025 - \lambda & -1.4513 \\ 0.078 & -1.3518 - \lambda \end{bmatrix} = 0 \tag{103}$$

Calculate the determinant:

$$(-2.9025 - \lambda)(-1.3518 - \lambda) - (-1.4513)(0.078) = 0 \quad (104)$$

So:

$$\lambda^2 + 4.2543\lambda + 4.0368 = 0. \quad (105)$$

$$\lambda_1 \approx -1.4286, \quad (106)$$

$$\lambda_2 \approx -2.8257. \quad (107)$$

Both eigenvalues are negative real numbers:

$$\lambda_1 \approx -1.4286, \quad \lambda_2 \approx -2.8257. \quad (108)$$

Since both eigenvalues have negative real parts, the equilibrium point $E_2 = (2.9025, 0.1950)$. is locally asymptotically stable.

## Graphs Analysis

The graphs capture the interplay between financial corruption and society within a fractional Lotka–Volterra framework. Numerical simulations under two parameter regimes of the nonlinear term $Y(\tau, \Phi, \Psi)$, reveal that, although the fractional order is fixed, the system produces distinct dynamical behaviors, reflecting its sensitivity to nonlinear effects.

Figure 1 shows numerical simulation of the fractional BVP system over $\tau \in [2000, 2025]$, with parameters $\alpha = 1.2, \xi_1 = 2, \xi_2 = 1, \alpha = 0.5, \rho_1 = 1, \rho_2 = 1, \beta = 0.6$, and boundary values $\Phi(T) = 1, \Psi(T) = 1$. The results show sustained oscillatory dynamics in both $\Phi(\tau)$ and $\Psi(\tau)$, converging toward equilibrium However, a closer inspection reveals that $\Phi(\tau)$ and $\Psi(\tau)$ also display persistent, bounded oscillations throughout the interval [2000, 2025], with no observable decay in amplitude. This pattern suggests the emergence of a nonlinear limit cycle, wherein the reciprocal influence between corruption and societal response gives rise to sustained periodic dynamics. Such behavior indicates a state of neutral stability, which is characteristic of systems with significant hereditary behaviors. The use of Caputo fractional derivatives introduces hereditary dynamics, whereby past states exert continuous influence on the present, effectively inhibiting convergence to equilibrium. This phenomenon mirrors real-world socio-political systems, where phases of corruption and reform tend to recur cyclically. In these cases, internal feedback mechanisms may be inadequate to achieve long-term stability, necessitating external intervention to alter the system's trajectory.

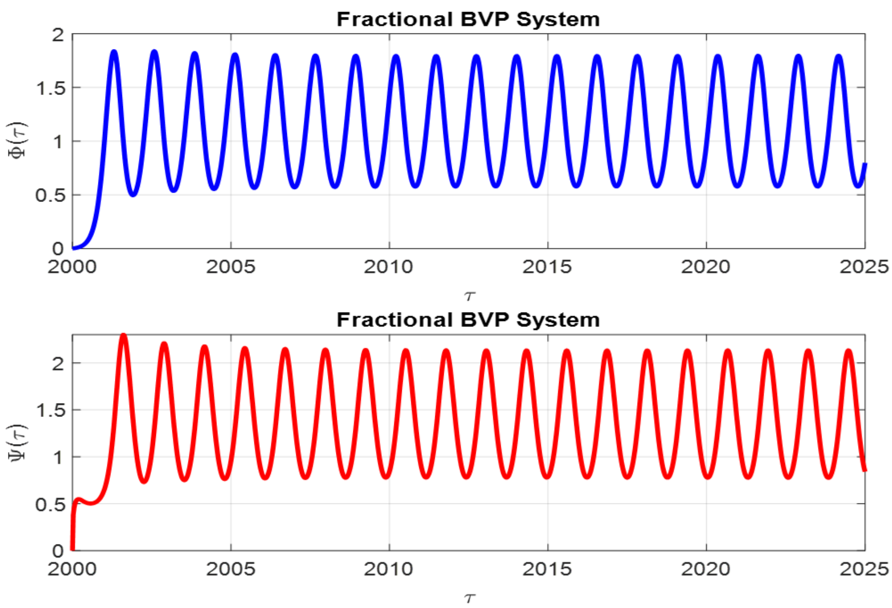


**Figure 1**. Fractional boundary value problem (BVP) with sustained oscillations

Figure 2 shows a simulation of the fractional BVP system over $\tau \in [2000, 2025]$, with parameters $\alpha = 1.2, \xi_1 = 2, \xi_2 = 0.9, a = 0.5, \rho_1 = 1.6, \rho_2 = 0.9$, and $\beta = 0.3$. The trajectories of $\Phi(\tau)$ and $\Psi(\tau)$ initially exhibit oscillatory behavior; however, their amplitudes progressively decline. By approximately $\tau \approx 2015$, the system stabilizes, indicating asymptotic stability. In this configuration, the system functions as a damping mechanism, gradually absorbing disturbances and converging toward equilibrium. Such stabilization may arise from weaker nonlinear effects, a lower Lipschitz constant, or differences in initial and boundary conditions. The resulting trajectory reflects a more resilient socio-economic structure capable of regulating the corruption-response feedback loop more effectively.

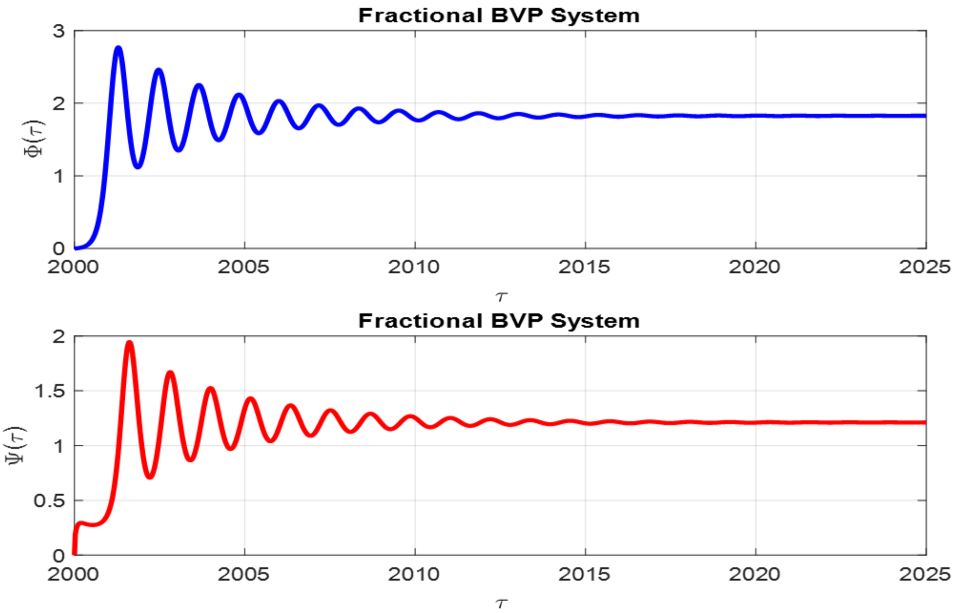


**Figure 2**. Fractional boundary value problem (BVP) with damped oscillations

The theoretical foundation of the fractional Lotka-Volterra framework is further substantiated by these simulation results. The model effectively characterizes complex socio-economic systems by capturing both sustained oscillatory regimes and stable equilibria. Thus, it demonstrates its versatility. The model provides a realistic and potent framework for comprehending the long-term evolution of corruption and its interaction with societal mechanisms by incorporating fractional-order dynamics.

## CONCLUSION

In order to accurately represent the changing dynamics between financial malfeasance and society, we implemented a fractional-order Lotka–Volterra model in this investigation. The model incorporates characteristics that are common in socio-economic systems through the use of Caputo derivatives. We discovered an explicit version of the solution by converting the problem into an analogous integral equation. To prove that solutions exist and are unique, we introduced appropriate operators and showed that these operators are both compact and contractive under standard Lipschitz continuity and boundedness assumptions. The existence of solutions was established by applying the Arzelà–Ascoli theorem, while uniqueness was guaranteed through Banach's fixed-point theorem. These findings provide a firm mathematical foundation for studying boundary value problems in fractional differential systems. The methods developed here can also be applied to other real-world models in areas such as physics, engineering, and economics, where hereditary effects play a key role.

## SUPPLEMENTARY MATERIAL

*No supplementary material is provided for this study.*

## AUTHOR CONTRIBUTIONS

*Dilbar M. Sharif Abdullah: Conceptualization, methodology, software, formal analysis, investigation, writing – original draft preparation, and visualization. Faraj Y. Ishak: Supervision, validation, writing – review and editing, and project administration.*

## FUNDING

*This research received no external funding.*

## DATA AVAILABILITY STATEMENT

*The MATLAB program was used to generate and analyze simulation graphs. Since no empirical data were available for this study, synthetic data produced through simulation were employed to demonstrate the dynamic behavior and interactions between the model variables. Therefore, no real-world datasets were generated or analyzed in this research.*

## ACKNOWLEDGMENTS

*The authors acknowledge the valuable support and assistance provided by the University of Duhok.*

## CONFLICTS OF INTEREST

*The authors declare no conflicts of interest.*

## DECLARATION OF GENERATIVE AI USE

*The authors declare that no generative Al or Al-assisted technologies were used in the preparation of this manuscript.*

## REFERENCES

[1] A. S. Haning Puspita, S. Sutrima, R. Setiyowati, and S. Wibowo, "Stability analysis of cellular operating system market share in Indonesia with the competitive Lotka-Volterra model," *Barekeng: Jurnal Ilmu Matematika dan Terapan*, vol. 18, no. 1, pp. 0333–0340, 2024, doi: 10.30598/barekengvol18iss1pp0333-0340.

[2] A. Ogundero, S. Connelly, and W. T. Sloan, "Developing Lotka-Volterra based models to describe *Bdellovibrio* predation in a batch and Chemostat experimental system," *Environmental Microbiology Reports*, vol. 17, no. 4, Art no. e70141, 2025, doi: 10.1111/1758-2229.70141.

[3] M. H. Al- Towaiq, "An analysis of a numerical solution of the Lotka-Volterra predator-prey model," *The Eurasia Proceedings of Science Technology Engineering and Mathematics*, vol. 34, pp. 352–358, Aug. 2025, doi: 10.55549/epstem.1755507.

[4] J. Ripoll and J. Font, "Numerical approach to an age-structured Lotka-Volterra model," *Mathematical Biosciences and Engineering*, vol. 20, no. 9, pp. 15 603–15 622, 2023, doi: 10.3934/mbe.2023696.

[5] A. Bondesan, M. Menale, G. Toscani, and M. Zanella, "Lotka-Volterra-type kinetic equations for interacting species," *Nonlinearity*, vol. 38, no. 7, Art no. 075026, 2025, doi: 10.1088/1361-6544/addfa1.

[6] J. D. Davis, D. V. Olivença, S. P. Brown, and E. O. Voit, "Methods of quantifying interactions among populations using Lotka-Volterra models," *Frontiers in Systems Biology*, vol. 2, Art no. 1021897, Oct. 2022, doi: 10.3389/fsysb.2022.1021897.

[7] S. Kekulthotuwage Don, K. Burrage, K. J. Helmstedt, and P. M. Burrage, "Stability switching in Lotka-Volterra and ricker-type predator-prey systems with arbitrary step size," *Axioms*, vol. 12, no. 4, Art no. 390, 2023, doi: 10.3390/axioms12040390.

[8] X. Cai, "The research history of population models," *Mathematical Modeling and Algorithm Application*, vol. 2, no. 1, pp. 57–60, 2024, doi: 10.54097/nz548x82.

[9] A. Genda, A. Fidlin, S. Lenci, and O. Gendelman, "Effects of seasonal hunting and food quantity variations on the Lotka-Volterra model," *Nonlinear Dynamics*, vol. 113, no. 10, pp. 12 351–12 369, 2024, doi: 10.1007/s11071-024-10667-y.

[10] É. Diz-Pita and M. V. Otero-Espinar, "Predator–prey models: A review of some recent advances," *Mathematics*, vol. 9, no. 15, Art no. 1783, 2021, doi: 10.3390/math9151783.

[11] A. Al-Saffar and E.-j. Kim, "Fisher information approach to understand the Gompertz model," *Mathematics in Applied Sciences and Engineering*, vol. 3, no. 4, pp. 269–279, 2022, doi: 10.5206/mase/15447.

[12] J. Wang and W. Jiang, "Impulsive perturbations in a predator–prey model with dormancy of predators," *Applied Mathematical Modelling*, vol. 38, no. 9–10, pp. 2533–2542, 2014, doi: 10.1016/j.apm.2013.11.004.

[13] S. Das and P. Gupta, "A mathematical model on fractional Lotka–Volterra equations," *Journal of Theoretical Biology*, vol. 277, no. 1, pp. 1–6, 2011, doi: 10.1016/j.jtbi.2011.01.034.

[14] H. Abdelhamid, G. Stamov, M. S. Souid, and I. Stamova, "New results achieved for fractional differential equations with Riemann-Liouville derivatives of nonlinear variable order," *Axioms*, vol. 12, no. 9, Art no. 895, 2023, doi: 10.3390/axioms12090895.

[15] A. Ganesh, S. Deepa, D. Baleanu, S. S. Santra, O. Moaaz, V. Govindan, and R. Ali, "Hyers-Ulam-Mittag-Leffler stability of fractional differential equations with two Caputo derivative using fractional Fourier transform," *AIMS Mathematics*, vol. 7, no. 2, pp. 1791–1810, 2022, doi: 10.3934/math.2022103.

[16] A. Alsaedi, A. F. Albideewi, S. K. Ntouyas, and B. Ahmad, "On Caputo–Riemann–Liouville type fractional integro-differential equations with multi-point sub-strip boundary conditions," *Mathematics*, vol. 8, no. 11, Art no. 1899, 2020, doi: 10.3390/math8111899.

[17] A. Chikrii and I. Matychyn, "Riemann–Liouville, caputo, and sequential fractional derivatives in differential games," in *Advances in Dynamic Games*. Birkhäuser Boston, Nov. 2010, pp. 61–81, doi: 10.1007/978-0-8176-8089-3_4.

[18] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent–II," *Geophysical Journal International*, vol. 13, no. 5, pp. 529–539, 1967, doi: 10.1111/j.1365-246x.1967.tb02303.x.

[19] N. A. Hamal, "Analysis of mixed differential equations involving the Caputo and Riemann-Liouville fractional derivatives," *Miskolc Mathematical Notes*, vol. 26, no. 1, Art no. 291, 2025, doi: 10.18514/mmn.2025.4630.

[20] S. Ryehan, "Numerically unveiling hidden chaotic dynamics in nonlinear differential equations with Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu fractional derivatives," *arXiv*, pp. 1–11, 2023, doi: 10.48550/ARXIV.2307.0 3251.

[21] P. A. Naik, S. Naveen, V. Parthiban, S. Qureshi, M. Alquran, and M. Senol, "Advancing Lotka-Volterra system simulation with variable fractional order Caputo derivative for enhanced dynamic analysis," *Journal of Applied Analysis & Computation*, vol. 15, no. 2, pp. 1002–1019, 2025, doi: 10.11948/20240243.

[22] M. Yamamoto, "Fractional calculus and time-fractional differential equations: Revisit and construction of a theory," *Mathematics*, vol. 10, no. 5, Art no. 698, 2022, doi: 10.3390/math10050698.

[23] D. M. Abdullah and F. Y. Ishak, "A novel Lotka-Volterra model for analyzing the dynamic relationship between financial corruption and society.," *AL-Rafidain Journal of Computer Sciences and Mathematics*, vol. 19, no. 1, pp. 144–153, 2025, doi: 10.33899/csmj.2025.158560.1183.

[24] Fatmawati, M. A. Khan, M. Azizah, Windarto, and S. Ullah, "A fractional model for the dynamics of competition between commercial and rural banks in Indonesia," *Chaos, Solitons & Fractals*, vol. 122, pp. 32–46, May 2019, doi: 10.1016/j.chaos.2019.02.009.

[25] J. Alebraheem, "Dynamics of a predator-prey model with the effect of oscillation of immigration of the prey," *Diversity*, vol. 13, no. 1, Art no. 23, 2021, doi: 10.3390/d13010023.

[26] V. Kiryakova and Y. Luchko, "Riemann-Liouville and Caputo type multiple Erdélyi-Kober operators," *Open Physics*, vol. 11, no. 10, pp. 1314–1336, 2013, doi: 10.2478/s11534-013-0217-1.

[27] F. Y. Ishak, "Existence and uniqueness solution for three-point Hadamard-type fractional Volterra BVP," *E-Jurnal Matematika*, vol. 11, no. 1, Art no. 70, 2022, doi: 10.24843/mtk.2022.v11.i01.p363.

[28] B. Ahmad and J. J. Nieto, "Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions," *Boundary Value Problems*, vol. 2009, Art no. 708576, 2009, doi: 10.1155/200 9/708576.