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Fractional Diffusion Equations Solved via Log–Laplace Residual Power Series

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ABSTRACT: In this paper, fractional diffusion equations of Caputo-Hadamard presented as interesting classes of fractional differential equations have not studied before. For the first time, the solution to a diffusion equation of this type was calculated. Therefore, it is considered a new addition to the diffusion equation with the presence of this form of fractional derivatives, with its initial conditions. Additionally, this paper presents a suitable Laplace logarithmic formula for the given fractional equation, derived from the generalized Laplace formula and its established conditions. This also relies on the formula for the fractional derivative as a transformation. The first time this type of fractional derivative was established, it helped in finding an approximate formula for the solution. The residual power series method used is effective in solving many fractional equations and was even more effective when used with the Laplace logarithmic formula. The $\text{Log}(t+1)$ –Laplace residual power series method combines the two concepts, formulating and evaluating the method, shown that from any changing of the values of the parameters involving in formulation of derivative as well as the values of fractional order which found in tables and figures for illustrative examples which illustrates the presented method.

Keywords: Caputo-Hadamard, Residual Power Series, $Log\ (t+1)-$ Laplace, Diffusion Equation.

1. INTRODUCTION

Fractional differential equations of fractional order were applied in many interesting applications was arise in many problems, such as the processing of single systems and diffusion reactions, as well as systems of electrical network, see [1]. The effectiveness of these fractional equations drew attention and attracted the authors through a study involving these equations in the field of applications modeling. For long time ago, the differential equations of fractional derivatives which extended of deferential equations of ordinary orders have been used in many branches of interesting scientific. moreover, through these years there are different methods as approaches described as analytically or numerically to solve many classes of differential equations involving fractional orders for many applications or models it had a great and important impactand open a more problems to study their solvability. The authors in the past decades have been made a great effort for develop many methods were some of them are new to complete the solution for modeling of fractional differential formulations was needed analytic and numerical approach for solvability. All methods was efficient analytically and numerically and solved many presented and proposal problems such as series method, homotopy analysis and perturbation methods, iterative method, A domain decomposition method, differential transform method, non-integer power series, Taylor's method [3], [4], [5], [6], [7], [8], and some other methods such as physical model, risk analysis, novel hybrid method, biological engineering image dispensation [9], [10], [11], [12], [13], and in [14] studied how to solve some classes of types which is linear or nonlinear differential equations with fractional derivatives. The method of residual part with power series is one of effective approach and accuracy methods for taking the differential equations with fractional order to explain their solutions, [15]. To find the series solution of nonlinear formulations involving of fractional deferential has need more challenging technical when working in this method. Moreover, the transformed functions also used to obtain the recurrence of the relations for solutions from using a residual power series technique by

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computing the coefficients in iterative form. The residual power series explained as a technique which contained differentiation of nth of partial sum power series (n-1) times to compute terms of solution including in nth order of their combination. So, we can say the general of differential equations has ordinary derivatives is fractional derivatives as various nonlinear fractional differential equations which solved in [16]. There are many transformations such as Sumudu, Laplace, Elzaki, and others all are used to studied and introduced the solutions of differential equations involving fractional represented a order, see [17], [18], [19], [20], [21]. The BBM-Burger equations with fractional order as a model have been solved and studied and relaxation-oscillation in many articles, [22], [23], [24]. The studies have proven the residual Laplace power series it is very successful method for fractional equations under studied.

The diffusion equation with fractional order is one of topics in partial calculus which sometimes refer to the heat, mass, or particles the evolution of the temporal evolution for a variable. In [3] the classical diffusion equation opposed by fractional order diffusion equation and happened that in the temporal domain. This model is particularly useful by this property from make the studying of phenomena depended on complex temporal or interactions by long-range since it is allowed and satisfy accurately reflect for the model of non-local and behaviors of the memory-dependent actually in processes of diffusion, see [25], [26]. Moreover, still the solutions expressed analytical and numerical approaches are used for fractional diffusion equations of linear and nonlinear combination.

In this paper, the article is organized as follows. Section 2 contains interesting definitions has big issue establish of fractional calculus and presented definitions of generalized Laplace transformations such as g-Laplace transform and log(t+1)-Laplace transform, including properties of this transform for certain fractional derivatives to satisfy the technique of certain method for certain problems, the method including a constriction between a generalized Laplace transforms of several fractional operators with respect to log(t+1). Section 3 included the $L_{Log(t+1)}-$ Residual Power Series Method, and their explained in detail by algorithm steps as well as how this approach is applied to the Caputo-Hadamard fractional equation including diffusion term. In section has number 4 explain the technical of applicability for proposal generalized Laplace transform with residual power series and their activity for appearing then explicitly solving for some classes of Caputo-Hadamard fractional diffusion introduced as initial value problems. Also, sectionhas number 5 have been observed the results in some tables and figures. Finally, Section 6 makes concluding statements about this manuscript.

2. PRELIMINARIES

In this section introduce the definitions are introduced, playing a good role in this article, and to understand the process of the definitions and concepts, as well as some interesting results for the presented technique.

Definition [26]. The Let a be positive number and α a be a real number with $k = [\alpha] + 1$. The left and right CH-FDE of order $\alpha > 0$ defined by

$$ {}^{CH}_{a}D_t^{\alpha,\mu} x(t) = \frac{1}{\Gamma(k-\alpha)}\left(t\frac{d}{dt}\right)^k \int_a^t \left(\log\frac{t}{s}\right)^{k-\alpha-1} \frac{x(s)-x(a)}{s} ds \; s > a $$

$$ {}^{CH}_{b}D_t^{\alpha,\mu} x(t) = \frac{1}{\Gamma(k-\alpha)}\left(-t\frac{d}{dt}\right)^k \int_t^b \left(\log\frac{t}{s}\right)^{k-\alpha-1} \frac{x(s)-x(a)}{s} ds, \; s < b $$

${}^{CH}_{a}D_t^{k,\mu} x(t) = \left(t\frac{d}{dt}\right)^k x(t)$ and

${}^{CH}_{b}D_t^{k,\mu} x(t) = \left(-t\frac{d}{dt}\right)^k x(t)$ where $\alpha = k$.

Definition, [27]. Let a, b are two real numbers, $0 < a < b$. The left and right C-HFIE of order $\alpha > 0$ for function $f: [a, b] \to R$ is defined by

$$ {}^{CH}_{a}D_t^{-\alpha,\mu} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{x(s)-x(a)}{\tau} d\tau, \; s > a, \quad (1) $$

$$ {}^{CH}_{b}D_t^{-\alpha,\mu} x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{x(s)-x(a)}{s} d\tau, \; s < b. \quad (2) $$

Definition, [28]. Let $f, g : [a, \infty) \to R$ are two real valued functions when $g(t)$ is continuous and $\acute{g}(t) > 0$ on $[a, \infty), a > 0$. If the generalized Laplace transformation exist, then

$$ \mathcal{L}_g\{f(t)\}(s) = \int_a^\infty e^{-s(g(t)-g(a))} f(t) g`(t) dt. \quad (3) $$

Remark, [28]. If $g(t) = \log(t+1)$ then (3) become $\log(t+1) -$ Laplace transformation that is,

$$ \mathcal{L}_{\log(t+1)}\{f(t)\}(s) = \int_0^\infty e^{-s(\log(t+1))} f(t) \frac{dt}{t}. \quad (4) $$

Lemma The $log(t+1) -$ Laplace Transformation of Caputo-Hadamard fractional derivative ${}^{CH}_{0}D_t^{\beta,\rho} x(t)\}(s)$ with $0 < \beta < 1$ and $\rho > 0$, and $x(t) \in \acute{C}[0, \infty)$. Then

$$ \mathcal{L}_{log(t+1)}\{{}^{CH}_{0}D_t^{\beta,\rho} x(t)\}(s) = s^\beta \mathcal{L}_{log(t+1)}\{x(t)\}(s) - s^{\beta-1} x(0) \quad (5) $$

Proof: By $\mathcal{L}_{log(t+1)}\{f(t)\}(s) = \int_a^\infty \frac{f(t)}{(t+1)^s}dt.$

$\mathcal{L}_{log(t+1)}\{f(t)\}(s) = \int_a^\infty \frac{f(t)}{(t+1)^s}$

$\mathcal{L}_{log(t+1)}\{f(t)\}(s) = \frac{1}{\Gamma(1-\beta)}\int_0^t \acute{x}(\tau)(\tau+1)^{1-\rho}\left[\int_0^t \left(log\frac{t+1}{\tau+1}\right)^{-\beta}\frac{1}{(t+1)^{s+1}}dt\right]d\tau$

Let $u = log\left(\frac{t+1}{\tau+1}\right)$ then $t+1 = (\tau+1)e^u$ and $dt = (\tau+1)e^u$

Therefore

$\int_0^\infty u^{-\alpha}\frac{1}{[(\tau+1)e^u]^{s+1}}(\tau+1)e^u\,du = \frac{1}{(\tau+1)^s}\int_0^\infty u^{-\beta}e^{-su}du = \int_0^\infty u^{-\beta}e^{-su}du = s^{\beta-1}\Gamma(1-\beta)$

Hence,

$\mathcal{L}_{log(t+1)}\{{}^{CH}_{0}D_t^{\beta,\rho}x(t)\}(s) = s^\beta\int_0^\infty \frac{\acute{x}(\tau)}{(\tau+1)^{s+\rho-1}}d\tau$

From, $\int_0^\infty \frac{\acute{x}(\tau)}{(\tau+1)^z}d\tau = -x(0)z\int_0^\infty \frac{\acute{x}(\tau)}{(\tau+1)^{z+1}}d\tau + \ldots.$

Thus,

$\mathcal{L}_{log(t+1)}\{x(\acute{t})\}(s) = s^\beta\mathcal{L}_{log(t+1)}\{x(t)\}(s) - x(0)$

Hence,

$\mathcal{L}_{log(t+1)}\{{}^{CH}_{0}D_t^{\beta,\rho}x(t)\}(s) = s^\beta\mathcal{L}_{log(t+1)}\{x(t)\}(s) - s^{\beta-1}x(0)$

Lemma. The $log(t+1)$ – Laplace Transformation of Caputo-Hadamard fractional derivative $\{{}^{CH}_{0}D_t^{\beta,\rho}x(t)\}(s)$ with $1 < \beta < 2$ and $\rho > 0$, and $x(t) \in C^2[0,\infty)$

Then

$$\mathcal{L}_{log(t+1)}\{{}^{CH}_{0}D_t^{\beta,\rho}x(t)\}(s) = s^\beta\mathcal{L}_{log(t+1)}\{x(t)\}(s) - s^{\beta-1}x(0) - s^{\beta-2}\acute{x}(0) \quad (6)$$

Proof: From $\mathcal{L}_{log(t+1)}\{f(t)\}(s) = \int_a^\infty \frac{f(t)}{(t+1)^s}dt.$ We get that

$\mathcal{L}_{log(t+1)}\{f(t)\}(s) = \int_a^\infty \frac{f(t)}{(t+1)^s}\left[\frac{1}{\Gamma(2-\beta)}\int_0^t \left(log\frac{t+1}{\tau+1}\right)^{1-\beta}(\tau+1)^{2-\rho}\; x''(\tau)\,d\tau\right]\frac{dt}{t+1}$

$\mathcal{L}_{log(t+1)}\{f(t)\}(s) = \frac{1}{\Gamma(2-\beta)}\int_0^t x''(\tau)(\tau+1)^{1-\rho}\left[\int_0^t \left(log\frac{t+1}{\tau+1}\right)^{1-\beta}\frac{1}{(t+1)^{s+1}}dt\right]d\tau$

Now let $u = log\left(\frac{t+1}{\tau+1}\right)$ then $t+1 = (\tau+1)e^u$ and $dt = (\tau+1)e^u$

Thus,

$\int_0^\infty u^{-\beta}e^{-su}du = s^{\beta-1}\Gamma(1-\beta)$

Hence,

$\mathcal{L}_{log(t+1)}\{{}^{CH}_{0}D_t^{\beta,\rho}x(t)\}(s) = s^\beta\int_0^\infty \frac{x''(\tau)}{(\tau+1)^{s+\rho-2}}d\tau$

Then

$\mathcal{L}_{log(t+1)}\{{}^{CH}_{0}D_t^{\beta,\rho}x(t)\}(s) = s^\beta\mathcal{L}_{log(t+1)}\{x(t)\}(s) - s^{\beta-1}x(0) - s^{\beta-2}x`(0)$

Lemma, [28], [29]:

1) $$\int_0^\infty t^{-s-1}(\ln(t+1))^\alpha dt = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \quad (7)$$

2) $$\int_0^\infty t^{-slog(t+1)}\frac{dt}{t} = \frac{1}{s} \quad (8)$$

3. THE $L_{Log(t+1)}$ – RESIDUAL POWER SERIES METHOD

The interesting approach of this section is to approximate a solution of Caputo-Hadamard diffusion equation with fractional order by $L_{Log(t+1)}$ – Residual Power Series Method discussed by steps of technical algorithm which given their steps in details. Consider now the following Caputo- Hadamard diffusion equation has fractional derivatives.

$$ {}^{CH}_{a}D_t^{\alpha}u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t)\; with\; 0 < \alpha \le 1\; u(x,y,0) = coshx.coshy \quad (10)$$

Steps Algorithm:

Step 1 Applying log $(t+1)$ –Laplace transform on (9) as,

$$L_{\log(t+1)}({}^{KC}D_t^{\alpha,\rho}u(x,y,t) = L_{\log(t+1)}(u_{xx}(x,y,t) + u_{yy}(x,y,t)) \quad (11)$$

By lemma 2.5 we have that,

$L_{\log(t+1)}\,[{}^{KC}D_t^{\alpha,\rho}u(x,y,t)] = s^\alpha L_{\log(t+1)}u(x,y,t) - s^{\alpha-1}u(x,y,0)\; L_{\log(t+1)}\,[{}^{KC}D_t^{\alpha,\rho}u(x,y,t)]$

$= s^\alpha\{L_{\log(t+1)}u(x,y,t) - s^{-1}u(x,y,0)\}$

Hence,

$s^{\alpha}\{L_{\log(t+1)}u(x,y,t) - g(x,y,0)\} = L_{\log(t+1)}\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right)$ we get $L_{\log(t+1)}u(x,y,t) = g(x,y,0) + s^{-\alpha}L_{\log(t+1)}(u_{xx}(x,y,t) + u_{yy}(x,y,t))$ (12)

Step 2 From inverse of $\log(t+1)$ −Laplace transform over (12), we obtain,

$$u(x,y,t) = G(x,y,0) + L^{-1}_{\log(t+1)}(s^{-\alpha}L_{\log(t+1)}\left((u_{xx}(x,y,t) + u_{yy}(x,y,t)\right)) \quad (13)$$

Such that the initial condition is a function G (x, y, 0).

Step 3 By $L_{\log(t+1)}$-Residual power series method for a given equation (9), we get that

$$u(x,y,t) = \sum_{n=0}^{\infty} f_n(x,y)\frac{(\log(t+1))^{n\alpha}}{(n\alpha)!} \quad (14)$$

Therefore, finite summation (14), can be written as:

$$s_i = \sum_{n=0}^{i} u_n(x,y,t) = \sum_{n=0}^{i} f_n(x,y)\frac{(\log(t+1))^{n\alpha}}{(n\alpha)!} \quad (15)$$

Step 4 The log $(t+1)$ −Laplace residual function of (13) have the following formulation,

$$Res_i(x,y,t) = u_i(x,y,t) - G(x,y,0) - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left((u_{i-1}(x,y,t)_{xx} + u_{i-1}(x,y,t)_{yy}\right)\right) \quad (16)$$

The values of f_n(x, y) can be computed with n = 0,1,2,... in the following,

$$t^{-n\alpha}Res_1(x,y,t)_{t=0} = 0 \quad (17)$$

The pseudo code of the algorithm $L_{\mathrm{Log}(t+1)}$ − residual power series method description in the following.

4. L log(t+1) – RESIDUAL POWER SERIES METHOD APPLIED ON CAPUTO-HADAMARD FRACTIONAL DIFFUSION EQUATION

Using log $(t+1)$ − Laplace transform on (9), then $L_{\log(t+1)}({}^{CH}D_t^{\alpha}u(x,y,t) = L_{\log(t+1)}(u_{xx}(x,y,t) + u_{yy}(x,y,t))$ (18)

By lemma 2.5, we get that

$$L_{\log(t+1)}u(x,y,t) = g(x,y,0) + s^{-\alpha}(L_{\log(t+1)}\left((u_{xx}(x,y,t) + u_{yy}(x,y,t)\right)) \quad (19)$$

By using the inverse log $(t+1)$ −Laplace transform on (19), we get

$u(x,y,t) = G(x,y,0) + L^{-1}_{\log(t+1)}(s^{-\alpha}L_{\log(t+1)}\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right))$

Now u_i(x, y, t) have the following formulation (20)

$$s_i = \sum_{n=0}^{i} u_n(x,y,t) = \sum_{n=0}^{i} f_n(x,y)\frac{(\log(t+1))^{n\alpha}}{(n\alpha)!} \quad (21)$$

f_n(x, y) can be obtained by using

$$Res_i(x,y,t) = u_i(x,y,t) - G(x,y,0) - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{logt}\left((u_{i-1}(x,y,t)_{xx} + u_{i-1}(x,y,t)_{yy}\right)\right) \quad (22)$$

When $i = 0$ from (22), we have that

$Res_0(x,y,t) = u_0(x,y,t) - G(x,y,0)$ and from (21),

$0 = u_0(x,y,t) - G(x,y,0)$ $i.e$

$u_0(x,y,t) = G(x,y,0)$ $u_0(x,y,t) = G(x,y,0) = f_0(x,y) = coshx.coshy$

When $i = 1$ in (22), we have that (23)

$Res_1(x,y,t) = u_1(x,y,t) - G(x,y,0) - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left((u_0(x,y,t)_{xx} + u_0(x,y,t)_{yy}\right)\right)$ with equations

$u_1(x,y,t) = f_0(x,y) + f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$. Then

$$Res_1(x,y,t) = f_0(x,y) + f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - G(x,y,0) - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{logt}\left((u_0(x,y,t)_{xx} + u_0(x,y,t)_{yy}\right)\right)$$

$$= coshx.coshy + f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - coshx.coshy - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left(((coshx.coshy)_{xx} + (coshx.coshy)_{yy}\right)\right)$$

$$= f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}(coshx.coshy + coshx.coshy)\right)$$

$$= f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}(2coshxcoshy)\right)$$

$$= f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - 2coshx.coshy\, L^{-1}_{\log(t}\left(s^{-\alpha}L_{logt}(1)\right)$$

$= f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - 2coshx.coshy\, L^{-1}_{\log(t+1)}\left(s^{-\alpha}\frac{1}{s}\right)$
$= f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$
$-2coshx.coshy\, L^{-1}_{\log(t+1)}\left(\frac{1}{s^{\alpha+1}}\right)$
$= f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - 2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$
$= (f_1(x,y) - 2coshx.coshy\frac{1}{\Gamma(1+\alpha)})\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$

Then after solving $t^{-\alpha}Res_1(x,y,t)_{t=0} = 0$ gives that
$f_1(x,y) - 2oshx.coshy\frac{1}{\Gamma(1+\alpha)} = 0\ i.e$
$f_1(x,y) = 2oshx.coshy\frac{1}{\Gamma(1+\alpha)}$ (24)

When i=2 from (22), we have that
$Res_2(x,y,t) = u_2(x,y,t) - G(x,y,0) - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left((u_1(x,y,t)_{xx} + u_1(x,y,t)_{yy}\right)\right)$, with equations $u_1(x,y,t) = f_0(x,y) + f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$ and $u_2(x,y,t) = f_0(x,y) + f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}$, we get that,

$Res_2(x,y,t) = f_0(x,y) + f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} - G(x,y,0) - L^{-1}_{\log(t+1)}\frac{(logt)^{\alpha}}{(\alpha)!} -$
$L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left((coshx.coshy + 2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)})\frac{(\log(t+1))^{\alpha}}{(\alpha)!})_{xx} + \left(coshx.coshy +\right.\right.\right.$
$\left.\left.\left.2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!}\right)_{yy}\right)\right) = 2coshx.coshy\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}$
$-L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left(coshx.coshy + 2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + coshx.coshy +\right.\right.$
$\left.\left.2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!}\right)\right)$
$= 2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left(2coshx.coshy +\right.\right.$
$\left.\left.4coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!}\right)\right)$
$= 2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} -$
$L^{-1}_{\log(t+1)}\left(s^{-\alpha}\left(2coshx.coshyL_{\log(t+1)}(1) + 4coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}L_{\log(t+1)}(\frac{(\log(t+1))^{\alpha}}{(\alpha)!})\right)\right)$
$= 2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$
$+f_2(x,y)\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}L^{-1}_{\log(t+1)}\left(s^{-\alpha}\left(2coshx.coshy\left(\frac{1}{s}\right) + 4coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+\alpha)}{s^{1+\alpha}}\right)\right)\right)$
$= 2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} - 2coshx.coshyL^{-1}_{\log(t+1)}\left(\frac{1}{s^{1+\alpha}}\right)$
$\quad - 4coshx.coshyL^{-1}_{\log(t+1)}\left(\frac{1}{s^{1+2\alpha}}\right)$
$= 2coshx.coshy\,\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} - 2coshx.coshy\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$
$\quad - 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}$
$= f_2(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}$
$= (f_2(x,y) - 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)})\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}$

From $t^{-2\alpha}Res_2(x,y,t)/_{t=0} = 0$ we get that,
$\left(f_2(x,y) - 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\right) = 0$ i.e $f_2(x,y) = 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}$

Now, the approximate of second solution is,

$$u_2(x,y,t) = coshx.coshy + 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}$$

When $i = 3$ from (2.34), we get that

$Res_3(x,y,t) = u_3(x,y,t) - G(x,y,0) - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left((u_2(x,y,t)_{xx} + u_2(x,y,t)_{yy}\right)\right)$, with equations

$u_3(x,y,t) = f_0(x,y) + f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1)^{2\alpha}}{(2\alpha)!}$ and

$u_3(x,y,t) = f_0(x,y) + f_1(x,y)\frac{(\log(t+1)^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1)^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1)^{3\alpha}}{(3\alpha)!}$, we get

$$Res_3(x,y,t) = f_0(x,y) + f_1(x,y)\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(\log(t+1)^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1)^{3\alpha}}{(3\alpha)!} - G(x,y,0) - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left(\left(f_0(x,y) + f_1(x,y)\frac{(logt)^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(logt)^{2\alpha}}{(2\alpha)!}\right)_{xx} + \left(f_0(x,y) + f_1(x,y)\frac{(logt)^{\alpha}}{(\alpha)!} + f_2(x,y)\frac{(logt)^{2\alpha}}{(2\alpha)!}\right)_{yy}\right)\right)$$

$$= 2coshx.coshy\frac{1}{\Gamma(1+\alpha)}\frac{(logt)^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(logt)^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(logt)^{3\alpha}}{(3\alpha)!} - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left((coshx.coshy + 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)})\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!})_{xx} + \left(coshx.coshy + 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)})\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}\right)_{yy}\right)\right)$$

$$= 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!} - L^{-1}_{\log(t+1)}\ (s^{-\alpha}L_{\log(t+1)}\left(coshx.coshy + 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}\right) + (coshx.coshy + 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}))$$

$$= 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!} - L^{-1}_{\log(t+1)}\left(s^{-\alpha}L_{\log(t+1)}\left(2coshx.coshy + 4coshx.coshy\ \frac{1}{\Gamma(1+\alpha)})\frac{(logt\log(t+1))^{\alpha}}{(\alpha)!}8coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}\right)\right)$$

$$= 2coshx.coshy\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!} - L^{-1}_{\log(t+1)}\left(s^{-\alpha}\left(2coshx.coshyL_{\log(t+1)}(1) + 4coshx.coshy\ \frac{1}{\Gamma(1+\alpha)})L_{\log(t+1)}(\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 8coshx.coshy\frac{1}{\Gamma(1+2\alpha)}L_{log}(\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!})\right)\right)$$

$$= 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!} - L^{-1}_{\log(t+1)}\left(s^{-\alpha}\left(2coshx.coshy\left(\frac{1}{s}\right) + 4coshx.coshy\ \frac{1}{\Gamma(1+\alpha)})\left(\frac{\Gamma(1+\alpha)}{s^{1+\alpha}}\right) + 8coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2\alpha)}{s^{1+2\alpha}}\right)\right)\right) = 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$$

$$+4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!} - L^{-1}_{\log(t+1)}(2coshx.coshy\left(\frac{1}{s^{1+\alpha}}\right) + 4coshx.coshy\ \left(\frac{1}{s^{1+2\alpha}}\right) + 8coshx.coshy\left(\frac{1}{s^{1+3\alpha}}\right)) = 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!}$$

$$+4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!} - 2coshx.coshyL^{-1}_{\log(t+1)}\left(\frac{1}{s^{1+\alpha}}\right) - 4coshx.coshy\ L^{-1}_{\log(t+1)}\left(\frac{1}{s^{1+2\alpha}}\right) - 8coshx.coshyL^{-1}_{\log(t+1)}\left(\frac{1}{s^{1+3\alpha}}\right) = 2sinxsiny\ \frac{1}{\Gamma(1+\alpha)}\frac{(logt\log(t+1))^{\alpha}}{(\alpha)!}$$

$$+4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} + f_3(x,y)\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!} - 2coshx.coshy\frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} - 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!} - 8coshx.coshy\frac{1}{\Gamma(1+3\alpha)}\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!}$$

$$= f_3(x,y)\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!} - 8coshx.coshy\frac{1}{\Gamma(1+3\alpha)}\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!}$$

$= (f_3(x,y) - 8coshx.coshy\frac{1}{\Gamma(1+3\alpha)})\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!}$

Therefore, from $t^{-3\alpha}Res_3(x,y,t)/_{t=0}= 0$ we have,

$$\left(f_3(x,y) - 8coshx.coshy\frac{1}{\Gamma(1+3\alpha)}\right) = 0\ i.e\ (f_3(x,y) = 8coshx.coshy\frac{1}{\Gamma(1+3\alpha)}) \quad (25)$$

Now, to compute approximate of third solution is,

$u_3(x,y,t) = coshx.coshy + 2coshx.coshy\ \frac{1}{\Gamma(1+\alpha)}\frac{(\log(t+1))^{\alpha}}{(\alpha)!} + 4coshx.coshy\frac{1}{\Gamma(1+2\alpha)}\frac{(\log(t+1))^{2\alpha}}{(2\alpha)!}$

$+8coshx.coshy\frac{1}{\Gamma(1+3\alpha)})\frac{(\log(t+1))^{3\alpha}}{(3\alpha)!}$

Similarly, the n^{th} coefficient of $u(x,y,t)$is

$f_n(x,y) = (2)^n coshx.coshy\frac{1}{\Gamma(1+n\alpha)}$

at last the n^{th} $L_{\log(t+1)} -$ RPSM approximate solution of $u(x,y,t)$ is

$u_n(x,y,t) = coshx.coshy\sum_{n=0}^{i}\frac{(2(\log(t+1))^{\alpha})^{n}}{\Gamma(1+n\alpha)(n\alpha)!}$

5. (C-HDFE) AND ITS GRAPHS RESPECT TO NUMERICAL SIMULATIONS

The following tables and figures explained the simulation of the presented numerical method from throughout of values represented as decreasing or increasing monotonic that refer that the method is efficient and active for solving C-HDFE and shown that the parameters and fractional order made a good role for achieved the purpose of the method.

Table 1: Solution at t = 0.1 when value of α = 0.1 and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.0
0.2	152	154	158	163	170.7	179.5	190.0	202.5	216.9	233.6	233.6
0.4	161	163	167	173	180.9	190.2	201.4	214.6	229.9	247.6	233.6
0.6	176	179	183	190	198.4	208.6	220.8	235.3	252.1	271.5	247.6
0.8	199.	202	207	214	223.8	235.3	249.1	265.5	284.5	306.3	271.5
1.0	230	233	239	247	258.2	271.5	287.5	306.3	328.2	353.4	306.3

Table 2: Solution at t = 0.2 when value of α = 0.2 and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.0
0.2	22.5	22.8	23.4	24.2	25.2	26.5	28.1	29.9	32.1	34.5	34.59
0.4	23.8	24.8	25.6	26.7	28.1	29.8	31.7	34.0	36.6	36.6	
0.6	26.1	26.5	27.2	28.1	29.3	30.8	32.7	34.8	37.3	40.2	40.20
0.8	29.5	29.9	30.7	31.7	33.1	34.8	36.9	39.3	42.1	45.3	
1.0	34.0	34.5	35.4	36.6	38.2	40.2	42.5	45.3	48.6	52.3	42.13

Table 3: Solution at t = 0.3 when value of α = 0.3 and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	7.75	7.87	8.06	8.34	8.70	9.14	9.68	10.3	11.05	11.90
0.4	8.21	8.34	8.54	8.84	9.22	9.695	10.26	10.93	11.72	12.61
0.6	9.01	9.14	9.37	9.69	10.11	10.63	11.25	11.99	12.85	13.83
0.8	10.16	10.32	10.57	10.93	11.40	11.99	12.69	13.53	14.49	15.61
1.0	11.73	11.90	12.20	12.61	13.16	13.83	14.65	15.61	16.72	18.01

Table 4: Solution at t = 0.4 when value of α = 0.4 and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	4.57	4.646	4.76	4.92	5.13	5.39	5.71	6.09	6.52	.02
0.4	4.85	4.923	5.04	5.21	5.44	5.72	6.05	6.45	6.91	7.44
0.6	5.3	5.399	5.53	5.72	5.96	6.27	6.64	7.07	7.58	8.16
0.8	6.00	6.091	6.24	6.45	6.73	7.07	7.49	7.98	8.55	9.21
1.0	6.92	7.028	7.20	7.44	7.76	8.16	8.64	9.21	9.87	10.63

Table 5: Solution at t = 0.5 when value of α = 0.5 and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	3.36	3.41	3.49	3.61	3.77	3.96	4.20	4.47	4.79	5.16
0.4	3.56	3.61	3.70	3.83	4	4.20	4.45	4.74	5.08	5.47
0.6	3.91	3.96	4.06	4.20	4.38	4.61	4.88	5.20	5.57	6.00
0.8	4.41	4.47	4.58	4.74	4.94	5.20	5.50	5.87	6.29	6.77
1.0	5.08	5.16	5.29	5.47	5.71	6.00	6.35	6.77	7.25	7.81

Table 6: Solution at t = 0.6 when value of α = 0.6 and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	2.73	2.77	2.84	2.94	3.06	3.22	3.41	3.63	3.89	4.19
0.4	2.89	2.94	3.01	3.11	3.25	3.41	3.61	3.85	4.13	4.44
0.6	3.17	3.22	3.30	3.41	3.56	3.74	3.96	4.22	4.53	4.87
0.8	3.58	3.63	3.72	3.85	4.02	4.22	4.47	4.77	5.11	5.50
1.0	4.13	4.19	4.30	4.44	4.64	4.87	5.16	5.50	5.89	6.34

Table 7: Solution at t = 0.7 when value of α = 0.7 i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	2.34	2.38	2.44	2.52	2.63	2.76	2.93	3.12	3.34	3.60
0.4	2.48	2.52	2.58	2.67	2.79	2.93	3.10	3.30	3.54	3.81
0.6	2.72	2.76	2.83	2.93	3.05	3.21	3.40	3.62	3.88	4.18
0.8	3.07	3.12	3.2	3.30	3.45	3.62	3.84	4.09	4.38	4.72
1.0	3.54	3.60	3.69	3.81	3.98	4.18	4.43	4.72	5.06	5.44

Table 8: Solution at t = 0.8 when value of α = 0.8 and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	2.082	2.11	2.16	2.24	2.33	2.45	2.60	2.77	2.96	3.19
0.4	2.207	2.24	2.29	2.37	2.47	2.60	2.75	2.93	3.14	3.38
0.6	2.42	2.45	2.51	2.60	2.71	2.85	3.02	3.22	3.45	3.71
0.8	2.73	2.77	2.84	2.93	3.06	3.22	3.41	3.63	3.89	4.19
1.0	3.15	3.19	3.27	3.38	3.53	3.71	3.93	4.19	4.49	4.83

Table 9: Solution at t = 0.9 when value of α = 0.9 and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.0
0.2	1.89	1.918	1.96	2.03	2.12	2.22	2.36	2.51	2.69	2.90	2.901
0.4	2.00	2.033	2.08	2.15	2.24	2.36	2.50	2.66	2.85	3.07	075
0.6	2.19	2.229	2.28	2.3	2.46	2.59	2.74	2.92	3.13	3.37	3.372
0.8	2.47	2.515	2.57	2.66	2.78	2.92	3.09	3.29	3.53	3.80	3.804
1.0	2.85	2.901	2.97	3.07	3.20	3.37	3.57	3.80	4.07	4.38	4.389

Table 10: Solution at t = 1 when value of α =1and i=10.

x/y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	1.74	1.76	1.81	1.87	1.95	2.05	2.17	2.31	2.48	2.67
0.4	1.84	1.87	1.92	1.98	2.07	2.17	2.30	2.45	2.63	2.83
0.6	2.02	2.05	2.10	2.17	2.27	2.38	2.52	2.69	2.88	3.10
0.8	2.28	2.31	2.37	2.45	2.56	2.69	2.85	3.03	3.25	3.50
1.0	2.63	2.67	2.74	2.83	2.95	3.10	3.29	3.50	3.75	4.04

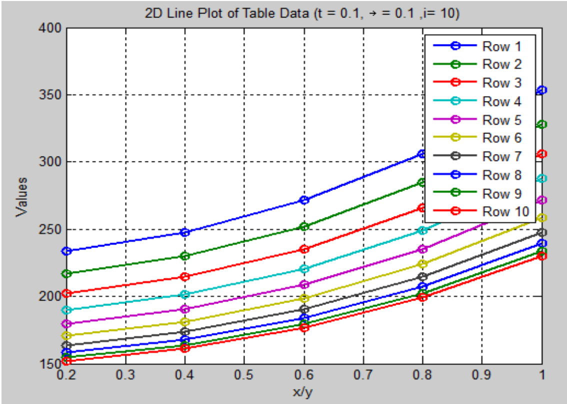

Figure 1: Solution when value of α = 0.1, t = 0.1, **ρ** = 1 and i = 10.

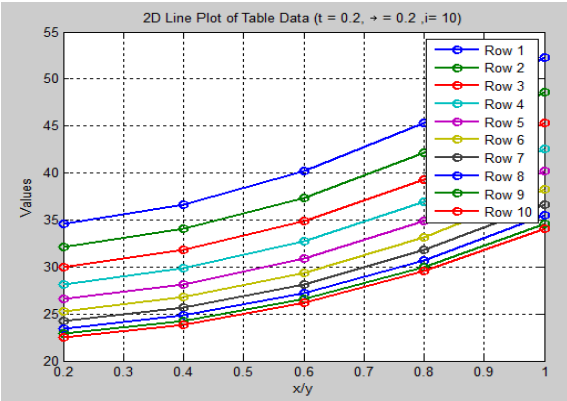

Figure 2: Solution when value of α = 0.2, t = 0.2 and i = 10.

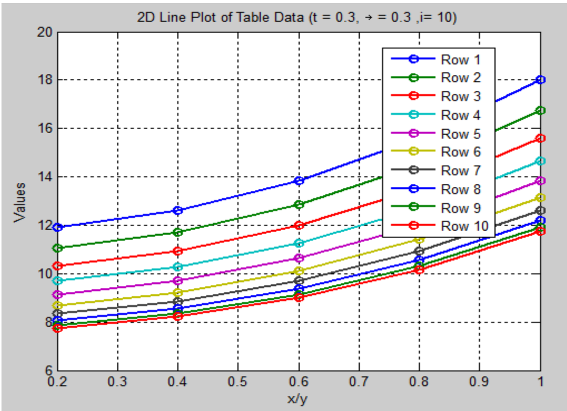

Figure 3: Solution when value of α = 0.3, t = 0.3 and i = 10.

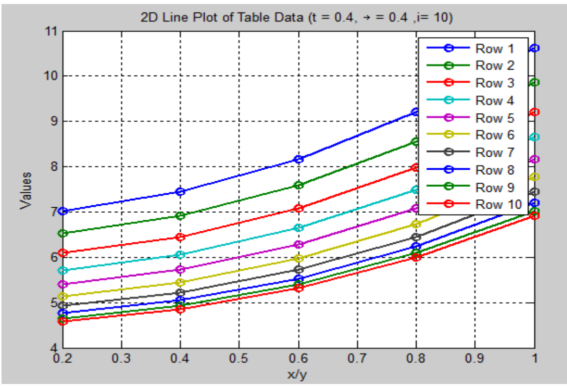

Figure 4: Solution when value of α = 0.4, t = 0.4 and i = 10.

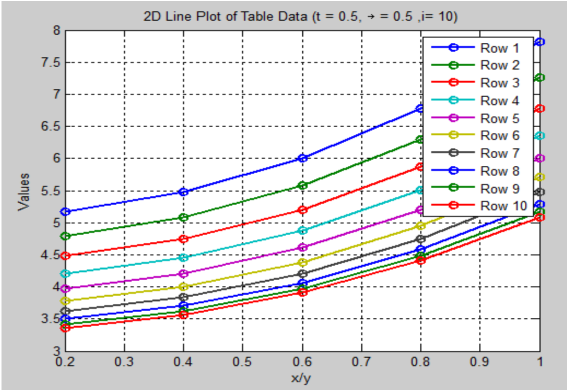

Figure 5: Solution when value of α = 0.5, t = 0.5 i = 10.

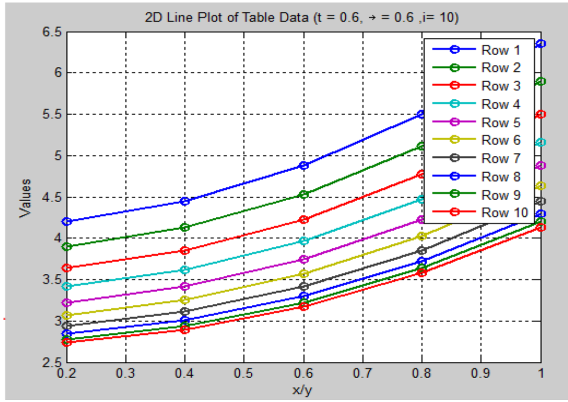

Figure 6: Solution when value of α = 0.16, t = 0.6 and i = 10.

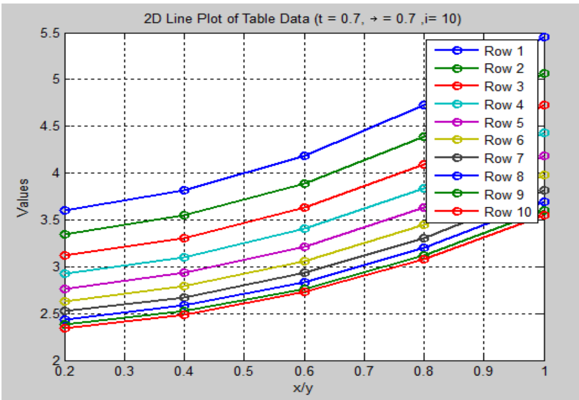

Figure 7: Solution when value of α = 0.7, t = 0.7 and i = 10.

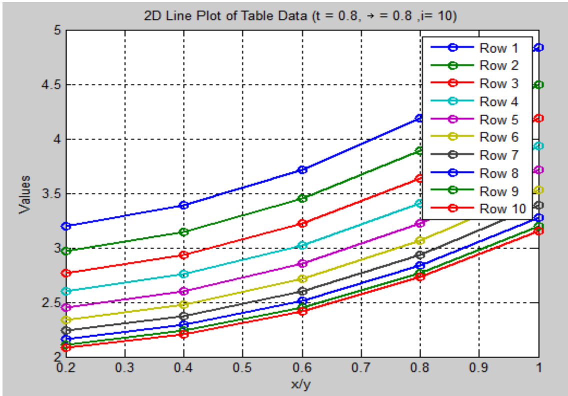

Figure 8: Solution when value of α = 0.8, t = 0.8 and i = 10.

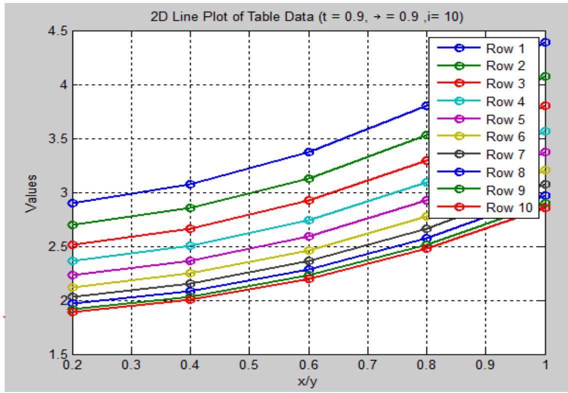

Figure 9: Solution when value of α = 0.9, t = 0.9 and i = 10.

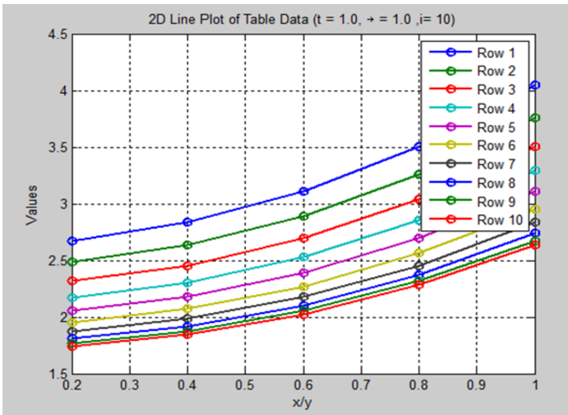

Figure 10: Solution when value of α = 1.0, t = 1.0 and i = 10.

6. CONCLUSIONS

We concluded through our work that working in this the $L_{\log(t+1)}-$ Residual Power Series Method on this for the solution of the Caputo--Hadamard fractional diffusion equation is highly effective, and this is clear from the tables and graphs that indicate the increase or decrease of values for these tables, also the analytic technique is very interest with assumption of residual series suitable with the fractional derivatives and assumption equation and this indicates the validity of the method.

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