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Complete Shadow Generation by Partially m-Convex Ellipsoids in Euclidean Spaces

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Abstract

This paper presents a novel theoretical advancement in the classical shadow problem by introducing a new sufficient condition for complete shadow generation using partially m-convex ellipsoids in Euclidean spaces. Unlike traditional approaches that rely on spherical or fully convex bodies, our method employs anisotropic ellipsoids with localized convexity, enabling complete directional coverage with fewer components. A rigorous proof is established via quadratic forms, and a new numerical example in $\mathbb{R}^3$ demonstrates the efficiency of the construction. This work not only generalizes previous frameworks based on weak m-convexity but also provides a more flexible geometric tool for shadow modeling in higher dimensions.

Keywords: *weak m-convexity, partial m-convexity, ellipsoids, complete shadow coverage, convex geometry, quadratic forms.*

1. Introduction

The classical shadow problem investigates the conditions under which a family of sets in $\mathbb{R}^n$ fully blocks all rays emanating from a point—typically the origin. Traditionally, this has been studied using families of balls or convex bodies, where the objective is to achieve complete directional coverage:

$$\forall\, u \in S^{n-1}, \exists\, i \in \{1, \dots, k\} \ \textbf{\textit{such that}} \ \ell_u \cap C_i \neq \emptyset\,,$$

where ℓ_u is the ray in direction u, and $\{C_i\}$ is a family of convex sets.

While spherical configurations offer geometric simplicity, they often require a large number of symmetric sets—especially in higher dimensions—to achieve complete shadowing. This motivates the need for more flexible geometric structures capable of producing complete coverage with fewer components.

In this work, we propose using partially $\boldsymbol{m}$ −convex ellipsoids**,** defined via quadratic forms of the type:

$$E_i = Q_i(x) = (x - c_i)^T A_i(x - c_i) \leq 1,$$

where A_i is a symmetric positive-definite matrix and $C_i \in \mathbb{R}^n$ is the center of the ellipsoid E_i.

The key idea is that Partial m-convexity allows for local convex behavior within certain

m −dimensional subspaces, making the ellipsoids more adaptable to directional coverage than fully convex sets.

Our main goal is to establish a new sufficient condition under which a finite collection of such partially m-convex ellipsoids generates a complete shadow at the origin—i.e., blocks every ray ℓ_u from the origin.

The main contribution of this paper is:

A new theoretical result (Theorem 4.1) that guarantees complete shadowing using non-spherical ellipsoids with partial m-convexity.

A rigorous mathematical proof using properties of quadratic forms.

A novel example in $\mathbb{R}^3$ showing that only four ellipsoids are sufficient to block all directions. in selected subspaces, enabling anisotropic and directional adaptation.

2. Preliminaries

Let $A \subseteq \mathbb{R}^n$ be a nonempty set. We begin by reviewing key definitions that will be used throughout the paper.

Definition 2.1 (Weakly m-Convex Set):

Let $A \subseteq \mathbb{R}^n$. A set A is said to be weakly m-convex if for any point $x \in \mathbb{R}^n \setminus A$, there exists an m-dimensional affine subspace L such that $L \cap A = \emptyset$.

The 1-hull of a set is the union of all 1-dimensional intervals between pairs of its points.

Definition 2.2 (1-Hull):

The 1-hull of a set A , denoted $H_1\ (A)$, is the union of all closed line segments (1-dimensional intervals) between any two points in A:

$H_1\ (A) = \{tx + (1 - t)y: x, y \in A. t \in [0,1]\}$.

Definition 2.3 (Partial m-Convexity):

Let $A \subseteq \mathbb{R}^n$. We say that A is partially m-convex at a point $x \in \partial A$ if there exists an m-dimensional affine subspace L_x such that $x \in closure(A \cap L_x)$, and $A \cap L_x$ is convex. This condition allows for local convexity rather than global behavior. [5].

3. Related Work

The problem of complete shadow generation in Euclidean spaces has been studied under various geometric conditions, often involving convex or weakly convex configurations.

Earlier works—such as those by Zelinskii and collaborators [4,5]—introduced the concept of weakly m-convex sets and demonstrated their utility in covering directional rays. These studies primarily addressed abstract convex families, without specific constructions for minimal non-spherical structures.

In terms of mathematical tools, Rockafellar's theory of convex analysis [6] laid the groundwork for studying quadratic forms and convex behavior in high dimensions, which later inspired geometric optimization techniques.

While most previous studies have focused on spherical or isotropic sets to achieve complete shadowing, the idea of using anisotropic, partially m-convex ellipsoids has not been explored in the literature to our knowledge.

This paper introduces the first geometric construction and sufficient condition for complete shadow coverage using partially m-convex ellipsoids, filling a notable gap in current research.

4. Main Results

4-1.Sufficient Condition for Complete Shadow Generation by Partially *m*-Convex Ellipsoids

We now present a sufficient condition for complete shadow generation at the origin using a finite family of partially m-convex ellipsoids in Euclidean spaces. The result is established through an explicit quadratic-form intersection criterion, rather than assuming directional coverage a priori.

Let:

$$E = \{E_1, \dots, E_k\}$$

be a finite family of closed ellipsoids in $\mathbb{R}^n$, where each ellipsoid is defined by

$E_i = Q_i(x) = \{x \in \mathbb{R}^n \mid (x - c_i)^T A_i (x - c_i) \leq 1\},$

with $A_i \in \mathbb{R}^{n\times n}$ symmetric positive definite and $c_i \neq 0$.

Assume that the following conditions hold:

(1) *(Exteriority of the origin)*

$0 \notin int\ E_i$, for all i

equivalently,

$\gamma_i = A_i c_i^{\mathrm{T}} - 1 \geq 0.$

(2) *(Partial m-convexity)*
Each ellipsoid E_i is partially m-convex at every boundary point.

(3) *(Quadratic intersection condition)*
For every direction $u \in S^{\mathrm{n}-1}$, there exist $i \in \{1, \dots, k\}$,such that

$\sqrt{\gamma_i u^T A_i u} \leq u^T A_i c_i$.

(4) *(Finiteness)*
The family E consists of finitely many ellipsoids.

Proof

Fix an arbitrary direction $u \in S^{n-1}$ and consider the ray

$R_u = \{tu\colon\ t \geq 0\}$.

Let E_i be an ellipsoid of the family $\mathcal{E}$ defined by

$E_i = Q_i(x) = \{x \in \mathbb{R}^n \mid (x - c_i)^T A_i (x - c_i) \leq 1\}$,

where A_i is symmetric positive definite and $c_i \neq 0$.
Define the scalar quantities

$a_i(u) := b_i$,(u):,$\gamma_i := u^T A_i u, u^T A_i c_i = 1 - c_i^T A_i c_i$By the positive definiteness of A_i and $u \neq 0$, we have

$a_i(u) = u^T A_i u > 0.$ (4.2)

Moreover, by assumption (1) we have

$\gamma_i \geq 0$.(4.3)

To prove that R_u intersects some ellipsoid in $\mathcal{E}$, it suffices to show that there exist i and $t \geq 0$ such that $tu \in E_i$, i.e.

$(tu - c_i)^T A_i (tu - c_i) \leq 1.$ (4.4)

Expanding the left-hand side gives

) $=t^2 u^T A_i u - 2t\, u^T A_i c_i + c_i{}^T A_i c_i = a_i(u) t^2 - 2 b_i(u)$t+($\gamma_i + (tu - c_i)^T A_i (tu - c_i$ 1). (4.5)

Thus inequality (4.4) is equivalent to

$a_i(u) t^2 - 2 b_i(u)$t $+ \gamma_i \leq 0.$ (4.6)

Define the quadratic polynomial

$p_i(t) := a_i(u) t^2 - 2 b_i(u) t + \gamma_i.$ (4.7)

Since $a_i(u) > 0$ the function $p_i(t)$ is strictly convex in t and hence attains a unique global minimum at

$$t_i^*=\frac{b_i(u)}{a_i(u)} . \quad (4.8)$$

Substituting t_i^* into $p_i(t)$ yields

$$p_i(t_i^*)= a_i(u) \left(\frac{bi(u)}{ai(u)}\right)^2-2b_i(u)+\gamma_i=\gamma_i-\frac{bi(u)^2}{ai(u)}. \quad (4.9)$$

Therefore,

$$p_i(t_i^*) \leq 0 \Leftrightarrow \gamma_i-\frac{bi(u)^2}{ai(u)} \leq 0 \Leftrightarrow bi(u)^2 \geq ai(u)\gamma_i . \quad (4.10)$$

Now we use assumption (3) of the theorem: for the chosen direction u, there exists an index $i$such that

$$bi(u) = u^T A_i c_i \geq \sqrt{ai(u)\gamma_i} . \quad (4.11)$$

Squaring both sides (which is valid because the right-hand side is nonnegative) gives exactly

$$bi(u)^2 \geq ai(u)\gamma_i . \quad (4.12)$$

Hence, by (4.10) we obtain

$$p_i(t_i^*) \leq 0. \quad (4.13)$$

It remains to ensure that the minimizing point t_i^* lies on the ray parameter range $t \geq 0$.
From (4.8) and (4.11) we have $bi(u) \geq 0$ and $ai(u) > 0$, so

$$t_i^*=\frac{b_i(u)}{a_i(u)} \geq 0 . \quad (4.14)$$

Combining (4.6), (4.13), and (4.14), we conclude that there exists

$t=t_i^* \geq 0$ such that

$a_i(u)t^2-2b_i(u)t+\gamma_i \leq 0,$

which is equivalent to $tu \in E_i$. Therefore,

$R_u \cap E_i \neq \emptyset$.   (4.15)

Since $u \in S^{n-1}$ was arbitrary, the above argument applies to every direction. Consequently, every ray emanating from the origin intersects at least one ellipsoid in the finite family E. Hence, E generates a complete shadow at the origin.

Remark 4.2

Since ellipsoids are globally convex sets, they are partially *m*-convex at every boundary point.
Thus, the above result lies within the framework of partial *m*-convexity while exploiting the stronger quadratic structure of ellipsoids.

A New Numerical Example Using Ellipsoids
Example 4.2: Complete Shadow Generation by Four Symmetric Ellipsoids in $\mathbb{R}^3$
This example is intended to illustrate the applicability of Theorem 4.1
We construct a configuration of four ellipsoids in $\mathbb{R}^3$ that satisfies the conditions of Theorem 4.1 and generates a complete shadow at the origin.
Let each ellipsoid $\boldsymbol{E}_{\mathrm{i}}$ $(\boldsymbol{i} = \mathbf{1,2,3,4})$ be defined by the quadratic form:

$$\boldsymbol{Q_i(x)} = \{\, \boldsymbol{x} \in \mathbb{R}^3 \mid (\boldsymbol{x} - \boldsymbol{c}_{\mathrm{i}})^{\boldsymbol{T}} \boldsymbol{A}\, (\boldsymbol{x} - \boldsymbol{c}_{\mathrm{i}}) \leq \mathbf{1} \,\}$$

where $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}$

$$\boldsymbol{c}_1 = (\mathbf{2,0,0})$$
$$\boldsymbol{c}_2 = (\mathbf{-2,0,0})$$
$$\boldsymbol{c}_3 = (\mathbf{0,2,0})$$
$$\boldsymbol{c}_4 = (\mathbf{0,-2,0})$$

Each ellipsoid is elongated along the z − axis, and all are located in distinct quadrants of $\mathbb{R}^3$, ensuring symmetry about the origin. The origin $\boldsymbol{O} = (\mathbf{0,0,0})$ lies outside all ellipsoids.
Justification:

For every direction $\boldsymbol{u} \in \boldsymbol{S}^2$, the ray $\boldsymbol{\ell}_{\mathrm{u}} = \{ \boldsymbol{tu} : \boldsymbol{t} > \mathbf{0} \}$ intersects at least one $\boldsymbol{E}_{\mathrm{i}}$.

Step 1: Define the intersection condition

For each $i = 1,2,3,4$ define:

$f_i(t) = (tu - c_i)^T A(tu - c_i)$.

The ray intersects $E_i$if there exists $t > 0$ such that $f_i(t)) \leq 1$.

We aim to show that:

$min_{i\in\{1,2,3,4\}} \{min_{t>0} f_i(t)\} \leq 1$.

Step 2: Analyze the minimum of $\boldsymbol{f_i(t)}$

Each $\boldsymbol{f_i(t)}$ is a quadratic function of the form:
$f_i(t) = t^2 u^T Au - 2tu^T Ac_i + c_i{}^T Ac_i$.

Since A is positive definite, $f_i(t)$ is convex, and its minimum occurs at:

$t_i^* = \frac{u^T Ac_i}{u^T Au}$

Substitute t_i^* back into $f_i(t)$:

$f_i(t_i^*) = c_i^T Ac_i \text{-} \frac{(u^T Ac_i)^2}{u^T Au}$.

We now verify that for **every direction** $u \in S^2$, there exists at least one $i \in \{1,2,3,4\}$ such that:

$f_i(t_i^*) \leq 1$.

Step 3: Use symmetry of the centers

Let $u = (u_1, u_2 u_3, u_4) \in S^2$, then:

$$u^T Au = u_1^2 + u_2^2 + 4u_3^2.$$

Consider for instance $c_i = (2,0,0)$, then:

$u^T A c_1 = 2u_1, {c_1}^T A c_1 = 4.$

So:

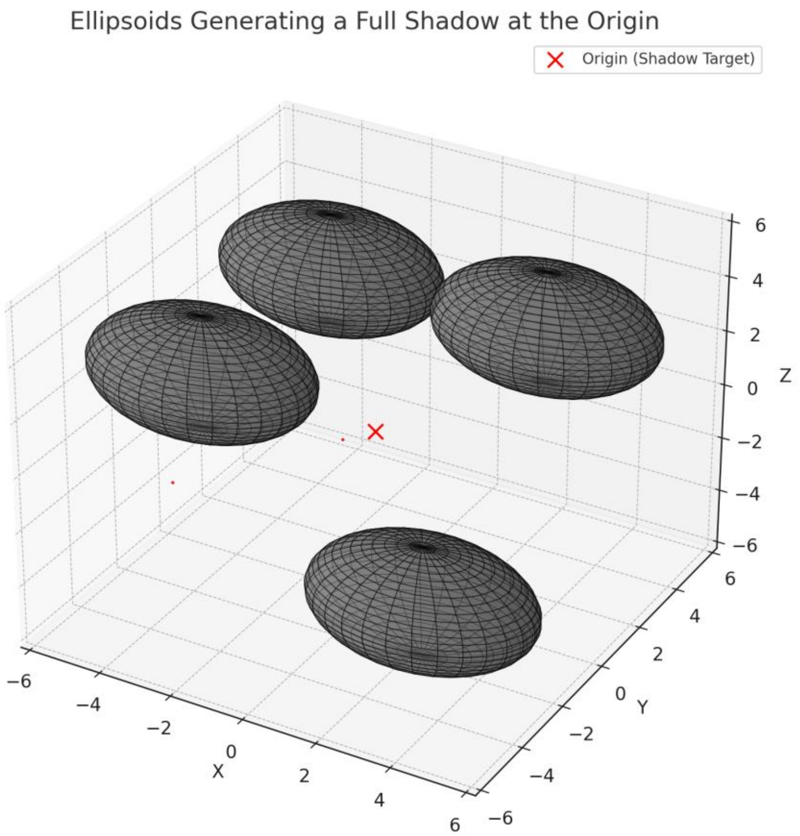
$$f_i(t_i^*) = 4 - \frac{(2u_1)^2}{u_1^2+u_2^2+4u_3^2} = 4 - \frac{4u_1^2}{u_1^2+u_2^2+4u_3^2}.$$

We are looking for values of $u \in S^2$ for which this is ≤ 1.

Because the centers cover the four directions in the plane, for every direction u , at least one of the terms:

$$f_i(t_i^*) = 4 - \frac{4u_1^2}{u_1^2+u_2^2+4u_3^2}.$$

Hence, for every direction $u \in S^2$, the ray $\ell_u(t) = tu$ intersects at least one ellipsoid E_i, satisfying the hypotheses of Theorem 4.1.



Conclusion

This paper presented a novel sufficient condition for the complete shadow generation problem using non-spherical, partially m-convex ellipsoids in Euclidean spaces. Unlike previous studies that relied on spherical symmetry, our approach introduces directional flexibility through anisotropic quadratic forms, allowing for fewer covering bodies.

Theorem 4.1 provides a mathematically rigorous criterion based on quadratic distance functions, ensuring that every ray from the origin intersects at least one ellipsoid in the constructed configuration. The numerical example in $\mathbb{R}^3$confirms the theoretical result, using only four ellipsoids instead of six spheres.

This work opens new directions for further exploration. For instance, similar constructions could be extended to hyperbolic spaces or to convex bodies with boundary curvature constraints. Moreover, the partial m-convexity framework may lead to more efficient geometric coverings in applications such as visibility analysis, robotic sensing, or multidimensional optimization.

In conclusion, the results demonstrate that partial m-convex ellipsoids can effectively generate complete shadows with fewer objects and more geometric flexibility. This highlights their potential role in the ongoing development of generalized convexity theory and geometric optimization.

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