

A Modified Semi-Analytical Method Using the Shehu Transform for Solving Fractional Volterra Integro-differential Equations

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ABSTRACT: *Background:* Finding an analytical solution to Volterra integro-differential equations (VIDEs), especially nonlinear types, often poses serious difficulties and is many times impossible, thus the need to provide a semi-analytical solution. *Objective:* This research focuses on the solutions of systems of linear and nonlinear fractional-order integro-differential equations with difference kernels. To achieve this, we exploited the advantage of integral transforms and one of the existing semi-analytical methods to develop the desired method of solution. *Methods:* One of the recently developed integral transforms, the Shehu transform, which generalizes Laplace and Sumudu transforms, is systematically integrated into the well-known Adomian Decomposition Method (ADM) to obtain a simplified approach to solving the class of problems considered. The Shehu transform is first applied to both sides of the given VIDEs with difference kernels, followed by the application of the convolution theorem. The ADM is then employed to handle the nonlinearities encountered. *Results:* The proposed method, the Modified Semi-analytical Method (MSM), is applied to selected problems in the literature and produces comparatively good results. The method also produces the exact solution whenever the solution is in closed form. The results are presented in tabular and 2D or 3D graphical forms for easy comparison. All computations are carried out using Mathematica 13.3, with the fractional-order derivative interpreted in the Caputo sense. *Conclusions:* Since MSM has been successfully used to solve linear and nonlinear VIDEs with difference kernels, the scope of the method can be expanded to cover Volterra-Fredholm integro-differential equations (VFIDEs) in future studies.

KEYWORDS: Shehu transform; Integro-differential equations; Fractional derivatives; Difference kernel; Adomian polynomials

INTRODUCTION

Integro-differential equations (IDEs) are the ones that consist of both integral and differential operators, which makes them an excellent tool for modeling real-life systems [1]. These equations are mostly used in various disciplines of science and engineering. Despite their usefulness, solving the nonlinear IDEs is often difficult using the known analytical methods. Volterra integro-differential equations (VIDEs) a class of IDEs, play a crucial role due to their ability to capture memory effects and hereditary properties [2], [3]. The VIDEs are highly relevant in practical applications in viscoelastic materials, population dynamics, heat conduction, and electrochemical processes [4]. Recent developments in research have underscored the relevance of VIDEs in modeling dynamic systems. A typical example is the work of [5] which analyzed the stability of VIDEs in control and biological models, demonstrating their effectiveness in handling impulsive effects and delay dynamics. Likewise, the work reported by [6] where the roles of VIDEs are explored in delay systems, emphasizing their applicability in engineering and theoretical sciences. Furthermore, Liu, Tao, and Zhang [7] proposed a spectral method to solve nonlinear VIDEs with weakly singular kernels, improving computational

efficiency and accuracy. All the foregoing studies highlight the growing significance of VIDEs in both theoretical research and practical problem-solving across various disciplines.

Fractional calculus provides the desired requirements in modeling practical phenomenon where precision is of essence [8]. The application of non-integer-order VIDEs enhances model accuracy in fields such as control theory and signal processing [9]. Expansion of the scope of VIDEs to fractional orders enhances the scope of their applications, since fractional calculus accurately describes real-world situations such as diffusion effects and so on [10]. Fractional order VIDEs provide the desired flexibility in capturing real-life phenomena compared to their integer-order counterparts, making them suitable for modeling processes in engineering, biomathematics, mathematical physics, and fluid dynamics [11], [12]. Several studies, including those by Wang [13] and Boulaaras, Jan, and Pham [14], [15], [16], [17], [18], have highlighted the advantages of fractional calculus in describing complex dynamical behaviors that classical integer-order models cannot effectively capture. More recent works by Guo, Yin, and Peng [19], as well as Alshammari, Iqbal, and Ntwiga [20], have demonstrated the increasing importance of fractional-order models in applications such as diffusion processes, viscoelasticity, and bioengineering, reinforcing the need for efficient solution methodologies.

Despite their extensive applications, solving FVIDEs remains a significant challenge due to memory effects and non-local dependencies [21], [22]. Many researchers have attempted to derive analytical or numerical solutions, though exact solutions remain scarce [23]–[25]. Several semi-analytical methods, including the ADM [26], [27], HPM, VIM, and HAM, have been widely applied, with studies specifically utilizing the VIM [28] and HAM [29], [30]. On the numerical front, researchers have explored methods based on the use operational matrices for nonlinear VIDEs [31] and the multi-wavelet Galerkin method, which employs operational matrices of integration and wavelet transforms [32]. While these methods have shown considerable effectiveness, they often face challenges such as slow convergence, computational instability, and difficulties in handling strong nonlinearities. Moreover, existing approaches frequently require extensive computational resources for higher-order nonlinear problems, underscoring the need for more efficient solution techniques.

The need for more easily accessible methods that are devoid of ambiguities that are inherent in some of the existing methods, such as small parameters to be determined along with solution to original problem. The present work therefore proposed Modified Semi-analytical Methods (MSM) which exploits the advantage in Shehu transform which is the fact that it handles both constant and variable coefficients problems, and the Adomian Decomposition Method. The essence of incorporating ADM is to overcome any nonlinear terms that may be encountered in the course of solving the problem. The proposed method has been successfully applied to both linear and nonlinear integro-differential equations with difference kernels as presented in the sequel.

Existing methods like VIM, HAM, HPM, PM, etc have one limitation or the other. For instance the success in the application of VIM rests with getting an appropriate correction functional, convergence of solution when using HAM is dependent on identifying correct value for the convergence control parameter in its algorithm, both HPM and PM required the use of a small positive parameter that will either come with the nonlinear problem or introduce in order to apply the method, and that increase the computation time and volume. The method being proposed here is free from all those limitations.

In this paper, since the Shehu transform as an integral transform like the earlier ones such as Fourier, Laplace, Sumudu, etc can not handle nonlinear problems, although it has the advantage of being able to solve variable coefficients problems which they can not solve. That necessitated integrating it with ADM which readily decomposes nonlinear terms into polynomials that Shehu transform then act on. The novelty in the present method is the fact that it reduces the computational volume and time, while producing exact solution when such exists in closed form, or gives truncated series with impressive degree of accuracy. It is a common knowledge that integral transforms generally give exact solutions due to the fact that they are analytical methods, integrating such transforms with a semi-analytical method such as ADM will automatically give improved approach to solution since no small parameter is required, nor linearization.

Adomian Polynomial

The Adomian polynomials, denoted by A_q , are used to decompose the nonlinear terms encountered in problems. It is given by

$$A_q = \frac{1}{\Gamma(q+1)} \left[\frac{d^q}{d\xi^q} N \left(\sum_{m=0}^q \xi^m u_m \right) \right]_{\xi=0}, \quad q = 0, 1, 2, \dots \quad (1)$$

For further reading on ADM and its applications see [33]–[35].

The Caputo Fractional Derivative

Let $\psi \in \mathfrak{R}_+$ and $\vartheta = \lceil \psi \rceil$. The operator \mathbb{D}_c^ψ is defined by

$$\mathbb{D}_c^\psi f(s) = J_c^{\vartheta-\psi} D^\vartheta f(s) = \frac{1}{(\vartheta - \psi - 1)!} \int_c^s (s - \rho)^{\vartheta-\psi-1} \left(\frac{d}{d\rho} \right)^\psi f(\rho) d\rho, \quad (2)$$

for $c \leq s \leq d$, is called the Caputo differential operator of order ψ . The definition (2) can equally be expressed as

$$D^\psi s^\vartheta = \frac{\Gamma(\vartheta + 1)}{\Gamma(\vartheta - \psi + 1)} s^{\vartheta-\psi}. \quad (3)$$

Shehu Transform

Definition

The Shehu transform of a function $g(t)$ which is of an exponential order is defined as [36]

$$\begin{aligned} \mathbb{S}[g(t)] &= \int_0^\infty \exp\left(\frac{-s}{u}t\right) g(t) dt = G(s, u). \\ \mathbb{S}[g(t)] &= \lim_{\eta \rightarrow \infty} \int_0^\eta \exp\left(\frac{-s}{u}t\right) g(t) dt; \quad s > 0, u > 0. \end{aligned} \quad (4)$$

If the limit of the integral in (4) exists, it converges, otherwise it diverges. The inverse of Shehu transform is given by

$$\mathbb{S}^{-1}[G(s, u)] = g(t), \quad \text{for } t \geq 0., \quad (5)$$

which can as well be stated as

$$v(x) = \mathbb{S}^{-1}[G(s, u)] = \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} \frac{1}{u} \exp\left(\frac{st}{u}\right) G(s, u) ds, \quad (6)$$

where s and u are the Shehu transform variables, and η is a real constant and the integral in (1) is taken along $s = \eta$ in the complex plane $s = x + iy$.

The present study explored the advantage of Shehu transform over some other transforms such as Laplace, due to the fact that it does not have restriction in application to constant coefficients problems. Another superiority of the transform adopted in this work is that it generalizes both Laplace and Sumudu transforms [34], [36].

MATERIALS AND METHODS

Application of Shehu Transform to Differential Coefficients

The Shehu transform of the differential coefficient $g'(s)$ is defined as [36]

$$\mathbb{S}\{g'(s)\} = \left(\frac{\psi}{\vartheta}\right) G(\psi, \vartheta) - f(0). \quad (7)$$

The n^{th} order derivative is given by

$$\mathbb{S}\{g^{(n)}(s)\} = \left(\frac{\psi}{\vartheta}\right)^\rho G(\psi, \vartheta) - \sum_{i=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(i+1)} g^{(i)}(0), \quad (8)$$

where $G(\psi, \vartheta)$ is the Shehu transform of the function $g(s)$.

Nature of the Problem Considered

In this work, the family of problem considered is

$$\mathbb{D}^\rho u(t) = g(t) + \mu u(t) + \int_0^t [g(\tau)u(\tau) + h(\tau)F(u(\tau))] d\tau, \quad 0 \leq t \leq b \tag{9}$$

where ρ is a fractional order derivative, $u(t)$ is the unknown function, $g(\tau)$ and $h(\tau)$ are degenerate kernels which in this study are taken to be difference kernel, $F(u(\tau))$ is the nonlinear term.

Implementation of the Modified Semi-analytical Method (MSM)

We shall apply the Shehu Transform to both sides of (9) as follows:

$$\mathbb{S}\{\mathbb{D}^\rho u(x)\} = \mathbb{S}\left\{g(t) + \mu u(t) + \int_0^t [g(\tau)u(\tau) + h(\tau)F(u(\tau))] d\tau\right\}. \tag{10}$$

Applying Shehu transform of derivatives to the term on the left hand side of (10), we have

$$\mathbb{S}\{\mathbb{D}^\rho u(t)\} = \frac{\psi^\rho}{\vartheta^\rho} U(\psi, \vartheta) - \sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) \tag{11}$$

We then substitute (11) back into (10) to get

$$\left(\frac{\psi}{\vartheta}\right)^\rho U(\psi, \vartheta) - \sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) = G(\psi, \vartheta) + \mu \mathbb{S}\{u(t)\} + \mathbb{S}\left\{\int_0^t k_1(t-\tau)u(\tau)d\tau + k_2(t-\tau)u(\tau)d\tau\right\}, \tag{12}$$

where $k_1(t - \tau)$ and $k_2(t - \tau)$ are difference kernel representing $g(\tau)$ and $h(\tau)$ respectively.

$$\begin{aligned} \left(\frac{\psi}{\vartheta}\right)^\rho U(\psi, \vartheta) &= \sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) + G(\psi, \vartheta) + \mu \mathbb{S}\{u(t)\} \\ &+ \mathbb{S}\{k_1(t - \tau)\} * \mathbb{S}\{u(s)\} + \mathbb{S}\{k_2(t - \tau)\} * \mathbb{S}\{F(u(\tau))\}. \end{aligned} \tag{13}$$

where $F(u(\tau))$ is the nonlinear term.

$$\begin{aligned} U(\psi, \vartheta) &= \frac{\vartheta^\rho}{\psi^\rho} \left(\sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) + G(\psi, \vartheta) + \mu \mathbb{S}\{u(t)\} \right. \\ &\left. + \mathbb{S}\{k_1(t - \tau)\} * \mathbb{S}\{u(s)\} + \mathbb{S}\{k_2(t - \tau)\} * \mathbb{S}\{F(u(\tau))\} \right). \end{aligned} \tag{14}$$

The initial approximation $u_0(t)$ is obtained from (14) as

$$\mathbb{S}\{u_0(t)\} = \frac{\vartheta^\rho}{\psi^\rho} \left[\sum_{j=0}^{\rho-1} \left(\frac{\psi}{\vartheta}\right)^{\rho-(j+1)} u^{(j)}(0) + G(\psi, \vartheta) + \mu \mathbb{S}\{u(t)\} + \mathbb{S}\{k_1(t - \tau)\} * \mathbb{S}\{u(s)\} \right], \tag{15}$$

while the recurrence relation is given by the remaining terms in (14) as

$$\mathbb{S}[u_{k+1}(t)] = \frac{\vartheta^\rho}{\psi^\rho} \left[\mathbb{S}\{k_2(t - \tau)\} * \mathbb{S}\left\{\sum_{k=0}^{\infty} A_k(s)\right\} \right], \quad k = 0, 1, 2, \dots \tag{16}$$

The Adomian polynomials corresponding to the nonlinear term in (1) are derived, substituted, and used as follows

$$F(u(t)) = \sum_{k=0}^{\infty} A_k(t), \tag{17}$$

where $A_k(t)$ represents the Adomian polynomials.

Numerical Experiment on Linear Problems

Problem 1

Consider the system of linear VIDEs [37]

$$\begin{aligned} \mathbb{D}_t^\zeta y_1(t) - 2t^2 - \int_0^t ((t - \tau)y_1(t) + (t - \tau)y_2(\tau)) d\tau &= 0, \\ \mathbb{D}_t^\zeta y_2(t) + 3t^2 + \frac{1}{5}t^5 - \int_0^t ((t - \tau)y_1(\tau) + (t - \tau)y_2(\tau)) d\tau &= 0. \end{aligned} \tag{18}$$

with the given initial conditions: $y_1(0) = y_2(0) = 1, \quad 0 < \zeta \leq 1.$

Solution to Problem 1:

$$\begin{aligned} \mathbb{D}_t^\zeta y_1(t) &= 2t^2 + \int_0^t ((t - \tau)y_1(\tau) + (t - \tau)y_2(\tau)) d\tau, \\ \mathbb{D}_t^\zeta y_2(t) &= -3t^2 - \frac{1}{5}t^5 + \int_0^t ((t - \tau)y_1(\tau) + (t - \tau)y_2(\tau)) d\tau. \end{aligned} \tag{19}$$

Taking the Shehu transform of both sides, we get

$$\begin{aligned} \mathbb{S}\{\mathbb{D}_t^\zeta y_1(t)\} &= 2\mathbb{S}\{t^2\} + \mathbb{S}\left\{\int_0^t ((t - \tau)y_1(\tau) + (t - \tau)y_2(\tau)) d\tau\right\}, \\ \mathbb{S}\{\mathbb{D}_t^\zeta y_2(t)\} &= -3\mathbb{S}\{t^2\} - \frac{1}{5}\mathbb{S}\{t^5\} + \mathbb{S}\left\{\int_0^t ((t - \tau)y_1(\tau) + (t - \tau)y_2(\tau)) d\tau\right\}. \end{aligned} \tag{20}$$

Implementing the Shehu transform of the derivatives and simplifying, we get

$$\begin{aligned} \left(\frac{\psi}{\vartheta}\right)^\zeta Y_1(\psi, \vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} y_1(0) &= 4\left(\frac{\vartheta}{\psi}\right)^3 + \mathbb{S}\{t\} * \mathbb{S}\{y_1(t)\} + \mathbb{S}\{t\} * \mathbb{S}\{y_2(t)\}, \\ \left(\frac{\psi}{\vartheta}\right)^\zeta Y_2(\psi, \vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} y_2(0) &= -6\left(\frac{\vartheta}{\psi}\right)^3 - 24\left(\frac{\vartheta}{\psi}\right)^6 + \mathbb{S}\{t\} * \mathbb{S}\{y_1(t)\} - \mathbb{S}\{t\} * \mathbb{S}\{y_2(t)\}. \end{aligned} \tag{21}$$

Applying initial conditions and simplification gives

$$\begin{aligned} \left(\frac{\psi}{\vartheta}\right)^\zeta Y_1(\psi, \vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} &= 4\left(\frac{\vartheta}{\psi}\right)^3 + \left(\frac{\vartheta}{\psi}\right)^2 Y_1(\psi, \vartheta) + \left(\frac{\vartheta}{\psi}\right)^2 Y_2(\psi, \vartheta), \\ \left(\frac{\psi}{\vartheta}\right)^\zeta Y_2(\psi, \vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} &= -6\left(\frac{\vartheta}{\psi}\right)^3 - 24\left(\frac{\vartheta}{\psi}\right)^6 + \left(\frac{\vartheta}{\psi}\right)^2 Y_1(\psi, \vartheta) - \left(\frac{\vartheta}{\psi}\right)^2 Y_2(\psi, \vartheta). \end{aligned} \tag{22}$$

$$\begin{aligned} \left(\frac{\psi}{\vartheta}\right)^\zeta Y_1(\psi, \vartheta) &= \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} + 4\left(\frac{\vartheta}{\psi}\right)^3 + \left(\frac{\vartheta}{\psi}\right)^2 Y_1(\psi, \vartheta) + \left(\frac{\vartheta}{\psi}\right)^2 Y_2(\psi, \vartheta), \\ \left(\frac{\psi}{\vartheta}\right)^\zeta Y_2(\psi, \vartheta) &= \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} - 6\left(\frac{\vartheta}{\psi}\right)^3 - 24\left(\frac{\vartheta}{\psi}\right)^6 + \left(\frac{\vartheta}{\psi}\right)^2 Y_1(\psi, \vartheta) - \left(\frac{\vartheta}{\psi}\right)^2 Y_2(\psi, \vartheta). \end{aligned} \tag{23}$$

$$\begin{aligned} Y_1(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right) + 4\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_1(\psi, \vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_2(\psi, \vartheta), \\ Y_2(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right) - 6\left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 24\left(\frac{\vartheta}{\psi}\right)^{6+\zeta} + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_1(\psi, \vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_2(\psi, \vartheta). \end{aligned} \tag{24}$$

From where we get the initial approximations as:

$$\begin{aligned} Y_{1,0}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right) + 4 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta}, \\ Y_{2,0}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right) - 6 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{6+\zeta}. \end{aligned} \quad (25)$$

Taking the inverse Shehu transform, we obtain

$$\begin{aligned} y_{1,0}(t) &= 1 + \frac{4t^{2+\zeta}}{\Gamma(3+\zeta)}, \\ y_{2,0}(t) &= 1 - \frac{6t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{24t^{5+\zeta}}{\Gamma(6+\zeta)}. \end{aligned} \quad (26)$$

For the recurrence relation, we have

$$\begin{aligned} Y_{1,k+1}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,k}(\psi, \vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,k}(\psi, \vartheta), \\ Y_{2,k+1}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,k}(\psi, \vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,k}(\psi, \vartheta). \end{aligned} \quad (27)$$

When $k = 0$:

$$\begin{aligned} Y_{1,1}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,0}(\psi, \vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,0}(\psi, \vartheta), \\ Y_{2,1}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,0}(\psi, \vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,0}(\psi, \vartheta). \end{aligned} \quad (28)$$

$$\begin{aligned} Y_{1,1}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} \left[\left(\frac{\vartheta}{\psi}\right) + 4 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} \right] + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} \left[\left(\frac{\vartheta}{\psi}\right) - 6 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{6+\zeta} \right], \\ Y_{2,1}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} \left[\left(\frac{\vartheta}{\psi}\right) + 4 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} \right] - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} \left[\left(\frac{\vartheta}{\psi}\right) - 6 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{6+\zeta} \right]. \end{aligned} \quad (29)$$

$$\begin{aligned} Y_{1,1}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} + 4 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 6 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}, \\ Y_{2,1}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} + 4 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} + 6 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 24 \left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}. \end{aligned} \quad (30)$$

$$\begin{aligned} Y_{1,1}(\psi, \vartheta) &= 2 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 2 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}, \\ Y_{2,1}(\psi, \vartheta) &= 10 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 24 \left(\frac{\vartheta}{\psi}\right)^{8+2\zeta}. \end{aligned} \quad (31)$$

Taking the inverse Shehu transform of both sides, we get

$$\begin{aligned} y_{1,1}(t) &= \frac{2t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} - \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)}, \\ y_{2,1}(t) &= \frac{10t^{4+2\zeta}}{\Gamma(5+2\zeta)} + \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)}. \end{aligned} \quad (32)$$

When $k = 1$:

$$\begin{aligned}
 Y_{1,2}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,1}(\psi, \vartheta) + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,1}(\psi, \vartheta), \\
 Y_{2,2}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{1,1}(\psi, \vartheta) - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} Y_{2,1}(\psi, \vartheta).
 \end{aligned}
 \tag{33}$$

$$\begin{aligned}
 Y_{1,2}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} \left[2 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 2 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{8+2\zeta} \right] \\
 &\quad + \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} \left[10 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 24 \left(\frac{\vartheta}{\psi}\right)^{8+2\zeta} \right], \\
 Y_{2,2}(\psi, \vartheta) &= \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} \left[2 \left(\frac{\vartheta}{\psi}\right)^{3+\zeta} - 2 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{8+2\zeta} \right] \\
 &\quad - \left(\frac{\vartheta}{\psi}\right)^{2+\zeta} \left[10 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 24 \left(\frac{\vartheta}{\psi}\right)^{8+2\zeta} \right].
 \end{aligned}
 \tag{34}$$

$$\begin{aligned}
 Y_{1,2}(\psi, \vartheta) &= 2 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 2 \left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{10+3\zeta} + 10 \left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} + 24 \left(\frac{\vartheta}{\psi}\right)^{10+3\zeta}, \\
 Y_{2,2}(\psi, \vartheta) &= 2 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 2 \left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{10+3\zeta} - 10 \left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} - 24 \left(\frac{\vartheta}{\psi}\right)^{10+3\zeta}.
 \end{aligned}
 \tag{35}$$

$$\begin{aligned}
 Y_{1,2}(\psi, \vartheta) &= 2 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} + 8 \left(\frac{\vartheta}{\psi}\right)^{7+3\zeta}, \\
 Y_{2,2}(\psi, \vartheta) &= 2 \left(\frac{\vartheta}{\psi}\right)^{5+2\zeta} - 12 \left(\frac{\vartheta}{\psi}\right)^{7+3\zeta} - 48 \left(\frac{\vartheta}{\psi}\right)^{10+3\zeta}.
 \end{aligned}
 \tag{36}$$

Taking the inverse Shehu transform of both sides, we get

$$\begin{aligned}
 y_{1,2}(\zeta) &= \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} + \frac{8t^{6+3\zeta}}{\Gamma(7+3\zeta)}, \\
 y_{2,2}(\zeta) &= \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} - \frac{12t^{6+3\zeta}}{\Gamma(7+3\zeta)} - \frac{48t^{9+3\zeta}}{\Gamma(10+3\zeta)}.
 \end{aligned}
 \tag{37}$$

The general solutions are obtained as:

$$\begin{aligned}
 y_1(t) &= y_{1,0}(t) + y_{1,1}(t) + y_{1,2}(t) + \dots, \quad k = 0, 1, 2, \dots, \\
 y_2(t) &= y_{2,0}(t) + y_{2,1}(t) + y_{2,2}(t) + \dots, \quad k = 0, 1, 2, \dots
 \end{aligned}
 \tag{38}$$

$$\begin{aligned}
 y_1(t) &= 1 + 4 \frac{t^{2+\zeta}}{\Gamma(3+\zeta)} + \frac{2t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} \\
 &\quad - \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)} + \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} + \frac{8t^{6+3\zeta}}{\Gamma(7+3\zeta)} + \dots
 \end{aligned}
 \tag{39}$$

$$\begin{aligned}
 y_2(t) &= 1 - \frac{6t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{24t^{5+\zeta}}{\Gamma(6+\zeta)} + \frac{10t^{4+2\zeta}}{\Gamma(5+2\zeta)} + \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)} \\
 &\quad + \frac{2t^{4+2\zeta}}{\Gamma(5+2\zeta)} - \frac{12t^{6+3\zeta}}{\Gamma(7+3\zeta)} - \frac{48t^{9+3\zeta}}{\Gamma(10+3\zeta)} + \dots
 \end{aligned}
 \tag{40}$$

$$y_1(t) = 1 + \frac{6t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)} + \frac{8t^{6+3\zeta}}{\Gamma(7+3\zeta)} + \dots \tag{41}$$

$$y_2(t) = 1 - \frac{6t^{2+\zeta}}{\Gamma(3+\zeta)} - \frac{24t^{5+\zeta}}{\Gamma(6+\zeta)} + \frac{24t^{7+2\zeta}}{\Gamma(8+2\zeta)} + \frac{12t^{4+2\zeta}}{\Gamma(5+2\zeta)} - \frac{12t^{6+3\zeta}}{\Gamma(7+3\zeta)} - \frac{48t^{9+3\zeta}}{\Gamma(10+3\zeta)} + \dots \tag{42}$$

Problem 2

Consider the system of linear VIDEs [37]

$$\begin{aligned} \mathbb{D}_t^\zeta y_1(t) &= 1 + t - \frac{t^3}{3} + \int_0^t ((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)) d\tau, \\ \mathbb{D}_t^\zeta y_2(t) &= 1 - t - \frac{t^4}{12} + \int_0^t ((t-\tau)y_1(\tau) + (t-\tau)y_2(\tau)) d\tau. \end{aligned} \tag{43}$$

With the given initial conditions: $y_1(0) = y_2(0) = 0$, and $0 < \zeta \leq 1$.

Solution to Problem 2:

Following the procedures as in problem 1, we have

$$\begin{aligned} y_1(t) &= \frac{t^\zeta}{\Gamma(1+\zeta)} + \frac{t^{1+\zeta}}{\Gamma(2+\zeta)} - 2\frac{t^{3+\zeta}}{\Gamma(4+\zeta)} + \frac{2t^{2+2\zeta}}{\Gamma(3+2\zeta)} \\ &\quad - \frac{2t^{5+2\zeta}}{\Gamma(6+2\zeta)} - \frac{2t^{6+2\zeta}}{\Gamma(7+2\zeta)} + \frac{2t^{4+3\zeta}}{\Gamma(5+3\zeta)} \\ &\quad - \frac{4t^{7+3\zeta}}{\Gamma(8+3\zeta)} + \frac{2t^{5+3\zeta}}{\Gamma(6+3\zeta)} + \dots \end{aligned} \tag{44}$$

$$\begin{aligned} y_2(t) &= \frac{t^\zeta}{\Gamma(1+\zeta)} - \frac{t^{1+\zeta}}{\Gamma(2+\zeta)} - 2\frac{t^{4+\zeta}}{\Gamma(5+\zeta)} + \frac{2t^{3+2\zeta}}{\Gamma(4+2\zeta)} \\ &\quad - \frac{2t^{5+2\zeta}}{\Gamma(6+2\zeta)} + \frac{2t^{6+2\zeta}}{\Gamma(7+2\zeta)} + \frac{2t^{4+3\zeta}}{\Gamma(5+3\zeta)} \\ &\quad - \frac{2t^{7+3\zeta}}{\Gamma(8+3\zeta)} - \frac{4t^{8+3\zeta}}{\Gamma(9+3\zeta)} + \dots \end{aligned} \tag{45}$$

Problem 3

Consider the system of linear VIDEs [38]

$$\begin{aligned} \mathbb{D}_t^\zeta u(t) + \frac{3t^{2\zeta}\zeta\Gamma(3\zeta)}{\Gamma(1+2\zeta)} - \int_0^t (t-\tau)u(\tau) d\tau - \int_0^t (t-\tau)v(\tau) d\tau &= 0, \\ \mathbb{D}_t^\zeta v(t) + \frac{2t^{3\zeta+2}}{2+9\zeta+9\zeta^2} + \frac{3t^{2\zeta}\zeta\Gamma(3\zeta)}{\Gamma(1+2\zeta)} + \int_0^t (t-\tau)u(\tau) d\tau + \int_0^t (t-\tau)v(\tau) d\tau &= 0. \end{aligned} \tag{46}$$

With the given initial conditions: $u(0) = v(0) = 0$, $0 \leq t \leq 1$, $0 < \zeta \leq 1$.

Solution to Problem 3:

Implementation of the earlier algorithm, we have

$$\begin{aligned} u(t) &= t^{3\zeta} - \frac{2\Gamma(3\zeta+1)t^{6\zeta+3}}{\Gamma(6\zeta+4)} + \frac{2\Gamma(3\zeta+1)t^{7\zeta+2}}{\Gamma(7\zeta+3)} + \dots \\ v(t) &= -\frac{2\Gamma(3\zeta+1)t^{4\zeta+2}}{\Gamma(4\zeta+3)} - t^{3\zeta} + \frac{2\Gamma(3\zeta+1)t^{5\zeta+1}}{\Gamma(5\zeta+2)} \\ &\quad + \frac{2\Gamma(3\zeta+1)t^{6\zeta+3}}{\Gamma(6\zeta+4)} - \frac{2\Gamma(3\zeta+1)t^{8\zeta+4}}{\Gamma(8\zeta+5)} - \frac{2\Gamma(3\zeta+1)t^{7\zeta+2}}{\Gamma(7\zeta+3)} + \dots \end{aligned} \tag{47}$$

Numerical Experiment on Nonlinear Problems

Problem 4

Consider the nonlinear VIDEs [39]

$$\mathbb{D}^\zeta y(t) - \int_0^t [y(\tau)]^2 d\tau = -1, \quad 0 \leq x \leq 1, \quad 0 < \zeta \leq 1. \quad (48)$$

Subject to the initial condition $y(0) = 0$.

Solution to Problem 4:

$$\mathbb{D}^\zeta y(t) = \int_0^t [y(\tau)]^2 d\tau - 1. \quad (49)$$

Taking the Shehu transform of both sides, we get

$$\mathbb{S}\{\mathbb{D}^\zeta y(t)\} = \mathbb{S}\left\{\int_0^t [y(\tau)]^2 d\tau - 1\right\}. \quad (50)$$

$$\left(\frac{\psi}{\vartheta}\right)^\zeta Y(\psi, \vartheta) - \left(\frac{\psi}{\vartheta}\right)^{\zeta-1} y(0) = \frac{\vartheta}{\psi} * \mathbb{S}\{y(t)^2\} - \frac{\vartheta}{\psi}. \quad (51)$$

Applying the initial condition, we get

$$\left(\frac{\psi}{\vartheta}\right)^\zeta Y(\psi, \vartheta) = \frac{\vartheta}{\psi} * \mathbb{S}\{y(t)^2\} - \frac{\vartheta}{\psi} \quad (52)$$

$$Y(\psi, \vartheta) = -\left(\frac{\vartheta}{\psi}\right)^{1+\zeta} + \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} * \mathbb{S}\{y(t)^2\} \quad (53)$$

The initial approximation and the recurrence relation are obtained from (50) as follows:

$$Y_0(\psi, \vartheta) = -\left(\frac{\vartheta}{\psi}\right)^{1+\zeta}. \quad (54)$$

$$y_0(x) = -\frac{t^\zeta}{\Gamma(\zeta + 1)}. \quad (55)$$

The recurrence relation becomes

$$Y_{q+1}(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\{A_q(t)\}, \quad q = 0, 1, 2, \dots \quad (56)$$

The Adomian polynomials corresponding to the nonlinear term $y(t)^2$ are derived from

$$A_q = \frac{1}{\Gamma(q+1)} \left[\frac{d^q}{d\xi^q} N \left(\sum_{i=0}^q \xi^i y_i \right) \right]_{\xi=0}, \quad q = 0, 1, 2, \dots \quad (57)$$

$N(y) = y^2$ is the nonlinear term.

When $q = 0, q = 1, q = 2$:

$$A_0 = y_0^2, \quad A_1 = 2y_0y_1, \quad A_2 = y_1^2 + 2y_0y_2, \quad \dots \quad (58)$$

When $q = 0$ is implemented in the recurrence relation, we have

$$Y_1(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\{A_0(t)\}. \quad (59)$$

$$Y_1(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\left\{\left(-\frac{t^\zeta}{\Gamma(\zeta+1)}\right)^2\right\}. \tag{60}$$

$$Y_1(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\left\{\frac{t^{2\zeta}}{\Gamma(\zeta+1)^2}\right\}. \tag{61}$$

$$Y_1(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \left[\frac{\Gamma(2\zeta+1)}{\Gamma(\zeta+1)^2} \left(\frac{\vartheta}{\psi}\right)^{2\zeta+1}\right]. \tag{62}$$

$$Y_1(\psi, \vartheta) = \frac{\Gamma(2\zeta+1)}{\Gamma(\zeta+1)^2} \left(\frac{\vartheta}{\psi}\right)^{3\zeta+2}. \tag{63}$$

Taking the inverse shehu transform of Y_1 , we get

$$y_1(t) = \frac{\Gamma(2\zeta+1)t^{3\zeta+1}}{\Gamma(\zeta+1)^2\Gamma(2\zeta+3)}. \tag{64}$$

When $k = 1$:

$$Y_2(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\{A_1(t)\}. \tag{65}$$

From Adomian polynomial, $A_1 = 2y_0y_1$

$$Y_2(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\left\{2\left[-\frac{t^\zeta}{\Gamma(\zeta+1)}\right]\left[\frac{\Gamma(2\zeta+1)t^{3\zeta+1}}{\Gamma(\zeta+1)^2\Gamma(3\zeta+2)}\right]\right\} \tag{66}$$

$$Y_2(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\left\{-2\frac{\Gamma(2\zeta+1)t^{4\zeta+1}}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)}\right\} \tag{67}$$

$$Y_2(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \left[\frac{-2\Gamma(4\zeta+2)\Gamma(2\zeta+1)}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)} \left(\frac{\vartheta}{\psi}\right)^{4\zeta+2}\right]. \tag{68}$$

$$Y_2(\psi, \vartheta) = \frac{-2\Gamma(4\zeta+2)\Gamma(2\zeta+1)}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)} \left(\frac{\vartheta}{\psi}\right)^{5\zeta+3}. \tag{69}$$

Taking the inverse Shehu transform of Y_2 , we get

$$y_2(x) = \frac{-2\Gamma(4\zeta+2)\Gamma(2\zeta+1)t^{2+5\zeta}}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)\Gamma(5\zeta+3)}. \tag{70}$$

When $k = 2$:

$$Y_3(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\{A_2(t)\}. \tag{71}$$

From Adomian polynomial,

$$A_2 = 2y_0y_2 + y_1^2. \tag{72}$$

$$Y_3(\psi, \vartheta) = \left(\frac{\vartheta}{\psi}\right)^{1+\zeta} \mathbb{S}\left\{2\left[-\frac{t^\zeta}{\Gamma(\zeta+1)}\right]\left[\frac{-2\Gamma(4\zeta+2)\Gamma(2\zeta+1)t^{2+5\zeta}}{\Gamma(\zeta+1)^3\Gamma(3\zeta+2)\Gamma(5\zeta+3)}\right] + \left[\frac{\Gamma(2\zeta+1)t^{3\zeta+1}}{\Gamma(\zeta+1)^2\Gamma(3\zeta+2)}\right]^2\right\}. \tag{73}$$

$$Y_3(\psi, \vartheta) = \frac{4\Gamma(4\zeta+2)\Gamma(2\zeta+1) \left(\frac{\vartheta}{\psi}\right)^{4+7\zeta}}{\Gamma(\zeta+1)^4\Gamma(3\zeta+2)\Gamma(5\zeta+3)\Gamma(6\zeta+3)} + \frac{\Gamma(2\zeta+1)^2 \left(\frac{\vartheta}{\psi}\right)^{4+7\zeta}}{\Gamma(\zeta+1)^4\Gamma(3\zeta+2)^2\Gamma(6\zeta+3)}. \tag{74}$$

Taking the inverse Shehu transform of Y_3 , we get

$$y_3(t) = \frac{4\Gamma(4\zeta + 2)\Gamma(2\zeta + 1)t^{3+7\zeta}}{\Gamma(\zeta + 1)^4\Gamma(3\zeta + 2)\Gamma(5\zeta + 3)\Gamma(6\zeta + 3)\Gamma(7\zeta + 4)} + \frac{\Gamma(2\zeta + 1)^2t^{3+7\zeta}}{\Gamma(\zeta + 1)^4\Gamma(3\zeta + 2)^2\Gamma(6\zeta + 3)\Gamma(7\zeta + 4)}. \tag{75}$$

The general solution is

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \dots \tag{76}$$

$$[b]y(t) = -\frac{t^\zeta}{\Gamma(\zeta + 1)} + \frac{\Gamma(2\zeta + 1)t^{3\zeta+1}}{\Gamma(\zeta + 1)^2\Gamma(2\zeta + 3)} - \frac{2\Gamma(4\zeta + 2)\Gamma(2\zeta + 1)t^{2+5\zeta}}{\Gamma(\zeta + 1)^3\Gamma(3\zeta + 2)\Gamma(5\zeta + 3)} + \frac{4\Gamma(4\zeta + 2)\Gamma(2\zeta + 1)t^{3+7\zeta}}{\Gamma(\zeta + 1)^4\Gamma(3\zeta + 2)\Gamma(5\zeta + 3)\Gamma(6\zeta + 3)\Gamma(7\zeta + 4)} + \frac{\Gamma(2\zeta + 1)^2t^{3+7\zeta}}{\Gamma(\zeta + 1)^4\Gamma(3\zeta + 2)^2\Gamma(6\zeta + 3)\Gamma(7\zeta + 4)}. \tag{77}$$

Problem 5

Consider the nonlinear VIDEs [39]

$$\mathbb{D}^\zeta y(t) - \int_0^t \exp^{-\tau} [y(\tau)]^2 d\tau = 1, \quad 0 \leq t \leq 1, \quad 3 < \zeta \leq 4. \tag{78}$$

Subject to the initial condition $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

Solution to Problem 5:

Implementing the algorithm as we did in the earlier problems yields

$$y(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^\zeta}{\Gamma(\zeta + 1)} + \frac{t^{\zeta+1}}{\Gamma(\zeta + 2)} + \frac{2t^{\zeta+2}}{\Gamma(\zeta + 3)} + \frac{4t^{\zeta+3}}{\Gamma(\zeta + 4)} + \frac{8t^{\zeta+4}}{\Gamma(\zeta + 5)} + \frac{14t^{\zeta+5}}{\Gamma(\zeta + 6)} + \frac{20t^{\zeta+6}}{\Gamma(\zeta + 7)} + \frac{20t^{\zeta+7}}{\Gamma(\zeta + 8)} + \frac{\Gamma(2\zeta + 1)t^{3\zeta+1}}{\Gamma(\zeta + 1)^2\Gamma(2\zeta + 3)} + \frac{2t^{2\zeta+1}}{\Gamma(2\zeta + 2)} + \frac{2\Gamma(\zeta + 2)t^{2\zeta+2}}{\Gamma(\zeta + 1)\Gamma(2\zeta + 3)} + \frac{\Gamma(\zeta + 3)t^{2\zeta+3}}{\Gamma(\zeta + 1)\Gamma(2\zeta + 4)} + \frac{\Gamma(2\zeta + 4)t^{2\zeta+4}}{3\Gamma(\zeta + 1)\Gamma(2\zeta + 5)} + \frac{t^{2\zeta+2}}{\Gamma(2\zeta + 3)} + \frac{\Gamma(\zeta + 3)t^{2\zeta+3}}{\Gamma(\zeta + 2)\Gamma(2\zeta + 5)} + \frac{\Gamma(\zeta + 3)t^{2\zeta+4}}{\Gamma(\zeta + 2)\Gamma(2\zeta + 5)} + \frac{\Gamma(\zeta + 5)t^{2\zeta+5}}{6\Gamma(2\zeta + 6)\Gamma(2\zeta + 2)} + \frac{\Gamma(2\zeta + 2)t^{2+3\zeta}}{6\Gamma(\zeta + 1)\Gamma(\zeta + 2)\Gamma(3\zeta + 3)} + \frac{2\Gamma(2)t^{3+2\zeta}}{\Gamma(\zeta + 3)\Gamma(2\zeta + 4)} + \frac{2\Gamma(\zeta + 4)t^{4+2\zeta}}{\Gamma(\zeta + 3)\Gamma(2\zeta + 5)}. \tag{79}$$

RESULTS AND DISCUSSION

In this study, five numerical examples are solved using the proposed Modified Semi-analytical Method (MSM). To validate the effectiveness of our method, these examples were carefully selected from recently published articles that employed different numerical techniques, highlighting the uniqueness and versatility of our approach.

Specifically, Osama & Adyan [37] applied the Homotopy Analysis Method (HAM) and the Variational Iteration Method (VIM) to solve Problems 1 and 2, respectively. Table 1 presents the results for the system of equations in Problem 1 for $y_1(t)$ and $y_2(t)$ with those obtained using our proposed method MSM alongside those reported in [37] where HAM was used. Comparing the two results with the exact solution reveals the MSM performs better for $y_2(t)$ than HAM. The results presented in

Table 2 for Problem 2 using MSM, VIM and the exact solution show the superiority of our proposed method over VIM. The results presented in Tables 1 and 2 are for $\eta = 1$ to enable comparison with the results in the literature.

Table 1. Exact and Approximate Solutions for $y_1(t)$ and $y_2(t)$ in Problem 1 when $\zeta = 1$

t	MSM $y_1(t)$	MSM $y_2(t)$	HAM $y_1(t)$	HAM $y_2(t)$	Exact $y_1(t)$	Exact $y_2(t)$
0.0	1.000	1.000	1.000	1.000	1.000	1.000
0.1	1.001	0.999	1.001	0.999	1.001	0.999
0.2	1.008	0.992	1.008	0.992	1.008	0.992
0.3	1.027	0.973	1.027	0.973	1.027	0.973
0.4	1.064	0.936	1.064	0.936	1.064	0.936
0.5	1.125	0.875	1.125	0.874	1.125	0.875
0.6	1.216	0.783	1.216	0.782	1.216	0.784
0.7	1.343	0.655	1.343	0.653	1.343	0.657
0.8	1.512	0.484	1.512	0.479	1.512	0.488
0.9	1.729	0.262	1.729	0.253	1.729	0.271
1.0	2.000	-0.017	2.000	-0.033	2.000	0.000

Table 2. Exact and Approximate Solutions for $y_1(t)$ and $y_2(t)$ in Problem 2 when $\zeta = 1$

t	MSM $y_1(t)$	MSM $y_2(t)$	VIM $y_1(t)$	VIM $y_2(t)$	Exact $y_1(t)$	Exact $y_2(t)$
0.0	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.105	0.095	0.105	0.095	0.105	0.095
0.2	0.220	0.180	0.220	0.180	0.220	0.180
0.3	0.345	0.255	0.345	0.255	0.345	0.255
0.4	0.480	0.320	0.480	0.320	0.480	0.320
0.5	0.625	0.375	0.625	0.374	0.625	0.375
0.6	0.780	0.420	0.780	0.419	0.780	0.420
0.7	0.945	0.455	0.945	0.452	0.945	0.455
0.8	1.120	0.480	1.120	0.475	1.120	0.480
0.9	1.305	0.495	1.305	0.485	1.305	0.495
1.0	1.500	0.500	1.500	0.483	1.500	0.500

Table 3 depicts exact solution and results obtained through MSM and those presented in [38] where OHAM is used. Our MSM and OHAM of the authors in the cited reference give exact solutions for $u(t)$ and $v(t)$ for $\eta = 1$. In Table 4, the results obtained for $y(t)$ for $\eta = 0.9$ and $\eta = 1$ for MSM and Bernoulli Pseudo-spectral Method (PSM) used in [39] are very close although the exact solution is not supplied for the Problem. Table 5 on the other hand presents results for $y(t)$ for three different values of η obtained through MSM and those of CAS wavelets method in the cited literature. Exact solution is equally not supplied for this problem too, but MSM performed excellently well. Additionally, our results for Problem 5 are in perfect agreement with those of [40], who employed the Wavelet Method.

In addition to the fact that results obtained for each of the problems are tabulated, graphical representations are equally presented. The 2D graphs are for Problems 1, 2, 3, and 4, while solutions for Problem 5 is presented in 3D, as shown in Figures 1-8. It is obvious from the graphs that results from the present study are in tandem with those in the existing literature, and even better in some other instances. The 3D graph plotted for Problem 5 is feasible because $y(t)$ is plotted against two independent variables t and η .

Table 3. Exact and Approximate Solutions for $u(t)$ and $v(t)$ in Problem 3 when $\zeta = 1$

t	Exact u(t)	Exact v(t)	MSM [u(t)]	MSM [v(t)]	OHAM [u(t)]	OHAM [v(t)]
0.0	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.001	-0.001	0.001	-0.001	0.001	-0.001
0.2	0.008	-0.008	0.008	-0.008	0.008	-0.008
0.3	0.027	-0.027	0.027	-0.027	0.027	-0.027
0.4	0.064	-0.064	0.064	-0.064	0.064	-0.064
0.5	0.125	-0.125	0.125	-0.125	0.125	-0.125
0.6	0.216	-0.216	0.216	-0.216	0.216	-0.216
0.7	0.343	-0.343	0.343	-0.343	0.343	-0.343
0.8	0.512	-0.512	0.512	-0.512	0.512	-0.512
0.9	0.729	-0.729	0.729	-0.729	0.729	-0.729
1.0	1.000	-1.000	1.000	-1.000	0.999	-1.000

Table 4. Results for $y(t)$ in Problem 4 when $\zeta = 1.0$ and 0.9

t	MSM ($\zeta = 1$)	MSM ($\zeta = 0.9$)	BPM ($\zeta = 1$)	BPM ($\zeta = 0.9$)
0.0000	0.00000	0.00000	0.00000	0.00000
0.0625	-0.06250	-0.08574	-0.06250	-0.08576
0.1250	-0.12498	-0.15996	-0.12498	-0.15997
0.1875	-0.18740	-0.23024	-0.18740	-0.23025
0.2500	-0.24968	-0.29790	-0.24968	-0.29791
0.3125	-0.31171	-0.36342	-0.31171	-0.36344
0.3750	-0.37336	-0.42699	-0.37336	-0.42702
0.4375	-0.43446	-0.48862	-0.43446	-0.48866
0.5000	-0.49482	-0.54824	-0.49482	-0.54829
0.5625	-0.55423	-0.60572	-0.55423	-0.60576
0.6250	-0.61243	-0.66086	-0.61243	-0.66089
0.6875	-0.66917	-0.71347	-0.66917	-0.71347
0.7500	-0.72416	-0.76330	-0.72115	-0.76327
0.8125	-0.77711	-0.81013	-0.77709	-0.81007
0.8750	-0.82771	-0.85373	-0.82767	-0.85360
0.9375	-0.87565	-0.89386	-0.87557	-0.89363

Table 5. Results for $y(t)$ in Problem 5 when the values of ζ are 3.25, 3.50 and 3.75

t	MSM ($\zeta = 3.25$)	MSM ($\zeta = 3.5$)	MSM ($\zeta = 3.75$)	CAS ($\zeta = 3.25$)	CAS ($\zeta = 3.5$)	CAS ($\zeta = 3.75$)
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.1052	1.1052	1.1052	1.1053	1.1052	1.1052
0.2	1.2220	1.2217	1.2215	1.2219	1.2216	1.2216
0.3	1.3521	1.3509	1.3502	1.3523	1.3510	1.3510
0.4	1.4975	1.4945	1.4928	1.4968	1.4941	1.4941
0.5	1.6604	1.6545	1.6509	1.6635	1.6565	1.8334
0.7	2.0484	2.0319	2.0210	2.0444	2.0283	2.0293
0.8	2.2793	2.2544	2.2374	2.2776	2.2537	2.2537
0.9	2.5390	2.5032	2.4781	2.5265	2.4949	2.4949

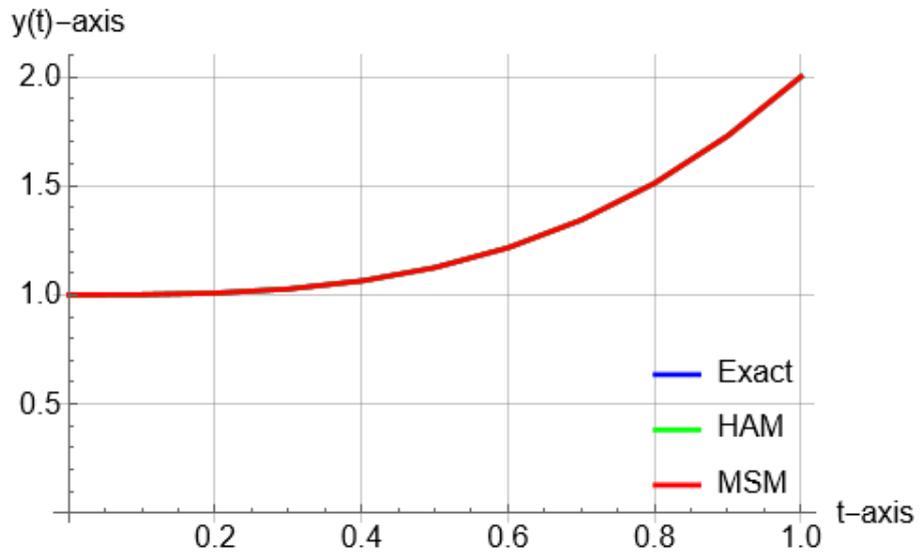


Figure 1. Graph of $y_1(t)$ for Problem 1

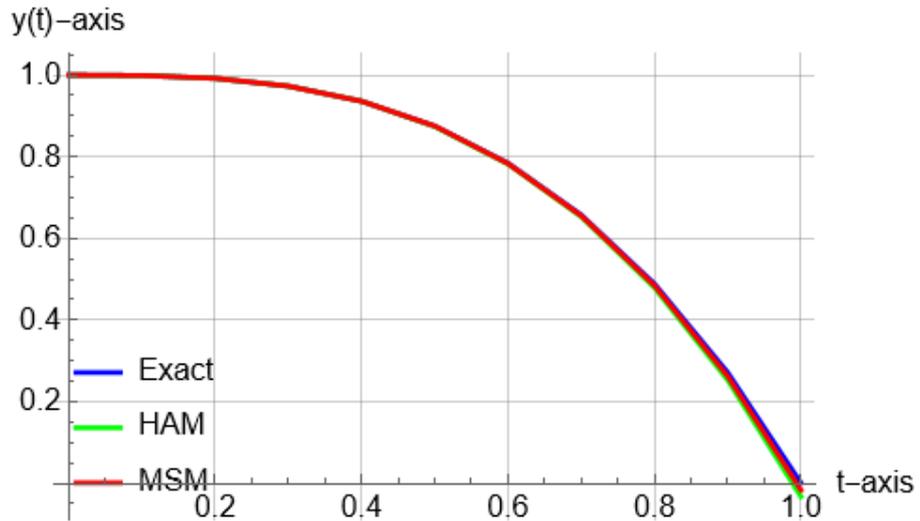


Figure 2. Graph of $y_2(t)$ for Problem 1

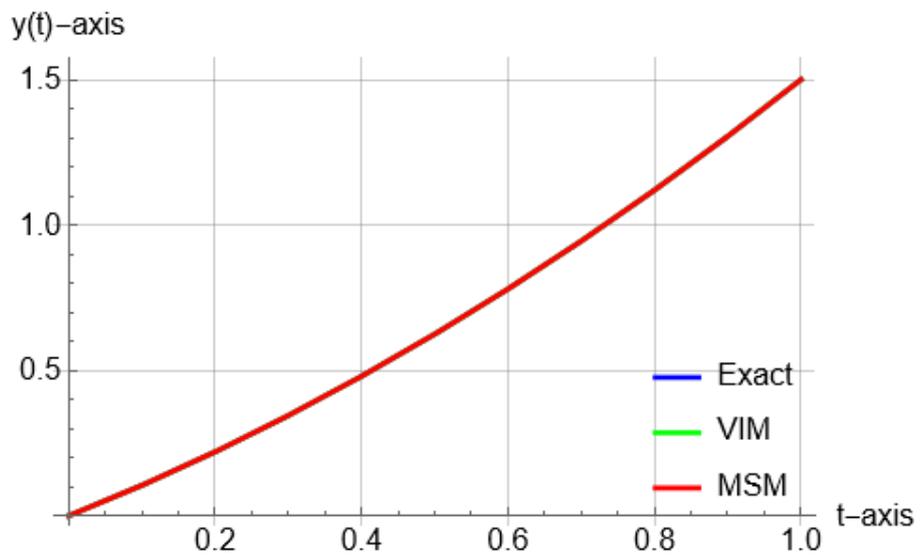


Figure 3. Graph of $y_1(t)$ for Problem 2

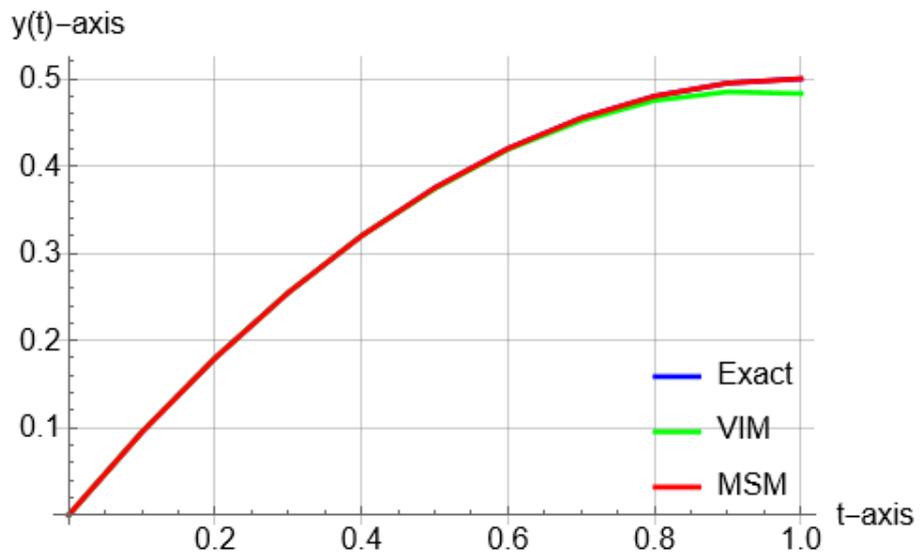


Figure 4. Graph of $y_2(t)$ for Problem 2

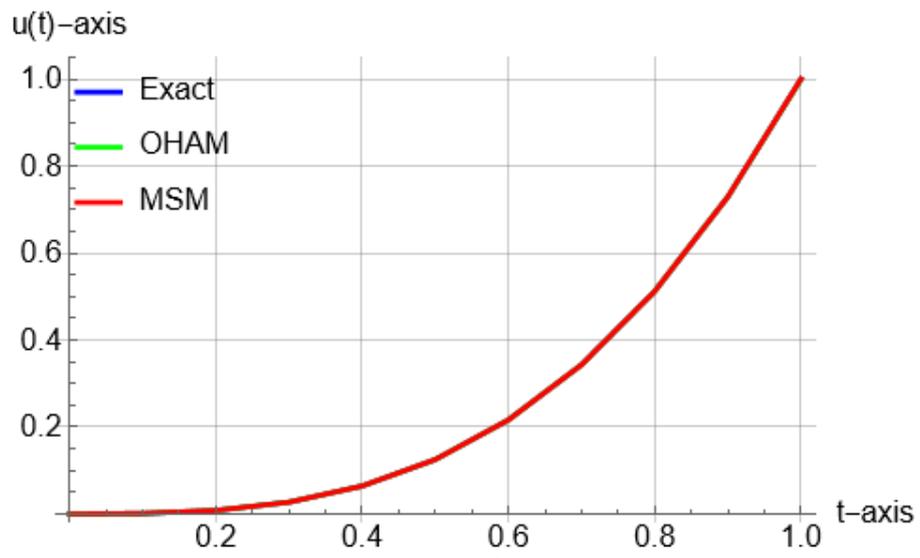


Figure 5. Graph of $u(t)$ for Problem 3

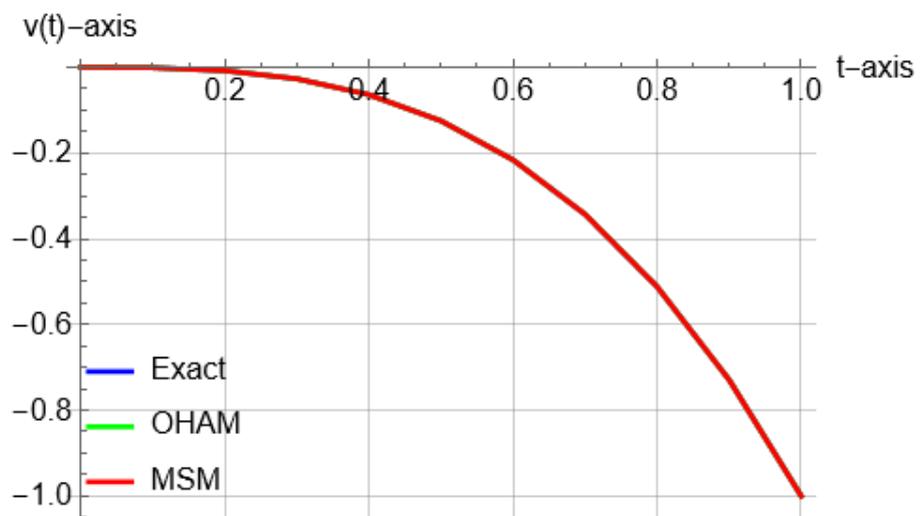


Figure 6. Graph of $v(t)$ for Problem 3

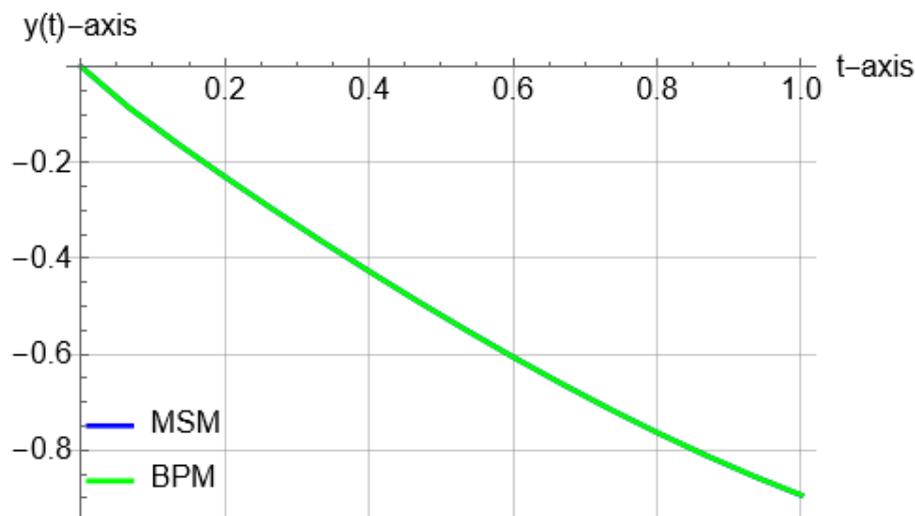


Figure 7. 2D Graph for Problem 4

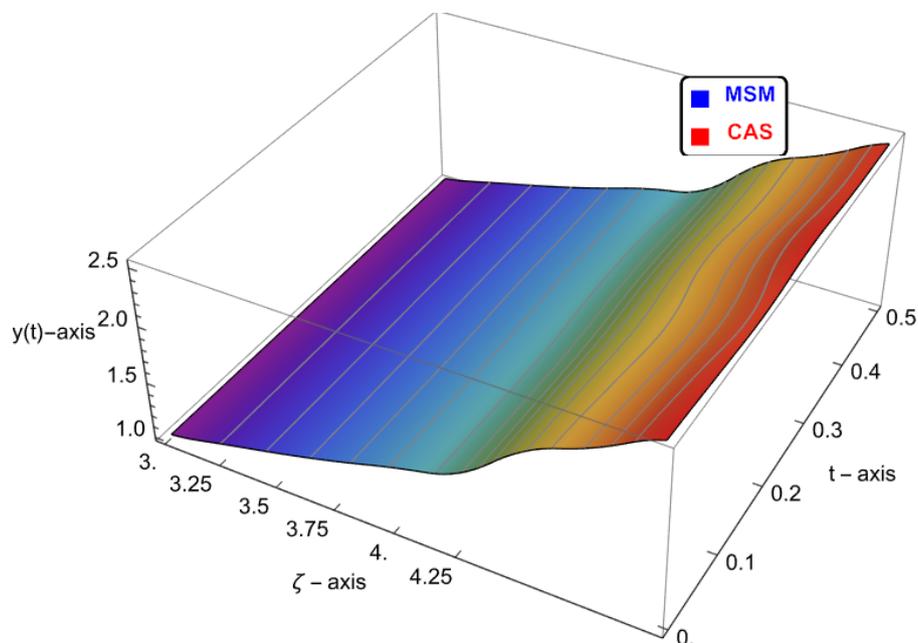


Figure 8. 3D Graph of $y(t)$ for MSM and CAS for Problem 5

CONCLUSION

It can be seen from the tabulated results and graphs that MSM is a reliable alternative to the existing methods in literature for the solution of linear and nonlinear fractional order VIDEs and system of the family of problems considered. This method has thus proven to be reliable. All computations were performed using Mathematica 13.3. It should equally be noted that the application of MSM is restricted to Volterra integro-differential equations with difference kernels such as $k(x - t)$ and $e^{(x-t)}$, while its applications to other types of kernels are yet to be established. The method reported in the present research can be studied with the view of extending it to Volterra-Fredholm Integro-differential Equations (VFIDEs). This thought is premised on the fact that VFIDEs is an extension of VIDEs, that have been successfully discussed in the present work.

SUPPLEMENTARY MATERIAL

None.

AUTHOR CONTRIBUTIONS

Babatunde M. Yisa: Conceptualization, methodology, writing—review and editing. Nicholas A. Onah: Software, validation, writing—original draft preparation, visualization. Lateefat O. Aselebe: Conceptualization, validation, writing—review and editing.

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DATA AVAILABILITY STATEMENT

Data are available from the authors upon reasonable request.

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CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

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