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A 4 parameters generalized Gamma probability density function
دالة الكثافة الاحتمالية لتوزيع غاما المعمّم ذات الأربع معلمات

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1. Abstract:

This paper deal with the probability density function where the data require to be transformed by subtract a value (say θ) from each observation and divided the result by another value (say β), arising the result to a power (say λ) where the degree of freedom (say n) play an important role in the analysis.

Keywords: cumulative p. d. f., hazard function, permutation j from n P_j^n, moment, survival function.

المستخلص :

تتناول هذه الورقة دالة الكثافة الاحتمالية في الحالات التي تتطلب فيها البيانات إجراء تحويل، وذلك بطرح قيمة معينة) ولتكن (θ من كل مشاهدة، ثم قسمة الناتج على قيمة أخرى) ولتكن(β ، وبعد ذلك رفع الناتج إلى قوة معينة) ولتكن(λ ، حيث تلعب درجات الحرية) ولتكن (n دورًا مهمًا في عملية التحليل.

الكلمات المفتاحية : دالة التوزيع التراكمي، دالة الخطر، التبديل ، العزم، دالة البقاء.

2. The proposed probability density function

A random sample of size n is said to fit a 4 parameters generalized Gamma probability density function f(y\θ, β, λ, n) if it pass the formula defined by equation 1,

$$f(y) = \frac{\lambda}{\beta\Gamma n} e^{-(\frac{y-\theta}{\beta})^{\lambda}} (\frac{y-\theta}{\beta})^{n\lambda-1}, \quad y > \theta, \qquad 1$$

Let $x_1, x_2, \dots, x_n$ be the final transformation data having a standard gamma distribution gamma(n,1) [1]

$$f(x) = \frac{e^{-x}\, x^{n-1}}{\Gamma n} \quad . \qquad x > 0, \qquad 2$$

and suppose that, $x = (\frac{y-\theta}{\beta})^{\lambda}$, then, in order to drive the p. d. f. of y, we have,

$$dx = \frac{\lambda}{\beta}(\frac{y-\theta}{\beta})^{\lambda-1}$$

Therefore

$$f(y) = \frac{\lambda}{\beta\Gamma n} e^{-(\frac{y-\theta}{\beta})^{\lambda}} (\frac{y-\theta}{\beta})^{n\lambda-1}, \quad y > \theta, \qquad 3$$

It is easily to poof that f(y) as a p. d. f.

3. The moments of y

To drive the moments of y, let us drive the moments of $\frac{y-\theta}{\beta}$, and then, it is easy to find the moments of y.

From $x = (\frac{y-\theta}{\beta})^{\lambda}$,

We have

$$\frac{y-\theta}{\beta} = x^{1/\lambda}$$

Therefore

$$E(\frac{y-\theta}{\beta})^{j} = Ex^{j/\lambda}$$

then

$$Ex^{j/\lambda} = \frac{1}{\Gamma n}\int_0^{\infty} x^{j/\lambda}\, e^{-x} x^{n-1} dx = \frac{\Gamma(n+\frac{j}{\lambda})}{\Gamma n}, \qquad 4$$

Therefore,

$$E(\frac{y-\theta}{\beta})^{j} = \frac{\Gamma(n+\frac{j}{\lambda})}{\Gamma n}$$

Now, for J=1, 2, 3, …. ; the general form of Ey^{k} is

$$Ey^{k} = \sum_{j=0}^{k} C_j^k\, \theta^j \beta^{k-j} \frac{\Gamma(n+\frac{k-j}{\lambda})}{\Gamma n}, \qquad 4$$

Which can be easily written as follows

$$Ey^{k} = \frac{(\beta + a\theta)^{k}}{\Gamma n}, \text{where } a^{i} = \Gamma\left(n + \frac{k-i}{\lambda}\right), \text{for } i = 0,1,2,\dots,k$$

Or

$$Ey^{k} = \frac{(a\beta + \theta)^{k}}{\Gamma n}, \text{where } a^{i} = \Gamma\left(n + \frac{i}{\lambda}\right), \text{for } i = 0,1,2,\dots,k$$

Remark that

$$EY^{1} = \frac{(\beta + a\theta)^{1}}{\Gamma n} = \frac{\beta^1 a^0 \theta^0 + \beta^0 a^1 \theta^1}{\Gamma n} = \frac{\beta\Gamma\left(n+\frac{1}{\lambda}\right) + \theta\Gamma n}{\Gamma n}$$

$$Ey = \frac{\beta\Gamma\left(n+\frac{1}{\lambda}\right)}{\Gamma n} + \theta$$

4. The cumulative distribution function:

The cumulative distribution function can be derived as follows:

Since the p. d. f. is defined as in equation 2, and for more simply , let $Z = \frac{y-\theta}{\beta}$, therefore

$$f(Z) = \lambda e^{-Z^{\lambda}} Z^{n\lambda-1}, \qquad Z > 0$$

Now, for n=1, 2,3,…., and with serial integrations using u.dv integration rule, the CDF is given by

$$F(T) = 1 - \frac{e^{-T^{\lambda}}\left[\sum_{j=0}^{n-1} P_j^{n-1}\, T^{(n-1-j)\lambda}\right]}{\Gamma n}$$

Where P_j^n is the permutation of j from n.

it is easily to proof that

$$F(T) = 1 - \frac{e^{-\left(\frac{T-\theta}{\beta}\right)^{\lambda}}\left[\sum_{j=0}^{n-1} P_j^{n-1}\left(\frac{T-\theta}{\beta}\right)^{(n-1-j)\lambda}\right]}{\Gamma n}$$

or

$$F(T) = 1 - \frac{e^{-\left(\frac{T-\theta}{\beta}\right)^{\lambda}}\left[\sum_{j=0}^{n-1} P_j^{n-1}\, \beta^{j\lambda}(T-\theta)^{(n-1-j)\lambda}\right]}{\beta^{(n-1)\lambda}\Gamma n}$$

Since $\frac{P_j^{n-1}}{\Gamma n} = \frac{1}{(n-1-j)!}$, it is easy to prove that

$$F(T) = 1 - e^{-\left(\frac{T-\theta}{\beta}\right)^{\lambda}} \sum_{j=0}^{n-1} \frac{\left(\frac{T-\theta}{\beta}\right)^{j\lambda}}{j!}, \qquad 5$$

Figure 1 illustrate the behavior of the CDF for a variate values of n.

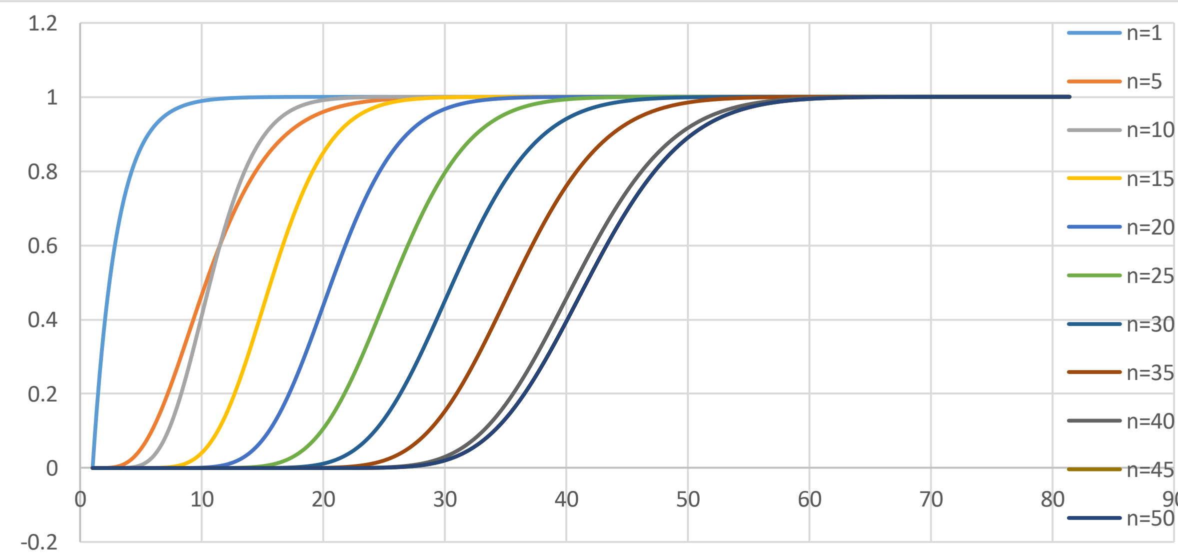

Fig. 1 CDF for variate values of n

5. The moment generating function

The moment generating function of Y is defined by Ee^{yt}.

Let $= \frac{y-\theta}{\beta}$, this implies that $y = x\beta + \theta$, and then;

$Ee^{yt} = Ee^{(x\beta+\theta)t} = e^{\theta t}Ee^{x\beta t}$

$e^{x\beta t} = \sum_{i=0} \frac{(\beta t)^i}{i!} x^i$

Suppose that $a_i = \frac{\Gamma(n+\frac{i}{\lambda})}{\Gamma n}$, for $= 0,1,2,3, \ldots$, then

$$M_y(t) = e^{\theta t} \sum_{i=0} \frac{(\beta t)^i}{i!} a_i, \qquad 6$$

There was a serial integrations and many transformations to reach another formula, that is;

$$M_y(t) = 1 + \sum_{i=1} \frac{(a\beta + \theta)^i}{i!.\,\Gamma n} t^i \text{, where } a^i = \Gamma\left(n + \frac{i}{\lambda}\right), \text{for } i = 1,2,3 \ldots$$

Or

$$M_y(t) = \sum_{i=0} \frac{(a\beta + \theta)^i}{i!.\,\Gamma n} t^i \text{, where } a^i = \Gamma\left(n + \frac{i}{\lambda}\right), \text{for } i = 0,1,2,3 \ldots,$$

Hint

$$(a\beta + \theta)^0 = C_0^0 a^0 \beta^0 \theta^0 = a^0 = \Gamma\left(n + \frac{0}{\lambda}\right) = \Gamma n$$

6. Survival function and hazard function

From F(T), the Survival function is

$$S(T) = F(x > T) = e^{-(\frac{T-\theta}{\beta})^\lambda} \sum_{j=0}^{n-1} \frac{(\frac{T-\theta}{\beta})^{j\lambda}}{j!}, \qquad 7$$

And the hazard function is

$$h(x) = \frac{f(x)}{S(x)} = \frac{\frac{(\frac{X-\theta}{\beta})^{n\lambda-1}}{\Gamma n}}{\sum_{j=0}^{n-1} \frac{(\frac{X-\theta}{\beta})^{j\lambda}}{j!}}, \qquad 8$$

7. Some parameters variates

If θ=0 and λ=1

$$p(X < T) = F(T) = 1 - e^{-\frac{T}{\beta}} \sum_{j=0}^{n-1} \left[\frac{T}{\beta}\right]^j /j!, \qquad 10$$

If θ=0 ,β=1 and λ=1

$$p(X < T) = F(T) = 1 - e^{-T} \sum_{j=0}^{n-1} T^j / j!, \qquad 11$$

8. Summary:

In this paper, some properties of a 4 parameters p. d. f. are states, that are the p. d. f. , the moments of the distribution, the cumulative distribution function and the moment generating function of the distribution.

References:

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