Zero- Stability and Convergence Analysis for Numerical Solutions of Initial Value Problem

تحليل الصفر للاستقرار والتقارب للحلول العددية لمشكلة القيمة الأولية

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Abstract

Numerical methods for initial value problems (IVPs) are foundational in scientific computing, yet their reliability hinges on rigorous theoretical guarantees. This paper establish a comprehensive framework for analyzing the convergence of linear multistep methods (LMMs) through the interplay of Zero-stability and consistency as formalized by Daglquist equivalence theorem. We demonstrate that convergence-defined as the numeral solution approaching that exact solution as the size $h \to 0$ is achievable only when the method is both consistent (local truncation error vanishing with $|h|$) and zero-stable (controlled error propagation under infinitesimal perturbations). Through analytical proofs and numerical experiments, we validate Dahlquist s theorem and illustrate how violating zero-stability, even in consistent methods, leads to divergent solutions. Case studies on Euler's method and Adams-Bashforth schemes best part the practical implications of these theoretical principle. Furthermore, we clarify the distinction between zero-stability (a theoretical necessity for convergence) and absolute stability (a practical requirement for finite $|h|$), emphasizing their complementary roles in solving stiff and non-stiff IVPs. By extending the analysis to nonlinear systems via Lipchitz continuity and Gronwall's inequality, this work bridges theoretical rigor with algorithmic design, offering actionable insights for developing robust numerical solvers.

Keywords: Euler method, convergence, zero-stability,initil value problem, local truncation error, global error.

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الملخص

طُرق الحساب العددية للمسائل ذات القيمة الابتدائية (IVPs) تُعد أساسية في الحوسبة العلمية، إلا أن موثوقيتها تعتمد على ضمانات نظرية صارمة. تقدم هذه الورقة إطاراً شاملاً لتحليل تقارب طرق الخطوات المتعددة الخطية (LMMs) من خلال التفاعل بين **الاستقرار-صفر** و **الاتساق** كما هو مُنظَّم من خلال **نظرية معادلة داهلكويست** .نوضح أن **التقارب** - المُعرف كاقتراب الحل العددي من الحل الدقيق عندما يحجم حجم الخطوة - (h→0) لا يمكن تحقيقه إلا عندما تكون الطريقة **متسقة**) خطأ القطع المحلي يتلاشى مع |h| و **مستقرة-صفر**) انتشار خطأ مُتحكَّم به تحت اضطرابات متناهية الصغر). من خلال البراهين التحليلية والتجارب العددية، نتحقق من صحة نظرية داهلكويست ونوضح كيف أن انتهاك الاستقرار-صفر، حتى في الطرق المتسقة، يؤدي إلى حلول متباعدة. تبرز دراسات الحالة على **طريقة أويلر** ومخططات **آدامز-باشفورث** الآثار العملية لهذه المبادئ النظرية. علاوة على ذلك، نوضح التمييز بين **الاستقرار-صفر**) ضرورة نظرية للتقارب) و **الاستقرار المطلق**) متطلب عملي لقيم محددة لـ|h| ، مؤكدين على أدوارهما التكميلية في حل مسائل القيمة الابتدائية الصلبة وغير الصلبة. من خلال توسيع التحليل إلى الأنظمة غير الخطية عبر **استمرارية ليبشيتز** و **متباينة غرونوال**، يربط هذا العمل بين الدقة النظرية وتصميم الخوارزميات، مقدِّماً رؤى قابلة للتطبيق لتطوير حلول عددية قوية.

1. Introduction

Initial Value Problems (IVPs) in differential equations from the cornerstone of modeling scientific and engineering phenomena, such as electrical circuit analysis, mention dynamics, and control systems. Since exact analytical solutions are often unattainable for complex real-world systems, numerical methods have become indispensable tools for approximating solutions [1]. Among these methods, Euler's Method stands out for its simplicity. However, it faces significant limitations in accuracy and stability, particularly in nonlinear systems or long-term integrations [2].

This study investigates the Zero-stability and convergence of Euler's Method when applied to IVPs, focusing on two critical metrics :

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i. Local Truncation Error (LTE): measuring accuracy at each computational step.
ii. Global Error (GE): reflecting cumulative errors over successive iterations [3].

Euler's Method defines the numerical solution through the iterative formula;

$$y_{n+1} = y_n + h.f(t_n, y_n)$$

Where h is the step size , and $f(t_n, y_n)$ represents the derivative of the function. While the method demonstrates efficiency for simple linear systems, its instability in nonlinear systems due to error amplification remains a key challenge [4].

This work builds upon prior research, including:

- Adesanya et al. (2017), which compared single-step (e.g. Euler) and multistep methods (e.g. Adams-Bashforth) [5].
- Zhang (2007), which analyzed the stability of numerical methods in rapidly changing systems [6].
- Amodes and Mazzia (1989), which highlighted the superior accuracy of Runge-Kutta methods at the cost of computational complexity [7].

The study provides novel contributions through:

- A detailed experimental analysis of global error accumulation in both linear and nonlinear systems.
- A systematic comparison between Euler's method and forth-order Runge-Kutta methods.
- Practical recommendations for enhancing Euler's method via adaptive step-size control and hybridization with high-precision technique

Objectives

- Establish theoretical guarantees for convergence.
- Link zero-stability to error propagation control.
- Highlight the role of Dahlquist‘s equivalence theorem.

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2. Preliminaries

Zero-Stability and Convergence for Initial Value Problems

In numerical analysis, ensuring the accuracy and reliability of methods for solving initial value problem (IVPs) involves understanding the concepts of zero-stability and convergence. Here 's a structured overview:

2.1. Convergence of Numerical methods

- **Definition of Convergence**:

  A numerical method for solving an initial value problem (IVP) is convergent if, as the step size $h \to 0$, the numerical solution y_n approaches the exact solution $y(t_n)$ at all discrete points (t_n) in interval $[t_n, T]$. Formally:

$$\lim_{h \to 0} \max_{0 \le n \le N} |y(t_n) - y_n| = 0$$

$$\text{Where } N = \frac{T - t_0}{h}$$

- **Global Error**: relationship between local and global errors (e.g., $O(h^p)$) for order- P methods).

2.2. Consistency and Local Truncation Error

- Local Truncation Error (LTE):

$$T_n = \frac{y(t_{n+1}) - y(t_n)}{h} - \Phi(t_n, y(t_n).h),$$

  Where Φ is the numerical scheme.

- Consistency Condition: $T_n \to 0 \; as \; h \to 0$
- LTE is $O(h^{p-1})$

2.3. Zero – Stability: Theoretical Foundations

- Definition: Bounded sensitivity to perturbations as $h \to 0$
- Root condition for (LMMs):
- Characteristic polynomial $P(r)$.
- All roots $|r_i| \le 1$, and simple if $|r_i| = 1$.

- Key Insight: Zero=Stability ensures controlled error propagation, even as the number of steps $\mathbb{N} \longrightarrow \infty$.

2.4. Dahlquist's Equivalence Theorem

"The Equivalent of Convergence Consistency, and Zero- stability for Linear Multistep Methods"

Statement:

For linear multistep methods (LMMs) applied to initial value problems (IVPs) the following three properties are equivalent:

1. Convergence: The numerical solution y_n approaches the exact solution$y(t_n)$ $as\ h \to 0$.
2. Consistency: The local truncation error (LTE) vanishes $as\ h \to 0$.
3. Zero-Stability: the method satisfies the root condition (all roots of the characteristic polynomial $P(r)$ lie within or on the unit circle, with roots on the circle being simple).

- $\boldsymbol{Consistency + Zero\text{-}Stability \Leftrightarrow Convergence}$.
  (and vice versa)
- One-step methods(e.g., Euler, Runge-Kutta) are automatically zero-stable if consistent.

2.7. Absolute Stability vs. Zero-Stability

- Zero-Stability: Theoretical requirement as $h \to 0$.
- Absolute Stability: Practical requirement for fixed h (e.g., $|y' = \lambda y|$
- Regions of Stability: Plots for Euler, Runge-Kutta, and LMMs.

2.8. Nonlinear IVPs and Extensions

- Lipschitz Continuity**:** Ensuring
  $\boldsymbol{f(t,y)}$ satisfies $|\boldsymbol{f(t,y) - f(t,z)}| \le \boldsymbol{L}\,|\boldsymbol{y - z}|$.
- **Gronwall's Inequality:** Bounding global errors for nonlinear systems.

2.9. Numerical Experiments

- Experiment 1 : Solve $y' = -y$, $y(0) = 1$

Using Euler and a non- zero- stable LMM.

- 2 :Compare global errors for $h = 0.1, 0.01, 0.001$ to verify convergence rates.

2.10. Example

The Adams-Bashforth 2-step method:

$$y_{n+2} = y_{n+1} + \frac{h}{2}\left(3f\left(t_{n+1}, y_{n+1}\right) - f\left(t_n, y_n\right)\right)$$

- Consistent: (LTE) $\mathcal{O}(h^2)$.
- Zero-stable: Roots of $P(r) = r^2 - r \; are \; r = 0{,}1$ (satisfy root condition).
- Conclusion: convergence by Dahlquist‘s theorem.

2.11. Practical Implications and Example

- Case study 1: Euler's method
  (consistent, zero-stable, convergence).
- Case study 2: A non zero-stable, LMM
  (e.g., $|(y_{\{n+2\}} - 4y_{\{n+1\}} + 3y_n = 0)|$ showing divergence.
- Adams-Bashforth
  Vs. Adams-Moulton: stability-convergence trade-offs

2.5. Definition of Euler's Method

Euler's method is a first-order numerical procedure for solving ordinary differential equation (ODEs) with a given initial value. It approximates the solution by stepping through the domain using the derivative at the current point to estimate the next value.

- Initial Value Problem (IVP):
  $y' = f(t, Y), \quad y(t_0) = y_0, \quad t \in [t_0, T]$
  the Euler method generates approximations y_n at discrete points $t_n = t_n + nh$ (where h is the step size) via:

$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$

- Numerical Methods:

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Overview of one-step (e.g., Euler-Kutta) and multistep methods (e.g., Adams-Bashforth).

2.12. Example : Euler's Method, consider the IVP:

$y' = -y,\ y(0) = 1,$ over $t \in [0,2]$.

exact solution: $y(t) = e^{-t}$

Euler's Steps: $y_{n+1} = y_n + h \cdot f\,(t_n, y_n) = y_n - hy_n$

- $y_0 = 1,\ t_0 = 0$
- $t_1 = 0.5:\ y_1 = y_0 - h.\,y_0 = 1 - 0.5(1) = 0.5$
- $t_2 = 1:\ y_2 = y_1 - h.\,y_1 = 0.25$
- $t_3 = 0.3:\ y_3 = 1.133$

Exact Value :

- $y\,(0.1) = e^{-0.1} \approx 0.9048$
- $y\,(0.2) = e^{-0.2} \approx 0.8187$
- $y\,(0.3) = e^{-0.3} \approx 0.7408$

Global Errors:

- At $t = 0.3$: $|0.7408 - 1.331| = 0.5902$

Advantages and Limitations

Limitations	Advantage
Low accuracy for large h.	Simple to implement
Error accumulates rapidly	Computationally inexpensive
Unstable for stiff equations	Foundation for advanced method

Why Use Euler's Method ?

i. Educational Tool: Demonstrates core principles of numerical ODE solvers.

ii. Quick Prototyping: Useful for simple systems where high precision is not critical.

iii. Basis for Improvements: Modified Euler, Runge-kutta, and adaptive methods build on its logic.

2.13. Remark:

Euler's method trades simplicity for accuracy, making it a starting point for understanding numerical ODE techniques.

2.14. Definition of Runge-kutta Methods

Runge-Kutta (RK) methods are a family of interval numerical techniques used to solve ordinary differential equations (ODEs) with initial values. They approximate the solution by combining multiple intermediate "slope "estimates within each step, achieving higher accuracy than simpler methods like Euler 's.

1. **General Form:**

An s-stage Runge-Kutta method computes the next solution y_{n+1} using**:**

$$\boldsymbol{y_{n+1} = y_n + h \sum_{j=1}^{s} b_i K_i}$$

Where: $K_i = f\left(h \ \sum_{j=1}^{s} a_{ij} K_j + c_i h, y_n + t_n\right),$

a_{ij}, b_i, c_i are coefficients defined by the specific RK method

2. **Explicit vs. Implicit:**
   - Explicit RK: $a_{ij} = 0$ for $j \geq i$ (e.g., RK4)
   - Implicit RK: $a_{ij} \neq 0$ for $j \geq i$ (better for stiff equations)
3. **Order of Accuracy**:
   - the global error is $O\,(h^p)$, where P is the methods order.

Comparison Between Euler and Runge-Kutta (RK4) at $\boldsymbol{t = 1}$

RK4 Error	Euler Error	Step Size
0.0003	0.1179	0.5
0.000002	0.0192	0.52
0.0000001	0.0192	0.1

Why RK4 Is More Accurate?

1. Higher Order:

- RK4 has a global error of $O\,(h^4)$, compared to Euler's $O\,(h)$.

- At h= 0.5, RK4 is 400x more accurate than Euler.

2. Multi-Stage Calculation:
   - The four slope estimates(K_1, K_2, K_3, K_4) average out local error, reducing cumulative inaccuracies.
3. Stability:
   - RK4 handles stiff equations better due to its balanced error propagation.

2.15. Definition of Adams-Bashforth Methods

Adams-Bashforth (AB) methods are explicit linear multistep methods for solving initial value problems (IVPs). They approximate the solution y_{n+1} using past values $y_n, y_{n-1}, \ldots$ and their derivatives $f\ (t, y)$. The K-step (AB)method has order K, meaning the global error is $\mathcal{O}\ (h^K\ )$,
General formula (K-Step Adams-Bashforth)

$$y_{n+1} = y_n + h \sum_{j=0}^{K-1} b_j\ f\ (t_{n-j}, y_{n-j}\ )$$

Where b_j are coefficients derived from polynomial interpolation of past slopes.

2.16. Example: 2-step Adams-Bashforth (AB2)

Solve the IVP: $y' = -y,\ y(0),\quad t \in [0,2]$.
With the exact solution $y\ (t) = e^{-t}$
AB2 formula: $y_{n+1} = y_n + \frac{h}{2}\left(3\ f(\ t_n, y_n) - f\ (t_{n-1}\ , y_{n-1})\right)$.
Step-by-step Implementation, step Size:$h = 0.5$.
Starting Values: Use Euler 's method to compute y_1.
Step 0: $t_0 = 0,\ y_0 = 1$.
Step 1: $t_1 = 0.5$, compute y_1 using Euler: $y_1 = 0.5$.
Step 2: $t_2 = 0.1$, compute y_2 using AB2: $y_2 = 0.5375$
Exact Solution at t = 1: .

Comparison Table of the Three Methods: Euler,RK4, and AB2 at $t = 1$

Adams-Bashforth 2 (AB2)	Runge-kutta 4 (RK4)	Euler	Criterion
$\mathcal{O}\ (h^2)$	$\mathcal{O}\ (h^4)$	$\mathcal{O}(h)$	Order
0.0071	0.0003	0.1179	Error example (t=1 , h =0.5)
1 evaluation step (after initialization)	4 evaluations/ step	1 evaluations/ step	Computation Cost
### (Requires initial setup)	## (Moderately complex)	##### (very simple)	Ease of Implementation
Moderate (equation-d dependent)	Good for non-stiff equations	Poor for stiff equations	Stability

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Long-term integration with fixed steps	High-precision (physics, engineering)	Rapid prototyping	Best Use

Explanation of the # in the table.
The(#) in the " Ease of lamentation" column represent a visual rating system to indicate how easy or challenging if it is to practically implement each method. This system is inspired by rating scales used for hotels or apps and is based on the following factors:
#. (1 #): complex / difficult to implement.
##. (2 #): moderately complex.
###. (3 #): Fairly straightforward with minor challenges.
####. (4#): Easy to implement.
#####. (5 #): Extremely simple, requiring minimal effort.

Application to the three methods:

1. Euler (#####): Reason: Requires only a single step $y_{n+1} = y_n + hf(t_n, y_n)$ And no prior steps or complex initiation.
2. RK4 (##): Reason: Involves calculating four step evaluation (K_1, K_2, K_3, K_4) per step, leading to longer code and higher chance of programming error.
3. Adams- Bashforth 2 (###): Reason: Simple to implement once initial values are set up (using Euler or RK4), but requires memory management to store past steps.

2.17. Key Conclusions

1. **Accuracy:**

- RK4 is the most accurate $O\ (h^4)$, ideal for scientific applications requiring high precision.
- AB2 improves accuracy over Euler $O\ (h^2)$ but is less precise than RK4.
- Euler is the least accuracy $O(h)$ but useful for foundational understanding.

2. **Computational Efficiency:**

- Euler and AB2 are computationally efficient (1 evaluation/ step after initialization for AB2).

3. **Ease of Use:**

- Euler is the easiest to implement (single-step formula).
- AB2 requires initial setup (e.g., Euler or RK4 for starting values).
- RK4 is more complex due to its multi-stage calculations.

4. **Stability:**

- RK4 is more stable for non-stiff equations compared to Euler and AB2.
- Euler and AB2 may fail for stiff equations unless combined with implicit methods.

2.18. Remarks: recommendations by Scenario

- For Learning / Prototyping: Euler (simplicity and speed)
- High- Accuracy Solutions: RK4(ideal for most practical applications)

- Long-Term Fixed-Step integration:
  AB2 (balances accuracy and efficiency).
- Stiff Equations: Implicit methods (e.g., Backward Euler, Implicit RK).

Final Summary

There is no "one-size-fits-all" method. The choice depends on:

i. Required accuracy level.
ii. Available computational resources.
iii. Equation nature (stiff or non-stiff)
iv. Need for adaptive step size.

High-order methods (e.g., RK4) are preferred when precision is critical, while multistep methods (e.g., AB2) balance performance and accuracy for specific use cases.

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