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$\frac{t^{\rho}}{\rho}$ −Laplace transformation for Finite Time Stability for Caputo − Katugampola Composition and Riemann-Katugampola Fractional Integro-Differential Systems

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ABSTRACT: In this research, the finite-time stability of Multi-Composition Caputo-Katugampola fractional Integro- Differential Nonlinear system with many values of fractional derivatives is studied with some sufficient and necessary conditions as Lipchitz conditions for nonlinear functions involving Riemann– Katugampola fractional integral and the formulas derived from them with their respective bounded value as well as finding the solution under these conditions that contains Mittag Leffler functions which appeared through the use of the generalized Laplace formula which suitable with Caputo-Katugampola fractional derivative. Therefore, important conditions appeared that contain the parameters that played a good role in finding and computing the stability as in the attached tables of illustrative examples that explain the necessary time requirements for it.

Keywords: finite time stability, Caputo–Katugampola Composition fractional derivatives, Riemann-Katugampola

1. INTRODUCTION

The first of fractional calculus roots started in the last years of the seventeenth century, by Newton's work along with Leibniz's which established as a basis for the classical calculus. The operators of fractional calculus which is most classical is called the Riemann– Liouville fractional integral and derivative [1, 36]. Also, Caputo is introducing another definition of fractional derivatives was suitable for physical conditions, see [14,28]. Moreover, several other definitions of fractional operators have been explained and studied until now, such as Erd´elyi–Kober, Hadamard, Gr¨unwald–Letnikov, Hilfer, Marchaud, and Prabhakar, we can see [1, 35, 36].so since that are many definitions implied that
to suggested to establish some generalized fractional operators for the particular cases of definitions. This The motivation is important for fractional differential equations in models of physics, economics, engineering and other branches of sciences, see [1,14,35]. There are many methods are presented by many authors for solving FDEs analytically or numerically, and they developing interesting methods to obtain analytic solutions for some classes of fractional differential equations and was considered as a one of the most challenging tasks in the field of fractional calculus. In the last years, some authors have been interested to extensions of the fractional for classical integral transforms, such as Laplace and other transformations, [13]. In particular, the Laplace transform extensions appeared in [13, 14, 15] has been used as a effective tool for finding analytic of FDEs. The traditional stability, asymptotical stability and exponential stability in the sense of Lyapunov, which is property of a system was considered in an infinite-time interval. The stability analysis is important issues of control systems field, also the time- delay systems investigation needed this problem many years ago [31]. Many researches have been published for this Issues with particular application of Lyapunov's second method, see [2,37]. Also, another approach is presented of system stability from the non-Lyapunov point which is (finite and practical stability) is presented in [25,26]. Moreover, the linear time-delay systems analytically with finite and practical stability was studied, [8,14]. Recently there are many advances in control theory of fractional dynamical systems for stability explanting, see [10,11]. A condition based to guarantee the asymptotic stability of the fractional order system

some of them is robust stability or stabilization of fractional are presented and discussed in [7,31]. Recently, there are many papers studied finite-time stability analysis of fractional time-delay systems by using many methods such as linear matrix inequality, Lyapunov functions, and Gronwall's integral inequality, [19,30]. In [8,29], the authors introduced the stability in finite time for the system of fractional order with delay equation by using Mittag-Leffler delay type matrix or other suitable conditions. The authors in [23], studied the finite time stability result for nonlinear fractional order system involving discrete time delay. In [6], studied the finite time stability for nonlinear fractional system by using Mittag Leffler function for both orders $0 < \alpha 1 - \alpha 2 < 1$ *and* $1 \leq \alpha 1 - \alpha 2 < 2$. In [4], have been studies the fractional-order Black-Scholes equation with Katugampola Fractional Derivative is a one of Caputo type. The analytic solution of the time-fractional Black-Scholes equation by using the technique of the generalized Laplace homotopy perturbation method which this method is combination between homotopy perturbation method and generalized Laplace transform appeared in [13]. Many a branch of science such as control systems, neural networks, missile systems and other fields, the more practical method is the finite-time stability, see [34,12]. finite time stability focuses on the behavior of a solution of given system work in a fixed time interval [20]. The finite-time stability analysis of fractional differential systems Have been considered in [17]. our aim to specifically on the generalized Laplace transform, see [2], which be called Laplace transform with respect to functions. This issue of operator suitable with the fractional integrals and derivatives with respect to functions, and which used to solve differential equations. In [5] have been introduced finite time stability by Granwall's approach for interesting system time –delay fractional system. In [6, 29] studied finite time stabile of delay nonlinear fractional equations. In [17] provided the impulses and state time delay involving with nonlinear fractional system and explained all details of finite stability. In [33] the authors studied the nonlinear stochastic Ψ −Hilfer fractional system and their finite time stability. In this paper the finite time stability the article is organized as follows. Section 2 contains preliminary definitions from classical and fractional calculus. In Section 3, we consider the generalized Laplace transform with respect to functions, including from the viewpoint of operational calculus, and prove several important properties including an inversion formula and generalized Laplace transforms of several fractional operators with respect to functions. Sections 4 and 5 are devoted to fractional differential equations, firstly a regularity result to show applicability of the generalized Laplace transform, and then explicitly solving some Cauchy initial value problems. Finally, Section 6 makes concluding statements about this manuscript and future directions of research.

2. PRELIMINARIES

Definition 2.1[37]: The Euclidean norm of a vector $x = (x_1, x_2, \dots\dots\dots.., x_n) \in R^n$ is defined as:

$\|x\|^2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$

Definition 2.2[37]: The Maximum of vector $x = (x_1, x_2, \dots\dots\dots.., x_n) \in R^n$ is defined as:

$\|x\|_\infty = \max|x_i|, \qquad 1 \leq i \leq n$

Definition 2.3 [9]: The finite time stable with respect to $t \in [0, T]$ and $\delta, \varepsilon > 0$ if and only if $\sigma < \delta$ such that $\|x(t)\| < \varepsilon \; \forall \; t \in [0, T]$,where $\sigma = \max\{\|x(0)\|, \|x`(0)\|\}$ is initial time of the system differential

Definition 2.4 [22]: Let $\beta \in (0,1), \rho > 0$ and $0 < a < b < \infty$.

$${}^{ck}_{a}D_t^{\beta,\rho} f(t) = \frac{\rho^\beta}{\Gamma(1-\beta)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_t^b \frac{s^{\rho-1}}{(t^\rho - s^\rho)^\beta} [f(s) - f(a)] ds,$$ left Caputo- Katugampola fractional differential systems

$${}^{ck}_{t}D_b^{\beta,\rho} f(t) = \frac{-\rho^\beta}{\Gamma(1-\beta)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^\beta} [f(s) - f(b)] ds,$$ right Caputo- Katugampola fractional differential systems

Definition 2.5 [22]: Let $\beta > 0, \rho > 0$ and consider the interval$[a, b]$, Subset of R, where $0 < a < b < \infty$. The left and right Riemann-Katugampola fractional differential systems are Respectively defined by

$${}^{RK}_{a}D_{t,}^{\beta,\rho} h(t) = \frac{\rho^\beta}{\Gamma(1-\beta)} (t^{1-\rho} \frac{d}{dt}) \int_t^b \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\beta} h(\tau) d\tau,$$

$${}^{RK}_{t}D_{b,}^{\beta,\rho} h(t) = \frac{-\rho^\beta}{\Gamma(1-\beta)} (t^{1-\rho} \frac{d}{dt}) \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^\beta} h(\tau) d\tau,$$

Lemma 2.6 [3]: Let $\alpha > 0, \mu > 0$, and $x(t) \in L^1([a, b]), R), 0 < a < b < \infty$

$$\left\| {}^{RK}_{a}D_t^{-\alpha,\mu} x(t) - {}^{RK}_{a}D_t^{-\alpha,\mu} y(t) \right\| \leq W(t) \|x(t) - y(t)\|$$

When $W(t) = \frac{1}{\mu\alpha\Gamma(\alpha)} (\frac{t^\mu - a^\mu}{\mu})$

Definition 2.7 [7, 11]: The one parameter Mittag Leffler function is given by

$$E_{\beta_1}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\beta_1+1)}, \tag{1}$$

With $\beta_1 > 0$, $Re(\beta_1) > 0$ and $z \in C$. For parameters β_1 and β_2

$$E_{\beta_1,\beta_2}(z) \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\beta_1+\beta_2)}, \tag{2}$$

With $\beta_1, \beta_2 \in C$, $Re(\beta_1) > 0$, $Re(\beta_2) > 0$, $z \in C$. By choosing $\beta_2 = 1$, $E_{\beta_{1,1}}(z) = E_{\beta_1}(z)$.

Definition 2.8 [13]: (The generalized Laplace transformation**)**

Let $f, g : [a,\infty) \to R$ are two real valued functions when $g(t)$ is continuous and $g`(t) > 0$ on $[a,\infty)$, $a > 0$. If the generalized Laplace transformation is

$$\mathcal{L}_g\{f(t)\}(s) = \int_a^{\infty} e^{-s(g(t)-g(a))} f(t)\, g`(t)\, dt, \text{ for all values of s, the integral is valid.} \tag{3}$$

Theorem 2.9[3]: If $\boldsymbol{f(t) \in C^1([a,b]), 0 < a < b < \infty}$, then

$$\left({}^{CK}_{a}D^{\alpha,\mu}_{t}\, {}^{CK}_{a}D^{\beta,\mu}_{t} f\right)(t) = \left({}^{CK}_{a}D^{\alpha+\beta,\mu}_{t} f\right)(t), \qquad t \in [a,b],$$

where $\alpha,\beta,\mu \in \mathbb{R} \ni \rho, \alpha$ be a positive, $\beta < 1$ and $\alpha + \beta \leq 1$.

Lemma 2.10 [28], [22]:

$$1 - \mathcal{L}_{\frac{t^\rho}{\rho}}\{{}^{CK}_{0}D^{\delta,\rho}_{t} x(t)\}(s) = s^{\delta} L_{\frac{t^\rho}{\rho}}\{x(t)\}(s) - s^{\delta-1}x(0) \quad 0 < \delta \leq 1 \tag{4}$$

$$2 - \mathcal{L}_{\frac{t^\rho}{\rho}}\{{}^{CK}_{0}D^{\delta,\rho}_{t} x(t)\}(s) = s^{\delta} L_{\frac{t^\rho}{\rho}}\{x(t)\}(s) - s^{\delta-1}x(0) - s^{\delta-2}x`(0) \quad 1 < \delta \leq 2 \tag{5}$$

$$3\text{-} \mathcal{L}\left\{\left(\frac{t^\rho}{\rho}\right)^{v}\right\}(s) = \frac{\Gamma(1+v)}{s^{1+v}} \tag{6}$$

$$4\text{-} \mathcal{L}_{\frac{t^\rho}{\rho}}\left\{E_{\delta,1}\left\langle \pm\lambda\left(\frac{t^\rho}{\rho}\right)^{\delta}\right\rangle\right\}(s) = \frac{s^{\delta-1}}{s^{\delta}\pm\lambda} \qquad , Re(\delta) > 0 \tag{7}$$

$$5\text{-} \mathcal{L}_{\frac{t^\rho}{\rho}}\left\{E_{\delta,(\alpha+\beta)}\left\langle \pm\lambda\left(\frac{t^\rho}{\rho}\right)^{\delta}\right\rangle\right\}(s) = \frac{s^{\delta-(\alpha+\beta)}}{s^{\delta}\pm\lambda} \qquad Re(\delta) > 0, Re((\alpha+\beta)) > 0 \tag{8}$$

$$6 - \mathcal{L}_{\frac{t^\rho}{\rho}}\{{}^{CK}_{0}D^{\alpha+\beta,\rho}_{t} x(t)\}(s) = s^{\alpha+\beta} L_{\frac{t^\rho}{\rho}}\{x(t)\}(s) - s^{(\alpha+\beta)-1}x(0) \tag{9}$$

3. MULTI-COMPOSITION Caputo - Katugampola -(C − K) FRACTIONAL INTEGRO-DIFFERENTIAL NONLINEAR SYSTEM

Consider the following system involving the Riemann-Katugampola nonlinear fractional integro –differential system which explained as follows:

$$\begin{cases} {}^{CK}_{0}D^{\delta,\rho}_{t} x(t) - A\left({}^{CK}_{0}D^{\alpha,\rho}_{t}\, {}^{CK}_{0}D^{\beta,\rho}_{t} x(t)\right) = g\left(t, x(t), {}^{RK}_{0}I^{\eta,\rho}_{t} x(t)\right), t \in I & (10) \\ x(0) = x_0 \;, x`(0) = x_1 & (11) \end{cases}$$

Where ${}^{CK}_{0}D^{\delta,\rho}_{t}$ indicates the Caputo-Katugampola Fractional Derivatives with α, β, δ and $L = [0,T], x(t) \in C(L, R^n), A \in R^{n\times n}$ and $0 < \alpha,\beta < 1, \alpha+\beta \leq 1, 1 < \delta \leq 2, 0 < \eta < 1$, $g: I \times R^n \times R^n \to R^n$ is a continuous function.

Assumption 3.1:

1- By lemma (2.6), the function $g\left(t, x(t), {}^{RK}_{0}I^{\eta,\rho}_{t} x(t)\right)$ satisfies $||g\left(t, x_1(t), {}^{RK}_{0}I^{\eta,\rho}_{t} x_1(t)\right) - g\left(t, x_2(t), {}^{RK}_{0}I^{\eta,\rho}_{t} x_2(t)\right)|| \leq W||x_1(t) - x_2(t)|| + \dot{w}(t)||x_1(t) - x_2(t)||$, for W and $\dot{w}(t) > 0$.

2- The function $g\left(t, x(t), {}^{RK}_{0}I^{\eta,\rho}_{t} x(t)\right)$ satisfies the following $||\, g\left(t, x(t), {}^{RK}_{0}I^{\beta_3,\rho}_{t} x(t)\right)|| \leq (M + \dot{w}(t))||x(t)||$, for $t \in I, x \in R^n$ and $M > 0$.

3- the operator $Bx(t) + {}^{RK}_{0}I^{\eta,\rho}_{t} x(t)$ satisfies the following $\left\|Bx(t) + {}^{RK}_{0}I^{\beta_3,\rho}_{t} x(t)\right\| \leq (\|B\| + \dot{w}(t))\|x(t)\|$, where B is a constant matrix.

Lemma 3.2: The solution of Multi -Composition- Caputo- Katugampola fractional integro-differential nonlinear System (10) with $0 < \alpha + \beta < 1, 1 < \delta \leq 2$ explained as following:

$$x(t) = X_0\, E_{\delta-(\alpha+\beta)}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) - AX_0\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)E_{\delta-(\alpha+\beta),\delta-(\alpha+\beta)+1}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)+X_1 E_{\delta-(\alpha+\beta),2}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)$$
$$\int_0^t \left(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-(\alpha+\beta)-1}}\right) E_{\delta-(\alpha+\beta),\delta}, \ A\left(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) g\left(\theta, x(\theta), {}^{RK}_{0}I_t^{\eta,\rho} x(\theta)\right)) \, d\theta$$

Proof: Take $\frac{t^\rho}{\rho}$ Laplace Transform for two side of (10) then
$\mathcal{L}_{\frac{t^\rho}{\rho}}\{{}^{CK}_{0}D_t^{\delta,\rho}x(t)\}(s) - \mathcal{L}_{\frac{t^\rho}{\rho}}\{{}^{CK}_{0}D_t^{(\alpha+\beta),\rho}x(t)\}(s) = \mathcal{L}_{\frac{t^\rho}{\rho}}\{g(t, x(t), {}^{RK}_{0}I_t^{\eta,\rho}x(t))\}$, By using lemma (2.10) we get that
$s^\delta X_{\frac{t^\rho}{\rho}}(s) - s^{\delta-1}x(0) - s^{\delta-2}x`(0) - As^{(\alpha+\beta)}X_{\frac{t^\rho}{\rho}}(s) + As^{(\alpha+\beta)-1}x(0) = \mathcal{L}_{\frac{t^\rho}{\rho}}\{g(t, x(t), {}^{RK}_{0}I_t^{\eta,\rho}x(t))\}(s)$. From initial condition (11) we have that

$$s^\delta X_{\frac{t^\rho}{\rho}}(s) - s^{\delta-1}x_0 - s^{\delta-1}x_1 - As^{(\alpha+\beta)}X_{\frac{t^\rho}{\rho}}(s) + As^{(\alpha+\beta)-1}x_0 = \mathcal{L}_{\frac{t^\rho}{\rho}}\{g(t, x(t), {}^{RK}_{0}I_t^{\eta,\rho}x(t))\}(s)$$

$$X_{\frac{t^\rho}{\rho}}(s)[s^\delta - A\, s^{(\alpha+\beta)}] - X_0[s^{\delta-1} - As^{(\alpha+\beta)-1}] - X_1 s^{\delta-2} = \mathcal{L}_{\frac{t^\rho}{\rho}}\{g(t, x(t), {}^{RK}_{0}I_t^{\eta,\rho}x(t))\}(s) \tag{12}$$

multiply (12) by $\frac{1}{[s^\delta - As^{(\alpha+\beta)}]}$, we obtain that

$$X_{\frac{t^\rho}{\rho}}(s) - x_0\left[\frac{\frac{s^{\delta-1} - As^{(\alpha+\beta)-1}}{s^{(\alpha+\beta)}}}{\frac{s^\delta - As^{(\alpha+\beta)}}{s^{(\alpha+\beta)}}}\right] - x_1\left[\frac{\frac{s^{\delta-2}}{s^{(\alpha+\beta)}}}{\frac{s^\delta - As^{(\alpha+\beta)}}{s^{(\alpha+\beta)}}}\right] = \frac{\mathcal{L}_{\frac{t^\rho}{\rho}}\{g(t,x(t), {}^{RK}_{0}I_t^{\eta,\rho}x(t))\}(s)\frac{1}{s^{(\alpha+\beta)}}}{\frac{s^\delta - As^{(\alpha+\beta)}}{s^{(\alpha+\beta)}}}$$

$$X_{\frac{t^\rho}{\rho}}(s) - X_0\left[\frac{s^{\delta-(\alpha+\beta)-1} + As^{-1}}{s^{\delta-(\alpha+\beta)} - A}\right] - X_1\left[\frac{s^{\delta-(\alpha+\beta)-2}}{s^{\delta-(\alpha+\beta)} - A}\right] = \frac{\mathcal{L}_{\frac{t^\rho}{\rho}}\{g\left(t, x(t), {}^{RK}_{0}I_t^{\eta,\rho}x(t)\right)\}(s)\frac{1}{s^{(\alpha+\beta)}}}{s^{\delta-(\alpha+\beta)} - A}$$

Now by using invers $\frac{t^\rho}{\rho}$ Laplace Transform we have that

$$x(t)\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left\{X_0\left(\frac{s^{\delta-(\alpha+\beta)-1}}{s^{\delta-(\alpha+\beta)} - A}\right)\right\} - \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left\{X_0\left(\frac{As^{-1}}{s^{\delta-(\alpha+\beta)} - A}\right)\right\} + \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left\{X_1\left(\frac{s^{\delta-(\alpha+\beta)-2}}{s^{\delta-(\alpha+\beta)} - A}\right)\right\} + \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left\{\frac{\mathcal{L}_{\frac{t^\rho}{\rho}}\{g(t,x(t), {}^{RK}_{0}I_t^{\eta,\rho}x(t))\}(s)\frac{1}{s^{(\alpha+\beta)}}}{s^{\delta-(\alpha+\beta)} - A}\right\}$$

, By from lemma (2.10), we get that

$$x(t) = X_0\, E_{\delta-(\alpha+\beta)}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) - AX_0\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)E_{\delta-(\alpha+\beta),\delta-(\alpha+\beta)+1}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)+X_1 E_{\delta-(\alpha+\beta),2}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)$$
$$\int_0^t \left(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-(\alpha+\beta)-1}}\right) E_{\delta-(\alpha+\beta),\delta}, \ A\left(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) g\left(\theta, x(\theta), {}^{RK}_{0}I_t^{\eta,\rho} x(\theta)\right)) \, d\theta \tag{13}$$

Lemma 3.3: If $b(t) > 0$& $c(t) > 0$ is locally integrable on $[0, T), \delta > 0, d(t) \le W$ with
$b(t) \le c(t) + d(t)\int_0^t (\frac{(t-\alpha)^{\rho(\delta-1)}}{\rho^{\delta-1}}) b(\alpha)d\alpha, 0 \le t < T$. Then
$b(t) \le c(t) + d(t)\int_0^t [\sum_{n=1}^{+\infty} \frac{[d(t)\Gamma(\delta)]^n}{\Gamma(n\delta)} (\frac{(t-\alpha)^{\rho(n\delta-1)}}{\rho^{n\delta-1}}) c(\alpha)]d\alpha, 0 \le t < T$.

proof: Assume that $A\emptyset(t) = d(t)\int_0^t \left(\frac{(t-\alpha)^{\rho(\delta-1)}}{\rho^{\delta-1}}\right) b(\alpha)d\alpha, t > 0$, for locally integrable function $\emptyset$. Then $b(t) \le c(t) + Ab(t)$ implies $b(t) \le \sum_{k=0}^{n-1} A^k\, c(t) + A^n b(t)$.

Now to prove that $A^n b(t) \le \int_0^t [\frac{[d(t)\Gamma(\delta)]^n}{\Gamma(n\delta)} (\frac{(t-\alpha)^{\rho(n\delta-1)}}{\rho^{n\delta-1}}) b(\alpha)]d\alpha. \quad (14)$

and $A^n b(t) \to 0 \text{ as } n \to +\infty$ for each t in $0 \le t < T$

from this (14) is true for $n = 1$. Assume that it is true for some $n = k$.

If $n = k + 1$, then the induction hypothesis implies

$A^{k+1}b(t) = B\left(A^k b(t)\right) \leq d(t)\int_0^t \left(\frac{(t-\alpha)^{\rho(\delta-1)}}{\rho^{\delta-1}}\right)\left[\int_0^\alpha [\frac{[d(t)\Gamma(\delta]^k}{\Gamma(k\delta)}(\frac{(\alpha-\tau)^{\rho(k\delta-1)}}{\rho^{k\delta-1}})b(\tau)]d\tau,\right] d\alpha$

Since d(t) is nondecreasing, we get that

$A^{k+1}b(t) \leq A^{k+1}b(t)(d(t))^{k+1}\int_0^t \left(\frac{(t-\alpha)^{\rho(\delta-1)}}{\rho^{\delta-1}}\right)\left[\int_0^\alpha [\frac{[\Gamma(\delta]^k}{\Gamma(k\delta)}(\frac{(\alpha-\tau)^{\rho(k\delta-1)}}{\rho^{k\delta-1}})b(\tau)]d\tau\right] d\alpha$, we have that

$A^{k+1}b(t) \leq (d(t))^{k+1}\int_0^t \left[\int_0^\alpha [\frac{[\Gamma(\delta)]^k}{\Gamma(k\delta)}\left(\frac{(t-\alpha)^{\rho(\delta-1)}}{\rho^{\delta-1}}\right)(\frac{(\alpha-\tau)^{\rho(k\delta-1)}}{\rho^{k\delta-1}})b(\tau)]d\alpha\right] b(\tau)\, d\tau$

$=\int_0^t [\frac{[d(t)\Gamma(\delta)]^{k+1}}{\Gamma((k+1)\beta_1)}(\frac{(t-\alpha)^{\rho((k+1)\delta-1)}}{\rho^{k\delta-1}})b(\alpha)]d\alpha$, Hence

$= \int_\tau^t \left(\frac{(t-\alpha)^{\rho(\delta-1)}}{\rho^{\delta-1}}\right)(\frac{(\alpha-\tau)^{\rho(k\delta-1)}}{\rho^{k\delta-1}})\, d\alpha = \left(\frac{(t-\tau)^{\rho(k\delta+\delta-1)}}{\rho^{k\delta+\delta}}\right)\int_0^1 (1-Z)^{\delta-1} Z^{k\delta-1}dz$

$=\left(\frac{(t-\tau)^{\rho((k+1)\delta-1)}}{\rho^{k\delta+\delta-1}}\right)A(K\delta,\delta) = \frac{\Gamma(\delta)\Gamma(K\delta)}{\Gamma((K+1)\delta)}\left(\frac{(t-\tau)^{\rho((k+1)\delta-1)}}{\rho^{k\delta+\delta-1}}\right),$

Since, $A^n b(t) \leq \int_0^t \frac{(\Gamma(\delta))^n}{\Gamma(n\delta)}(t-\alpha)^{n\delta-1}b(\alpha)d\alpha \to 0 \; as\; n \to +\infty$, $t \in [0,T)$.Then

$b(t) \leq c(t) + d(t)\int_0^t [\sum_{n=1}^{+\infty}\frac{[d(t)\Gamma(\delta)]^n}{\Gamma(n\delta)}(\frac{(t-\alpha)^{\rho(n\delta-1)}}{\rho^{n\delta-1}})c(\alpha)]d\alpha, 0 \leq t < T.$

Remark 3.4: From above lemma (3.3) we obtain that, $b(t) \leq c(t)E_\delta\left(d(t)\Gamma(\delta)\right)(\frac{t^\rho}{\rho})^\delta)$ where $c(t)$ is a nondecreasing function.

Lemma 3.5:

(1) For $\omega_1, \omega_2 \geq 0$ such that, for $\delta, \alpha, \beta \in R^+$ and $\delta - (\alpha+\beta) < 1$,
$||E_{\delta-(\alpha+\beta),1}\left(A\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right)|| \leq \omega_1 ||e^{A\left(\frac{t^\rho}{\rho}\right)}||$, and $||E_{\delta-(\alpha+\beta),\delta-(\alpha+\beta)}(A(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})|| \leq \omega_2||e^{A\left(\frac{t^\rho}{\rho}\right)}||$,
where A is constant matrix.

(2) Suppose that $\delta, \alpha, \beta \in R^+$, $\delta - (\alpha+\beta) \geq 1$ then for $\gamma = 1,2,\delta$,
We have that,$||E_{\delta-(\alpha+\beta),\gamma}(A(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})|| \leq ||e^{A\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)}||$

(3) If A is a stability matrix, then $\exists$ a constant $K \geq 1$ such that $\left(\frac{t^\rho}{\rho}\right) > 0$, then
$||E_{\delta-(\alpha+\beta),\gamma}(A(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})|| \leq Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)}$for $0 < \delta - (\alpha+\beta) < 1$, and
$||E_{\delta-(\alpha+\beta),\gamma}(A\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)) \leq e^{-\eta\left(\frac{t^\rho}{\rho}\right)}$for $1 \leq \delta - (\alpha+\beta) < 2$,
When the greatest eigenvalue of A is η.

proof: Firstly, from definition (2.3) for $0 < \delta - (\alpha+\beta) < 1$, we get that

$E_{\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right) = \sum_{i=0}^\infty \frac{(A\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})^i}{\Gamma(i((\delta-(\alpha+\beta))+1)} = \sum_{i=0}^\infty \frac{K!}{\Gamma(i((\delta-(\alpha+\beta))+1)}\frac{A^i(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})^i}{K!} = \sum_{i=0}^\infty \frac{\frac{t^{i\rho(((\delta-(\alpha+\beta))-1)}}{\rho^{(\delta-(\alpha+\beta)}}K!}{\Gamma(i((\delta-(\alpha+\beta))+1)}\frac{A^i\frac{t^{i\rho}}{\rho}}{K!} =$
$\sum_{i=0}^\infty \frac{\frac{t^{i\rho(((\delta-(\alpha+\beta))-1)}}{\rho^{(\delta-(\alpha+\beta)}}K!}{\Gamma(i((\delta-(\alpha+\beta))+1)}\frac{(A\frac{t^\rho}{\rho})^i}{K!}$

$||E_{\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)|| \leq \sup_{t\in R^+}(\sup_{i\in Z^+}\left(\frac{K!}{\frac{t^{i\rho(1-((\delta-(\alpha+\beta)))}}{\rho^{(\delta-(\alpha+\beta)}}\Gamma(i((\delta-(\alpha+\beta))+1)}\right))||e^{A\frac{t^\rho}{\rho}}||$

$\limsup_{t\to\infty}\rho\, ||E_{(\delta-(\alpha+\beta)(\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)|| \leq \lim_{t\to\infty}\sup(\sup_{i\in Z^+}(\frac{K!}{\frac{t^{i\rho(1-(\delta-(\alpha+\beta)))}}{\rho^{\delta-(\alpha+\beta)}}\Gamma(i(\delta-(\alpha+\beta))+1)}))||\sum_i^\infty\frac{(A\frac{t^\rho}{\rho})^i}{i!}||$

$\limsup_{t\to\infty}\rho\, ||E_{(\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)|| \leq \limsup_{t\to\infty}\rho\, ||e^{A\frac{t^\rho}{\rho}}||$

Also note from definition (2.3) for $0 < \beta_1 - \beta_2(\epsilon R^+) < 1$

$$E_{(\delta-(\alpha+\beta),(\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)=\sum_{i=0}^{\infty}\frac{(A\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})^{i}}{\Gamma(i((\delta-(\alpha+\beta))+((\delta-(\alpha+\beta)))}=\sum_{i=0}^{\infty}\frac{K!}{\Gamma(i((\delta-(\alpha+\beta))+((\delta-(\alpha+\beta)))}\frac{A^{i}(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})^{i}}{K!}=$$

$$\sum_{i=0}^{\infty}\frac{\frac{t^{i\rho(((\delta-(\alpha+\beta))-1)}}{\rho^{(\delta-(\alpha+\beta)}}K!}{\Gamma(i((\delta-(\alpha+\beta))+((\delta-(\alpha+\beta)))}\frac{A^{i}\ \frac{t^{i\rho}}{\rho}}{K!}=\sum_{i=0}^{\infty}\frac{\frac{t^{i\rho(((\delta-(\alpha+\beta))-1)}}{\rho^{(\delta-(\alpha+\beta)}}K!}{\Gamma(i((\delta-(\alpha+\beta))+((\delta-(\alpha+\beta)))}\frac{(A\ \frac{t^{\rho}}{\rho})^{i}}{K!}$$

$$||E_{(\delta-(\alpha+\beta),(\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)||\leq\sup_{t\in R^{+}}\left(\sup_{i\in Z^{+}}\left(\frac{K!}{\frac{t^{i\rho(1-((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}}\Gamma(i((\delta-(\alpha+\beta))+((\delta-(\alpha+\beta)))}\right)\right)||e^{A\frac{t^{\rho}}{\rho}}||$$

$$\limsup_{t\to\infty}\rho\,||\,E_{(\delta-(\alpha+\beta),(\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)||\leq$$

$$\lim_{t\to\infty}sup(\sup_{i\in Z^{+}}(\frac{K!}{\frac{t^{i\rho(1-((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}}\Gamma(i((\delta-(\alpha+\beta))+((\delta-(\alpha+\beta)))}))||\sum_{i}^{\infty}\frac{(A\ \frac{t^{\rho}}{\rho})^{i}}{i!}||$$

$$\leq\limsup_{t\to\infty}\rho\,||e^{A\frac{t^{\rho}}{\rho}}||$$

$$\limsup_{t\to\infty}\rho\,||E_{(\delta-(\alpha+\beta),(\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)||\leq\limsup_{t\to\infty}\rho\,||e^{A\frac{t^{\rho}}{\rho}}||$$

On the other hand, if $(\delta-(\alpha+\beta)\in R^{+})\geq 1$ then for $\gamma=1,2,\delta$

$$E_{(\delta-(\alpha+\beta),\gamma}(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})=\sum_{i=0}^{\infty}\frac{(A\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})^{i}}{\Gamma(i((\delta-(\alpha+\beta))+\gamma)}=\sum_{i=0}^{\infty}\frac{K!}{\Gamma(i((\delta-(\alpha+\beta))+\gamma)}\frac{A^{i}(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})^{i}}{K!}$$

$$||E_{(\delta-(\alpha+\beta),\gamma}(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})||\leq\sup_{t\in R^{+}}\left(\sup_{i\in Z^{+}}\left(\sum_{i=0}^{\infty}\frac{K!}{\Gamma(i((\delta-(\alpha+\beta))+\gamma)}\right)\right)||e^{A\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}}}||$$

$$\limsup_{t\to\infty}\rho\,||\,E_{(\delta-(\alpha+\beta),(\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)||\leq\lim_{t\to\infty}sup(\sup_{i\in Z^{+}}(\sum_{i=0}^{\infty}\frac{K!}{\Gamma(i((\delta-(\alpha+\beta))+\gamma)}))\,||\sum_{i}^{\infty}\frac{(A\ \frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})^{i}}{i!}||\leq$$

$$\lim_{t\to\infty}sup||e^{A\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}}}||$$

$$\limsup_{t\to\infty}\rho\,||\,E_{(\delta-(\alpha+\beta),(\delta-(\alpha+\beta)}\left(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})\right)||\leq\limsup_{t\to\infty}||e^{A\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}}}||$$

In addition, if A is a stability matrix, then $\exists$ a constant $K\geq 1$ such that $\left(\frac{t^{\rho}}{\rho}\right)>0$

$$||E_{(\delta-(\alpha+\beta),\gamma}(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})||\leq Ke^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}\text{for }0<(\delta-(\alpha+\beta)<1$$

$$||E_{(\delta-(\alpha+\beta),\gamma}(A(\frac{t^{\rho((\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}})||\leq e^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}\text{for }1<(\delta-(\alpha+\beta)<2.$$ When η be the greatest eigenvalue of A

Theorem 3.6: Let $0<\delta-(\alpha+\beta)<1$ and $t\in I, x(.)\in R^{n}$ the $R-K$ nonlinear fractional integro –differential system (10), (11) is finite time stable provided that condition:

$$Ke^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}\left[1+||A||\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)+\left(\frac{t^{\rho}}{\rho}\right)\right]E_{\delta-(\alpha+\beta)}\left(K\left(M+\dot{w}(t)\right)\right)\Gamma\left(\delta-(\alpha+\beta)\right)\left(\frac{t^{(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)<\frac{\varepsilon}{\mu}\qquad(15)$$

proof: Taking norm function of solution equation (13),we obtain following,

$$\|x(t)\|=\|x_{0}\|\left\|E_{\delta-(\alpha+\beta)}(A(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})\right\|+\|A\|\|x_{0}\|\left\|(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})\right\|\left\|E_{\delta-(\alpha+\beta),\delta-(\alpha+\beta)+1}(A(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})\right\|+\|x_{1}\|$$

$$||E_{\delta-(\alpha+\beta),2}(A(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})||+\int_{0}^{t}|\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}|\ ||E_{\delta-(\alpha+\beta),\delta}\ A(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})||\ ||g(\theta,x(\theta),{}^{RK}_{0}I_{t}^{\eta,\rho}x(\theta))||\,d\theta\qquad(16)$$

by using lemma(3.5), equation (16), become that

$$\|x(t)\| = \|x_0\| Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|A\|\|x_0\| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|x_1\||\left(\frac{t^\rho}{\rho}\right)| Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)}$$

$$+ \int_0^t |\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}| Ke^{-\eta\left(\frac{(t-\theta)^\rho}{\rho}\right)} || \, || g\left(\theta, x(\theta), {}^{RK}_{0}I^{\eta,\rho}_{t} x(\theta)\right)|| \, d\theta \quad (17)$$

By using assumption (3.1), (2), the equation (17), become that

$$\|x(t)\| = \|x_0\| Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|A\|\|x_0\| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|x_1\||\left(\frac{t^\rho}{\rho}\right)| Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)}$$

$$+ \int_0^t |\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}| Ke^{-\eta\left(\frac{(t-\theta)^\rho}{\rho}\right)} \, ||(M + \dot{w}(\theta))||x(\theta)|| \, d\theta \quad (18)$$

Multiply equation (18) by $e^{\eta(\frac{t^\rho}{\rho})}$, we have that

$$e^{\eta\left(\frac{t^\rho}{\rho}\right)}\left\|x(t)\right\| = K[||x_0|| + ||A|| \, ||x_0|| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + ||x_1||] + \int_0^t |\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}| \, Ke^{\eta\theta}(M +$$

$$\dot{w}(\theta))\|x(\theta)\| d\theta \quad (19)$$

*A*ccording to lemma (3.3), and put $b(t) = e^{\eta(\frac{t^\rho}{\rho})}||x(t)||$, and put $c(t) = K[||x_0|| + ||A|| \, ||x_0|| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + ||x_1||]$, and put $d(t) = K(M + \dot{w}(\theta))$, By remark (3.4), the equation (19), become to

$$b(t) \le c(t) E_{\delta-(\alpha+\beta)}\left(d(t)\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right)$$

$$\le K[||x_0|| + ||A|| \, ||x_0|| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + ||x_1||] \, E_{\delta-(\alpha+\beta)}\left(K(M + \dot{w}(\theta))\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right) \quad (20)$$

then, $\|x(t)\| \le Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)}\left[||x_0|| + \|A\|\|x_0\| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + \|x_1\|\right]$

$$E_{\delta-(\alpha+\beta)}\left(K(M + \dot{w}(\theta))\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right) \quad (21)$$

From definition (2.3), we obtain $\|x_0\| \le \delta, \|x_1\| \le \delta$, then

$$||x(t)|| \le K\delta e^{-\eta\left(\frac{t^\rho}{\rho}\right)}[1 + ||A|| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + |\left(\frac{t^\rho}{\rho}\right)|] \, E_{\delta-(\alpha+\beta)}\left(K(M + \dot{w}(\theta))\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right),$$

Therefore, $||x(t)| \le \varepsilon$, for all $t \in [0, T)$. Which implies that formula (10), is a finite time stable.

Theorem 3.7: Let $1 \le \delta-(\alpha+\beta) < 2$ and $t \in I, x(.) \in R^n$. Then the $R-K$ nonlinear fractional integro – differential system (10), (11), is a finite time stable provided that

$$e^{-\eta\left(\frac{t^\rho}{\rho}\right)}\left[1 + ||A||\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) + \left(\frac{t^\rho}{\rho}\right)\right] E_{\delta-(\alpha+\beta)}\left(\left(M + \dot{w}(t)\right)\Gamma\left(\delta-(\alpha+\beta)\right)\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right) < \frac{\varepsilon}{\mu}, \text{ for any } t \in [o, T). \quad (22)$$

Proof: Taking norm function of solution equation (13),we obtain following

$$\|x(t)\| = \|x_0\| \left\|E_{\delta-(\alpha+\beta)}\left(A\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + \|A\|\|x_0\| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| \left\|E_{\delta-(\alpha+\beta),\delta-(\alpha+\beta)+1}\left(A\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + \|x_1\|$$

$$||E_{\delta-(\alpha+\beta),2}\left(A\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)|| + \int_0^t |\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}| \, ||E_{\delta-(\alpha+\beta),\delta} \, A\left(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)|| \, ||g\left(\theta, x(\theta), {}^{RK}_{0}I^{\eta,\rho}_{t} x(\theta)\right)|| \, d\theta \quad (23)$$

by using lemma (3.5), equation (23), become that

$$\|x(t)\| = \|x_0\| e^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|A\|\|x_0\| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| e^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|x_1\||\left(\frac{t^\rho}{\rho}\right)| e^{-\eta\left(\frac{t^\rho}{\rho}\right)}$$

$$+ \int_0^t |\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}| e^{-\eta\left(\frac{(t-\theta)^\rho}{\rho}\right)} \, ||g\left(\theta, x(\theta), {}^{RK}_{0}I^{\eta,\rho}_{t} x(\theta)\right)|| \, d\theta \, . \quad (24)$$

By using assumption (3.1), (2), the equation (24), become that

$$\|x(t)\| = \|x_0\| \, e^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|A\|\|x_0\| \left\| \left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) \right\| e^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|x_1\| |(\frac{t^\rho}{\rho})| \, e^{-\eta\left(\frac{t^\rho}{\rho}\right)}$$

$$+ \int_0^t \left| \frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}} \right| e^{-\eta\left(\frac{(t-\theta)^\rho}{\rho}\right)} \, ||(M + \dot{w}(\theta))||x(\theta)|| \, d\theta . \qquad (25)$$

Multiply equation (25) by $e^{\eta(\frac{t^\rho}{\rho})}$, we have that

$$e^{\eta(\frac{t^\rho}{\rho})}\|x(t)\| = \|x_0\| + \|A\|\|x_0\| \left\| \left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\delta-(\alpha+\beta)}\right) \right\| + \|x_1\| |(\frac{t^\rho}{\rho})| + \int_0^t \left| \frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}} \right| e^{\eta\theta} \, ||(M + \dot{w}(\theta))||x(\theta)|| \, d\theta . \qquad (26)$$

According to lemma (3.3), we put $b(t) = e^{\eta(\frac{t^\rho}{\rho})}||x(t)||$, and put $c(t) = [||x_0|| + ||A|| \, ||x_0|| \left\| \left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) \right\| + ||x_1|| \, |(\frac{t^\rho}{\rho})| \,]$, and $d(t) = (M + \dot{w}(\theta))$

by remark (3.4), the equation (26), become to

$$b(t) \le c(t) E_{(\delta-(\alpha+\beta)}\left(d(t)\Gamma((\delta-(\alpha+\beta))(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})\right) E_{(\delta-(\alpha+\beta)}\left((M + \dot{w}(\theta))\Gamma((\delta-(\alpha+\beta))(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})\right) \quad (27)$$

Then, $\|x(t)\| \le e^{-\eta\left(\frac{t^\rho}{\rho}\right)}[\|x_0\| + \|A\|\|x_0\| \left\| \left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) \right\| + ||x_1|| \,]$

$$E_{\delta-(\alpha+\beta)}\left((M + \dot{w}(\theta))\Gamma(\delta-(\alpha+\beta))(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})\right) \qquad (28)$$

From definition (2.3), we obtain $\left||x_0|\right| \le \delta$, $\left||x_1|\right| \le \delta$, then

$$\|x(t)\| \le \delta e^{-\eta\left(\frac{t^\rho}{\rho}\right)}[1 + \|A\| \left\| \left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) \right\| + |(\frac{t^\rho}{\rho})|] \, E_{(\delta-(\alpha+\beta)}\left((M + \dot{w}(\theta))\Gamma((\delta-(\alpha+\beta))(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}})\right).$$

Therefore, $\|x(t)\| \le \varepsilon$ for all $t \in [0, T)$. which implies that formula (10), is finite time stable.

Corollary 3.8: The R−K linear fractional integro−differentail system

$$\begin{cases} {}^{CK}_{0}D_t^{\delta,\rho}x(t) - A\left({}^{CK}_{0}D_t^{\alpha,\rho}\,{}^{CK}_{0}D_t^{\beta,\rho}x(t)\right) = Bx(t) + {}^{RK}_{0}I_t^{\eta,\rho}x(t)), \quad \text{for } t \in I & (29)\\ x(0) = x_0 \;, x`(0) = x_1 & (30)\end{cases}$$

Is finite time stable for $0 < \delta - (\alpha + \beta) < 1$, if

$$Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)}\left[1 + ||A|| \left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) + \left(\frac{t^\rho}{\rho}\right)\right] E_{\delta-(\alpha+\beta)}((\|B\| + \dot{w}(t))\|x(t)\| \; K \; \Gamma(\delta-(\alpha+\beta)) \; (\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}) \,) < \frac{\varepsilon}{\mu} \qquad (31)$$

proof: By using $\frac{t^\rho}{\rho}$ Laplace trance form on (29), we obtion

$$\mathcal{L}_{\frac{t^\rho}{\rho}}\{{}^{CK}_{0}D_t^{\delta,\rho}x(t)\}(s) - \mathcal{L}_{\frac{t^\rho}{\rho}}\left\{A({}^{CK}_{0}D_t^{\alpha,\rho}\,{}^{CK}_{0}D_t^{\beta,\rho}x(t))\right\}(s) = \mathcal{L}_{\frac{t^\rho}{\rho}}\{Bx(t) + {}^{RK}_{0}I_t^{\eta,\rho}x(t) \;\}(s)$$

$$s^\delta X(s) - s^{\delta-1}x(0) - s^{\delta-2}x`(0) - As^{(\alpha+\beta)}X(s) + As^{(\alpha+\beta)-1}x(0)$$

$= \mathcal{L}_{\frac{t^\rho}{\rho}}\{Bx(t) + {}^{RK}_{0}I_t^{\eta,\rho}x(t) \;\}(s)$, By using initial condition (30) ,we get that,

$$X_{\frac{t^\rho}{\rho}}(s)[s^\delta - As^{(\alpha+\beta)}] - x_0[s^{\delta-1} - As^{(\alpha+\beta)-1}] - x_1 s^{\delta-2} = \mathcal{L}_{\frac{t^\rho}{\rho}}\{Bx(t) + {}^{RK}_{0}I_t^{\eta,\rho}x(t) \;\}(s)$$

$$X_{\frac{t^\rho}{\rho}}(s) - x_0[\frac{s^{\delta-1}+As^{(\alpha+\beta)-1}}{s^\delta - As^{(\alpha+\beta)}}] - x_1\left[\frac{s^{\delta-2}}{s^\delta - As^{(\alpha+\beta)}}\right] = \frac{\mathcal{L}_{\frac{t^\rho}{\rho}}\{Bx(t)+{}^{RK}_{0}I_t^{\eta,\rho}x(t) \;\}(s)}{s^\delta - As^{(\alpha+\beta)}}$$

$$X_{\frac{t^\rho}{\rho}}(s) - x_0\left[\frac{\frac{s^{\delta-1}+As^{(\alpha+\beta)-1}}{s^{(\alpha+\beta)}}}{\frac{s^\delta - As^{(\alpha+\beta)}}{s^{(\alpha+\beta)}}}\right] - x_1\left[\frac{\frac{s^{\delta-2}}{s^{(\alpha+\beta)}}}{\frac{s^\delta - As^{(\alpha+\beta)}}{s^{(\alpha+\beta)}}}\right] = \frac{\mathcal{L}_{\frac{t^\rho}{\rho}}\left\{Bx(t)+{}^{RK}_{0}I_t^{\eta,\rho}x(t) \;\right\}(s)\frac{1}{s^{(\alpha+\beta)}}}{\frac{s^\delta - As^{(\alpha+\beta)}}{s^{(\alpha+\beta)}}}$$

$$X_{\frac{t^\rho}{\rho}}(s) - x_0\left[\frac{s^{\delta-(\alpha+\beta)-1}+As^{-1}}{s^{\delta-(\alpha+\beta)}-A}\right] - x_1\left[\frac{s^{\delta-(\alpha+\beta)-2}}{s^{\delta-(\alpha+\beta)}-A}\right] = \frac{\mathcal{L}_{\frac{t^\rho}{\rho}}\left\{Bx(t)+{}^{RK}_{0}I_t^{\eta,\rho}x(t) \;\right\}(s)\frac{1}{s^{(\alpha+\beta)}}}{s^{\delta-(\alpha+\beta)}-A}$$

By using inverse $\frac{t^\rho}{\rho}$ Laplace trance form we get that,

$$x(t)=\mathcal{L}^{-1}_{\frac{t^{\rho}}{\rho}}\left\{x_0\left(\frac{s^{\delta-(\alpha+\beta)-1}}{s^{\delta-(\alpha+\beta)}-A}\right)\right\}-\mathcal{L}^{-1}_{\frac{t^{\rho}}{\rho}}\left\{x_0\left(\frac{As^{-1}}{s^{\delta-(\alpha+\beta)}-A}\right)\right\}+\mathcal{L}^{-1}_{\frac{t^{\rho}}{\rho}}\left\{x_1\left(\frac{s^{\delta-(\alpha+\beta)-2}}{s^{\delta-(\alpha+\beta)}-A}\right)\right\}+\mathcal{L}^{-1}_{\frac{t^{\rho}}{\rho}}\left\{\frac{\mathcal{L}_{\frac{t^{\rho}}{\rho}}\{Bx(t)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(t)\}(s)\frac{1}{s^{(\alpha+\beta)}}}{s^{\delta-(\alpha+\beta)}-A}\right\},$$

By using lemma (2.10) we get

$$x(t)=x_0E_{\delta-(\alpha+\beta)}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)-Ax_0\left(\frac{t^{\rho\delta-(\alpha+\beta)}}{\rho^{\delta-(\alpha+\beta)}}\right)E_{\delta-(\alpha+\beta),\delta-(\alpha+\beta)+1}\left(A\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right)+x_1E_{\delta-(\alpha+\beta),2}\left(A\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right)$$
$$+\int_0^t\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}E_{\delta-(\alpha+\beta),\delta}\,A\left(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\left(Bx(\theta)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(\theta)\right)d\theta \quad (32)$$

Now taking norm function of solution equation (32), then

$$\|x(t)\|=\|x_0\|\left\|E_{\delta-(\alpha+\beta)}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|-\|A\|\|x_0\|\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|\,||E_{\delta-(\alpha+\beta),\delta-(\alpha+\beta)+1}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)||+\|x_1\|$$
$$\left\|E_{\delta-(\alpha+\beta),2}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|+\int_0^t\left|\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}\right|\,||E_{(\delta-(\alpha+\beta),\delta}\,A\left(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)||\left\|\left(Bx(\theta)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(\theta)\right)\right\|d\theta. \quad (33)$$

from lemma (3.5), equation (33) implies that,

$$\|x(t)\|=\|x_0\|Ke^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}-\|A\|\|x_0\|\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|Ke^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}+\|x_1\|\,Ke^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}+$$
$$\int_0^t\left|\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}\right|\,||\,Ke^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}\left\|\left(Bx(\theta)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(\theta)\right)\right\|d\theta. \quad (34)$$

Multiply equation (34) by $e^{\eta(\frac{t^{\rho}}{\rho})}$, implies that,

$$e^{\eta(\frac{t^{\rho}}{\rho})}\|x(t)\|=K\left[\|x_0\|+\|A\|\|x_0\|\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|+\|x_1\|\right]+\int_0^t\left|\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}\right|Ke^{\eta s}$$
$$\left\|\left(Bx(\theta)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(\theta)\right)\right\|d\theta \quad (35)$$

From lemma (3.3), put $b(t)=e^{\eta(\frac{t^{\rho}}{\rho})}||x(t)||$ and $c(t)=K[||x_0||+||A||\,||x_0||\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|+||x_1||]$ and $d(t)=\left\|\left(Bx(\theta)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(\theta)\right)\right\|K$, from the Remark (3.4), the equation (35), become

$$b(t)\le c(t)E_{(\delta-(\alpha+\beta)}\left(d(t)\Gamma((\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right)\le K[||x_0||+||A||\,||x_0||\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|+||x_1||]\,E_{\delta-(\alpha+\beta)}$$
$$\left(\left\|\left(Bx(\theta)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(\theta)\right)\right\|K\Gamma(\beta1-\beta2)\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right) \quad (36)$$

therefore, $||x(t)||\le Ke^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}[\|x_0\|+\|A\|\|x_0\|\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|+||x_1||]E_{\delta-(\alpha+\beta)}\left(\left\|\left(Bx(\theta)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(\theta)\right)\right\|K\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right)$, From definition (2.3), we have that $\|x_0\|\le\delta$, $\|x_1\|\le\delta$, thus

$$||x(t)||\le K\delta e^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}[1+||A||\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|+|\left(\frac{t^{\rho}}{\rho}\right)|]\,E_{\delta-(\alpha+\beta)}\left((\|B\|+\dot{w}(\theta))\,K\Gamma(\beta_1-\beta_2)\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right)$$

Hence. $||x(t)||\le\varepsilon$, for all $t\in[0,T)$. then system (29), (30) is finite time stable, for the interval $0<\delta-(\alpha+\beta)<1$.

Corollary 3.9: System(29) is finite time stable for $1\le\delta-(\alpha+\beta)<2$, if

$$e^{-\eta\left(\frac{t^{\rho}}{\rho}\right)}[1+||A||\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)+\left(\frac{t^{\rho}}{\rho}\right)]E_{\delta-(\alpha+\beta)}((\|B\|+\dot{w}(t))\,\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right))<\frac{\varepsilon}{\mu} \quad (37)$$

Proof: Now taking norm function of solution equation (32), then

$$\|x(t)\|=\|x_0\|\left\|E_{\delta-(\alpha+\beta)}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|-\|A\|\|x_0\|\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|\,||E_{\delta-(\alpha+\beta),\delta-(\alpha+\beta)+1}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)||+\|x_1\|$$
$$\left\|E_{\delta-(\alpha+\beta),2}\left(A\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\|+\int_0^t\left|\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}\right|\,||E_{(\delta-(\alpha+\beta),\delta}\,A\left(\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)||\left\|\left(Bx(\theta)+{}^{RK}_{0}I^{\eta,\rho}_{t}x(\theta)\right)\right\|d\theta. \quad (38)$$

From lemma (3.5), equation (38) implies that,

$$\|x(t)\| = \|x_0\| e^{-\eta\left(\frac{t^\rho}{\rho}\right)} - \|A\|\|x_0\| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| e^{-\eta\left(\frac{t^\rho}{\rho}\right)} + \|x_1\| e^{-\eta\left(\frac{t^\rho}{\rho}\right)} +$$

$$\int_0^t \left|\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}\right| e^{-\eta\left(\frac{t^\rho}{\rho}\right)} \left\|\left(Bx(\theta) + {}_0^{RK}I_t^{\eta,\rho} x(\theta)\right)\right\| d\theta . \quad (39)$$

Multiply equation (39) by $e^{\eta\left(\frac{t^\rho}{\rho}\right)}$, implies that

$$e^{\eta\left(\frac{t^\rho}{\rho}\right)}\|x(t)\| = \|x_0\| - \|A\|\|x_0\| \left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + \|x_1\| + \int_0^t \left|\frac{(t-\theta)^{\rho(\delta-(\alpha+\beta)-1)}}{\rho^{\delta-1}}\right| e^{\eta\theta}$$

$$\left\|\left(Bx(\theta) + {}_0^{RK}I_t^{\eta,\rho} x(\theta)\right)\right\| d\theta . \quad (40)$$

From lemma (3.3), put $b(t) = e^{\eta\left(\frac{t^\rho}{\rho}\right)}||x(t)||$ and $c(t) = [||x_0|| - ||A||\,||x_0||\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + ||x_1||]$ and $d(t) = \left\|\left(Bx(\theta) + {}_0^{RK}I_t^{\eta,\rho} x(\theta)\right)\right\|$, from Remark (3.4), the equation (40), become

$$b(t) \le c(t)E_{\delta-(\alpha+\beta)}\left(d(t)\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right) \le [||x_0|| + ||A||\,||x_0||\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + ||x_1||]$$

$$E_{\beta_1-\beta_2}\left(\left\|\left(Bx(\theta) + {}_0^{RK}I_t^{\eta,\rho} x(\theta)\right)\right\|\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right) \quad (41)$$

Therefore, $||x(t)|| \le e^{-\eta\left(\frac{t^\rho}{\rho}\right)}[||x_0|| + ||A||\,||x_0||\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{(\delta-(\alpha+\beta)}}\right)\right\| + ||x_1||]\ E_{\delta-(\alpha+\beta)}\left(\left\|\left(Bx(\theta) + {}_0^{RK}I_t^{\eta,\rho} x(\theta)\right)\right\|\Gamma(\delta -\right.$

$$\left.(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right) \quad (42)$$

From definition (2.3), we have that $||x_0|| \le \delta$, $||x_1|| \le \delta$,thus,

$$||x(t)|| \le \delta e^{-\eta\left(\frac{t^\rho}{\rho}\right)}[1 + ||A||\left\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right\| + |\left(\frac{t^\rho}{\rho}\right)|]E_{(\delta-(\alpha+\beta)}\left((\|B\| + \dot{w}(\theta))\,\Gamma((\delta-(\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right)\right)$$

Hence, $||x(t)|| \le \varepsilon$, for all $t\epsilon[0,T)$ Then system (29), (30) is finite time stable, for the interval $1 \le \delta - (\alpha+\beta) < 2$.

Example 3.10: Consider the R−C nonlinear fractional integro−differential system

$$\begin{cases} {}_0^{CK}D_t^{\delta,\rho}x_1(t) - {}_0^{CK}D_t^{(\alpha+\beta),\rho}x_2(t) = (x_1^2+2.9)^{\frac{1}{2}} + 0.05\ \tanh(x) + {}_0^{RK}I_t^{\beta_3,\rho}x_1(t) & (43)\\ {}_0^{CK}D_t^{\delta,\rho}x_2(t) - {}_0^{CK}D_t^{(\alpha+\beta),\rho}x_1(t) = 0 & (44)\end{cases}$$

Where $\delta = 1.75$ and$\alpha = 0.5, \beta = 0.75$, $(\delta-(\alpha+\beta) = 1.25$,by using consider system (10), for (43), (44), yield $A = \begin{bmatrix}1 & 0\\ 0 & 0.75\end{bmatrix}$ and $g(t, x(t)), {}_0^{RK}I_t^{\beta_3,\rho}x(t) = \begin{bmatrix}(x_1^2+2.9)^{\frac{1}{2}} + 0.05\ \tanh(x) + {}_0^{RK}I_t^{\beta_3,\rho}x_1(t)\\ 0\end{bmatrix}$Now to proof that, $g(t, x(t)), {}_0^{RK}I_t^{\beta_3,\rho}x(t) = \begin{bmatrix}(x_1^2+2.9)^{\frac{1}{2}} + 0.05\ \tanh(x) + {}_0^{RK}I_t^{\beta_3,\rho}x_1(t)\\ 0\end{bmatrix}$ satisfies the Lipchitz condition, ménage there exists a constant L such that:

$g(t, x(t)), {}_0^{RK}I_t^{\beta_3,\rho}x(t) - g(t, y(t), {}_0^{RK}I_t^{\beta_3,\rho}y(t))| \le (w + \grave{w}(t))|x(t) - y(t)|$ for all $x(.), y(.) \in R^+$.

Since $(x_1^2+2.9)^{\frac{1}{2}} + 0.05\ \tanh(x)$ has derivative $\frac{x}{(x_1^2+2.9)^{\frac{1}{2}}} + 0.05\ \mathrm{sech}^2(x)$

$\frac{|x_1|}{(x_1^2+2.9)^{\frac{1}{2}}+0.05\ \tanh(x)} \le \frac{|x_1|}{(x_1^2+2.9)^{\frac{1}{2}}+0.05\ \tanh(x)} \le |x_1|$, by assumption(3.1), we obtain $|g(t, x(t), {}_0^{RK}I_t^{\beta_3,\rho}x(t)| \le (1 + \grave{w}(t))\|x(t)\|$ for $t \in I, x \in R^2$

Now to compute the condition (11) w.r.t M =1 and $\dot{w}(t)$ =1.05, η= 1.5,1.6,1.7,1.8,1.9 and by definition (2.1) we find $||A|| = 1$ and choose $\mu = 0.2,0.1,0.01,0.4,0.5$ and $K = 1,1.2,1.3,1.4,1.5$,$\varepsilon = 4,3,4,2,1.5$, ρ = 1, then

$$Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)}\left[1 + \|A\|\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) + \left(\frac{t^\rho}{\rho}\right)\right]E_{\delta-(\alpha+\beta)}(K(M + \dot{w}(t)))\Gamma(\delta-(\alpha+\beta))\left(\frac{t^{(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) < \frac{\varepsilon}{\mu}$$

Hence the estimated finite time stable ofR − K nonlinear fractional integro- differential system T ≈ 0.4495,0.0723,1.4379,2.8075,0.0178

Table (1) The value of Finite time stability T, for $\delta = 0.05, \varepsilon = 1$

$\beta_1 =$	$\beta_2 =$	$\varepsilon =$	$\delta =$	$K =$	$\rho =$	$\eta =$	$M =$	$T =$
2.75	2.25	1	0.05	1.74	1	0.33	1	0.4495
2.75	2.25	1	0.05	1.20	1	1.44	1	0.0723
2.75	2.25	1	0.05	1.35	1	1.9	1	1.4379
2.75	2.25	1	0.05	0.88	1	1.63	1	2.8075
2.75	2.25	1	0.05	0.99	1	1.55	1	0.0178

Example 3.11: Consider the system (29) with the parameters $\delta = 1.75, \alpha = 0.75$, $\beta = 0.50$ $(\delta - (\alpha + \beta) = 0.5$, $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, be choosing many value of $K = 1,1.2,1.3,1.4,1.5$ and , $\varepsilon = 1$ and $\rho = 1$, and $\delta = 1.75$, and , $\eta = 0.2, 0.1, 0.01, 0.4, 0.5$, and $\mu = 0.05$ and by definition (2.2) we find $||A|| = 3.7417$ and by definition (2.2) we find $||B|| = 2$. With $Ke^{-\eta\left(\frac{t^\rho}{\rho}\right)}\left[1 + ||A||\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) + \left(\frac{t^\rho}{\rho}\right)\right] E_{\delta-(\alpha+\beta)}((\|B\| + \dot{w}(t))\ K\ \Gamma(\delta - (\alpha+\beta))\left(\frac{t^{\rho(\delta-(\alpha+\beta))}}{\rho^{\delta-(\alpha+\beta)}}\right) < \frac{\varepsilon}{\mu}$, with $\dot{w}(t) = \frac{1}{(\alpha+\beta)\delta\Gamma(\delta)}\left(\frac{t^{(\alpha+\beta)} - a^{(\alpha+\beta)}}{(\alpha+\beta)}\right)$ the estimated time stable of (29), (30) are $T \approx 0.6961, 0.6362, 1.3200, 0.1219, 0.0373$

Table (2) The value of Finite time stability T, for $\delta = 1.75, \varepsilon = 1$

$\alpha + \beta =$	$\alpha =$	$\eta =$	$\mu =$	$K =$	$\varepsilon =$	$\beta =$	$\delta =$	$T =$
1.25	0.75	0.2	0.05	1	1	0.5	1.75	0.6961
1.25	0.75	0.1	0.05	1.2	1	0.5	1.75	0.6362
1.25	0.75	0.01	0.05	1.3	1	0.5	1.75	1.3200
1.25	0.75	0.4	0.05	1.4	1	0.5	1.75	0.1219
1.25	0.75	0.5	0.05	1.5	1	0.5	1.75	0.0373

4. CONCLUSION

We concluded that each fractional derivative has a special function that is substituted into the generalized formula of Laplace to find the transformations of the functions required in calculating the results of the given problem. Also, the Grönwall's inequality with Mittag Leffler functions made a good role for compute the conditions of time stability. The time stability depended on the type of derivatives of the nonlinear fractional system and concepts of nonlinear functional analysis. The time stability of this problem of composite fractional derivative depended on a value of fractional orders of derivate and integrals

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