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RESEARCH ARTICLE

Truncated Exponentiated Ailamujia Exponential Distribution: Properties and Applications

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ABSTRACT

Probability distribution has shown its practicality in almost every area of human effort. This paper introduces a novel family of distributions called the [0, 1] Truncated Exponentiated Ailamujia-G family. The [0, 1] Truncated Exponentiated Ailamujia Exponential ([0, 1]TEAE) distribution, which is a sub-model of the recently created family, is completely constructed. The [0, 1]TEAE distribution is created by merging the [0, 1] Truncated and Exponentiated Ailamujia distributions. the mathematical features of the [0, 1]TEAE distribution, including moments, skewness, kurtosis, incomplete moments, renyi entropy, quantile function, and probability-weighted moments, are extensively examined. The quantiles for the chosen parameter values are clearly defined. The techniques used for parameter estimation include maximum likelihood, least squares, weighted least squares, and Anderson-Darling. This study assesses the efficacy of several estimators using a Monte Carlo simulation. In addition, the estimators were used on three actual datasets, and the Kolomogorov-Simirnov statistics were documented for each of them. The [0, 1] Truncated Exponentiated Ailamujia Exponential distribution was used on three real world datasets. Its effectiveness was evaluated by comparing it to other well-known extensions of the Exponential distribution, using criteria such as Hannan-Quinn information criterion, Bayesian information criterion, Akaike information criterion, and consistent AIC and goodness of fit tests such as Anderson-Darling statistic, Cramer-von Mises statistic, and P-value for the KS test.

Keywords: Ailamujia distribution, Estimation methods, Moments, [0, 1] truncated, Renyi entropy

Introduction

Even though statistical models are available, real-world circumstances frequently do not meet well-known probability models. It's important to develop probability models to fully comprehend real-world phenomena. For real-world data, new families of distributions provide flexible modeling choices. Adding new parameters to existing distributions improves their tail weights and makes them more applicable to real-world occurrences.

Numerous significant contributions to probability distributions on the interval [0, 1] have been made by researchers. For instance, the shortened Frechet-G family was described by Abid and Abdulrazak.[1] The truncated inverted Kumaraswamy-generated family was proposed by Bantan et al.,[2] and ZeinEldin et al.[3] presented the generalized truncated Frechet-Generated family. Almarashi and Associates[4] introduced the condensed Muth family. A new class of truncated Nadarajah-Haghighi-G distributions was proposed by Khalaf et al.[5] In addition, Khaleel et al.[6] introduced the [0, 1] truncated inverse Weibull Rayleigh distribution. The paper mentioned in reference[7] discusses a truncated random variable X that is defined on the interval [0, 1].

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The cumulative distribution function (CDF) of X is given in Eq. (1):

$$F(x) = \frac{\Phi(x;\xi) - \Phi(0)}{\Phi[1] - \Phi(0)},\ x \in [0, 1] \tag{1}$$

And the [0, 1] Truncated distribution's probability density function (PDF) is

$$f(x) = \frac{\phi(x;\xi)}{\Phi[1] - \Phi(0)} \tag{2}$$

The family introduced in this paper is built upon the Exponentiated Ailamujia (EA) distribution, which was originally proposed by Aafaq et al.[8] The CDF of the EA distribution plays a crucial role in the development of this new family.

The CDF and PDF for EA distribution are stated in Eq. (3) and Eq. (4) respectively.

$$\Phi(x;\beta,\lambda) = \left(1 - (1 + 2\lambda x)\, e^{-2\lambda x}\right)^{\beta} \tag{3}$$

$$\phi(x;\beta,\lambda) = 4\beta\lambda^2 x e^{-2\lambda x}\left(1 - (1 + 2\lambda x)\, e^{-2\lambda x}\right)^{\beta-1},\ x \geq 0,\ \beta,\lambda > 0 \tag{4}$$

Consider a random variable X that follows the [0, 1] Truncated Exponentiated Ailamujia ([0, 1] TEA) distribution with shape parameters β and λ. We can denote this as X $\sim$ [0, 1] TEA (β, λ). Then by substituting Eq. (3) into Eq. (1), we get the CDF of the ([0, 1] TEA distribution:

$$K(x) = \frac{\left(1 - (1 + 2\lambda x)\, e^{-2\lambda x}\right)^{\beta}}{\left(1 - (1 + 2\lambda)\, e^{-2\lambda}\right)^{\beta}} \tag{5}$$

By finding $\frac{dK(x)}{dx}$, the result is the PDF of the ([0, 1] TEA distribution

$$k(x) = \frac{4\beta\lambda^2 x e^{-2\lambda x}\left(1 - (1 + 2xt)\, e^{-2\lambda x}\right)^{\beta-1}}{\left(1 - (1 + 2\lambda)\, e^{-2\lambda}\right)^{\beta}},\ 0 < x < 1, \beta,\lambda > 0 \tag{6}$$

The purpose of this article is to provide a sub-model [0, 1]TEAE distribution that is more adaptable and superior to the [0, 1] TEA-G family. By use of [0, 1] TEA-G, the E distribution becomes more adaptable by means of the [0, 1]TEAE distribution. It gives better fits than competing models and creates a new model with developing hazard functions.

Derivation of [0, 1] truncated exponentiated Ailamujia-G ([0, 1] TEA-G) family

Using the methodology proposed by Alzaatreh et al.,[9] the CDF of the new [0, 1] Truncated Exponentiated Ailamujia-G ([0, 1] TEA-G) Family is defined by the following procedures:

$$F(x)_{[0,1]TEA-G} = \int_0^{G(x;\xi)} \frac{4\beta\lambda^2 t e^{-2\lambda t}\left(1 - (1 + 2\lambda t)\, e^{-2\lambda t}\right)^{\beta-1}}{\left(1 - (1 + 2\lambda)\, e^{-2\lambda}\right)^{\beta}} dt$$

The CDF is

$$F(x)_{[0,1]TEA-G} = \emptyset\left(1 - (1 + 2\lambda G(x;\xi))\, e^{-2\lambda G(x;\xi)}\right)^{\beta} \tag{7}$$

And the PDF of the [0, 1] TEA-G Family is

$$f(x)_{[0,1]TEA-G} = \emptyset 4\beta\lambda^2 g(x;\xi)\, G(x;\xi)\, e^{-2\lambda G(x;\xi)}\left(1 - (1 + 2\lambda G(x;\xi))\, e^{-2\lambda G(x;\xi)}\right)^{\beta-1} \tag{8}$$

Where $x \geq 0,\ \beta,\lambda > 0, and\ \emptyset = \frac{1}{(1-(1+2\lambda)e^{-2\lambda})^{\beta}}$.

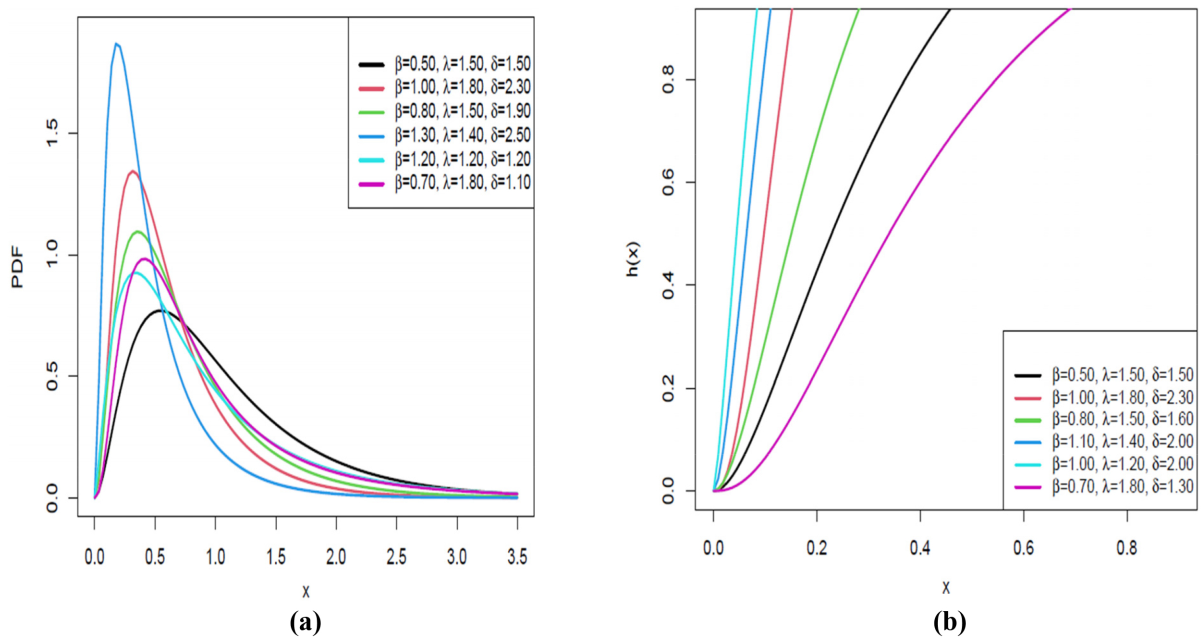


Fig. 1. (a) Graphs of the PDF of the [0, 1]TEAE. (b) Graphs of h(x) of the [0, 1]TEAE.

[0, 1] truncated exponentiated Ailamujia exponential ([0, 1]TEAE) distribution

By taking the baseline distribution of the [0, 1] TEA-G Family as the Exponential distribution with CDF

$$G(x,\delta) = 1 - e^{-\delta x} \tag{9}$$

and the PDF

$$g(x,\delta) = \delta e^{-\delta x},\ x > 0, \delta > 0 \tag{10}$$

The CDF, PDF, and hazard function of the [0, 1]TEAE distribution are given in Eqs. (11) to (13) respectively.

$$F(x)_{[0,1]TEAE} = \emptyset\left(1 - \left(1 + 2\lambda\left(1 - e^{-\delta x}\right)\right) e^{-2\lambda\left(1-e^{-\delta x}\right)}\right)^{\beta} \tag{11}$$

And

$$f(x)_{[0,1]TEAE} = \emptyset 4\beta\lambda^2\delta e^{-\delta x}\left(1 - e^{-\delta x}\right) e^{-2\lambda\left(1-e^{-\delta x}\right)}\left(1 - \left(1 + 2\lambda\left(1 - e^{-\delta x}\right)\right) e^{-2\lambda\left(1-e^{-\delta x}\right)}\right)^{\beta-1} \tag{12}$$

Where x $\geq$0, β, λ , δ>0
The hazard function is

$$h(x) = \frac{\emptyset 4\beta\lambda^2\delta e^{-\delta x}\left(1 - e^{-\delta x}\right) e^{-2\lambda\left(1-e^{-\delta x}\right)}\left(1 - \left(1 + 2\lambda\left(1 - e^{-\delta x}\right)\right) e^{-2\lambda\left(1-e^{-\delta x}\right)}\right)^{\beta-1}}{1 - \emptyset\left(1 - \left(1 + 2\lambda\left(1 - e^{-\delta x}\right)\right) e^{-2\lambda\left(1-e^{-\delta x}\right)}\right)^{\beta}} \tag{13}$$

Fig. 1(a) shows that the [0, 1]TEAE distribution has a pronounced positive skew. This indicates a larger concentration of values on the left side of the distribution, indicating that lower values occur more often than higher ones. Furthermore, Fig. 1(b) illustrates that the rate hazard escalates with rising values.

Statistical properties for [0, 1]TEAE distribution

Moment, Skewness and Kurtosis

The r^{th} moment of the [0, 1]TEAE distribution is derived by using the method given

$$\acute{\mu}_r = E(X^r)_{[0,1]\mathrm{TEAE}} = \int_0^\infty x^r f(x, \beta, \lambda, \delta)\, dx$$

First, expand Eq. (8) by using the generalized binomial series $[1+u]^{-q} = \sum_{z=0}^{\infty} (-1)^z \binom{q}{z} u^z$, $[1-u]^{-q} = \sum_{w=0}^{\infty} \frac{\lambda(q+w)}{w!\lambda(q)} u^w : |u| < 1,\ q > 0$ as given in[10] and by using the exponential expansion formula $e^{-q} = \sum_{z=0}^{\infty} \frac{(-1)^z}{z!} u^z$ employed by[11]

$$f(x)_{[0,1]TEA-G} = \sum_{w=h=s=0}^{\infty} \frac{(-1)^{w+s} 2^{s+h} \lambda^{s+h} (w+1)^s}{s!} \binom{\beta-1}{w} \emptyset 4\beta\lambda^2 g(x;\xi)\, G(x;\xi)^{1+h+s} \tag{14}$$

Now, by substituting Eqs. (11) and (12) into Eq. (14), we get

$$f(x)_{[0,1]TEAE} = \sum_{w=h=s=0}^{\infty} \frac{(-1)^{w+s} 2^{s+h} \lambda^{s+h} (w+1)^s}{s!} \binom{\beta-1}{w} \emptyset 4\beta\lambda^2 \delta e^{-\delta x} \left(1-e^{-\delta x}\right)^{1+h+s}$$

Using binomial expansion $\left(1-e^{-\delta x}\right)^{1+h+s} = \sum_{k=0}^{\infty} (-1)^k \binom{m+j}{k} e^{-\delta k x}$
Then

$$f(x)_{[0,1]TEAE} = ¥ e^{-\delta(k+1)x} \tag{15}$$

Where $¥ = \sum_{w=h=s=0}^{\infty} \frac{(-1)^{w+s+k} 2^{s+h} \lambda^{s+h} (w+1)^s}{s!} \binom{\beta-1}{w} \binom{m+j}{k} \emptyset 4\beta\lambda^2 \delta a$
Now, by substituting Eq. (15) into $E(X^r)_{[0,1]\mathrm{TEAE}}$,

$$\acute{\mu}_r = E(X^r)_{[0,1]\mathrm{TEAE}} = \int_0^\infty ¥ x^r e^{-\delta(k+1)x} dx$$

Let $y = \delta(k+1)x \Rightarrow x = \frac{y}{\delta(k+1)}$, then $\frac{dy}{dx} = \delta(k+1) \Rightarrow dx = \frac{dy}{\delta(k+1)}$
Then

$$\acute{\mu}_r = E(X^r)_{[0,1]\mathrm{TEAE}} = \frac{¥}{(\delta(k+1))^{r+1}} \int_0^\infty y^r e^{-y} dy$$

By utilizing the gamma integral, we can derive an expression for the r^{th} moment of the [0, 1]TEAE distributions.

$$\acute{\mu}_r = E(X^r)_{[0,1]\mathrm{TEAE}} = \frac{¥\lambda(r+1)}{(\delta(k+1))^{r+1}} \tag{16}$$

By employing Eq. (16), we can derive the $\acute{\mu}_1$, $\acute{\mu}_2$, $\acute{\mu}_3$ and $\acute{\mu}_4$of the [0, 1]TEAE distribution as follows:

$$\acute{\mu}_1 = E(X^1)_{[0,1]\mathrm{TEAE}} = \frac{¥\lambda(2)}{(\delta(k+1))^2} = \frac{¥}{(\delta(k+1))^2} \tag{17}$$

$$\acute{\mu}_2 = E(X^2)_{[0,1]\mathrm{TEAE}} = \frac{¥\lambda(3)}{(\delta(k+1))^3} = \frac{2¥}{(\delta(k+1))^3} \tag{18}$$

Table 1. Effects of Parameter Changes on Moments, Variance, Skewness, and Kurtosis.

β	λ	δ	$\acute{\mu}_1$	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$	V_1	Q_1	Q_2
Values of parameter			values of properties						
1	0.2	0.13	2.64	28	566	16281	21.03	3.820	20.76
		0.15	2.29	21	368	9185	15.75	3.824	20.82
	0.5	0.13	5.28	66	1375	40207	38.12	2.564	9.230
		0.15	4.57	49	895	22683	28.11	2.609	9.447
	0.8	0.13	7.15	99	2142	63590	47.87	2.174	6.488
		0.15	6.20	74	1394	35875	35.56	2.189	6.551
2	0.2	0.13	1.70	13	229	6134	10.11	4.885	36.29
		0.15	1.47	9	149	3460	6.83	5.518	42.71
	0.5	0.13	3.43	31	563	15233	19.23	3.261	15.85
		0.15	2.97	23	367	8594	14.17	3.327	16.24
	0.8	0.13	4.69	48	886	24219	26.00	2.664	10.51
		0.15	4.06	36	576	13663	19.51	2.666	10.54
3	0.2	0.13	1.13	6	80	1895	4.72	5.443	52.63
		0.15	0.98	4	52	1069	3.03	6.5	66.81
	0.5	0.13	2.27	14	198	4720	8.84	3.779	24.08
		0.15	1.96	10	128	2663	6.15	4.047	26.63
	0.8	0.13	3.09	21	312	7524	11.45	3.242	17.06
		0.15	2.68	16	203	4245	8.81	3.171	16.58

$$\acute{\mu}_3 = E\left(X^3\right)_{[0,1]\mathrm{TEAE}} = \frac{¥\lambda\,(4)}{\left(\delta\,(k+1)\right)^4} = \frac{6¥}{\left(\delta\,(k+1)\right)^4} \tag{19}$$

$$\acute{\mu}_4 = E\left(X^4\right)_{[0,1]\mathrm{TEAE}} = \frac{¥\lambda\,(5)}{\left(\delta\,(k+1)\right)^5} = \frac{24¥}{\left(\delta\,(k+1)\right)^5} \tag{20}$$

By employing Eqs. (17) to (20), we can derive the variance $V_1 = E(X^2) - (E(X))^2$, Skewness $Q_1 = \frac{\mu_3}{\mu_2^{(\frac{3}{2})}}$ and Kurtosis $Q_2 = \frac{\mu_4}{\mu_2^2}$ of the [0, 1]TEAE distribution as follows:

$$V_1 = \frac{2¥}{\left(\delta\,(k+1)\right)^3} - \left[\frac{¥}{\left(\delta\,(k+1)\right)^2}\right]^2 \tag{21}$$

$$\text{Skewness } Q_1 = \frac{\frac{6¥}{\left(\delta(k+1)\right)^4}}{\left[\frac{2¥}{\left(\delta(k+1)\right)^3}\right]^{\frac{3}{2}}} \tag{22}$$

$$\text{Kurtosis } Q_2 = \frac{\frac{24¥}{\left(\delta(k+1)\right)^5}}{\left[\frac{2¥}{\left(\delta(k+1)\right)^3}\right]^{2}} \tag{23}$$

Table 1 shows how different parameter values affect the distribution's moments, variance, skewness, and kurtosis. When we hold certain factors constant while increasing others, we see the following patterns: When β and λ remain constant, increasing δ results in a decrease in all moments and variances. Skewness and kurtosis rise. Increasing the λ value while maintaining constant β and δ results in an increase in all moments and variance. Skewness and kurtosis decrease. Increasing the β value while keeping λ and δ constant results in a decrease in all moments and variance. Skewness and kurtosis increase.

Figs. 2 and 3 show 3D plots of the skewness, and kurtosis of X for different values of β, λ , and δ. These plots illustrate a variety of non-monotonic patterns, emphasizing the measures' adaptability and versatility.

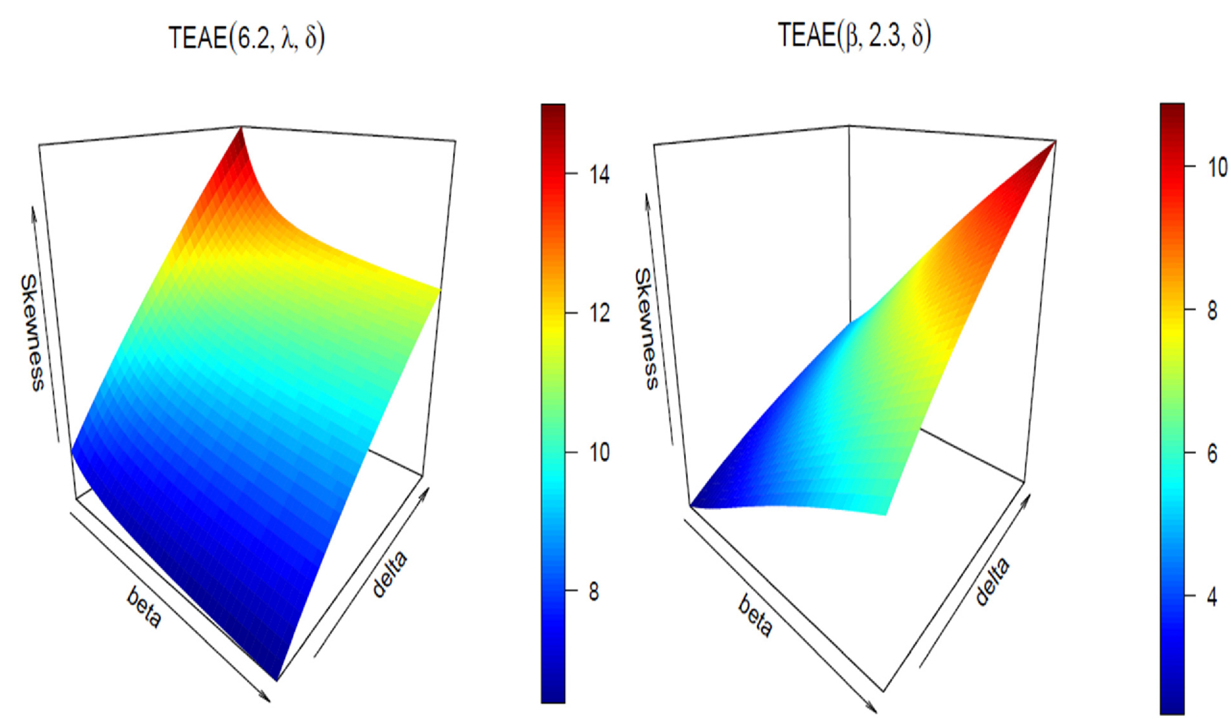


Fig. 2. Three-dimensional graphs illustrate the skewness values for different parametric variables.

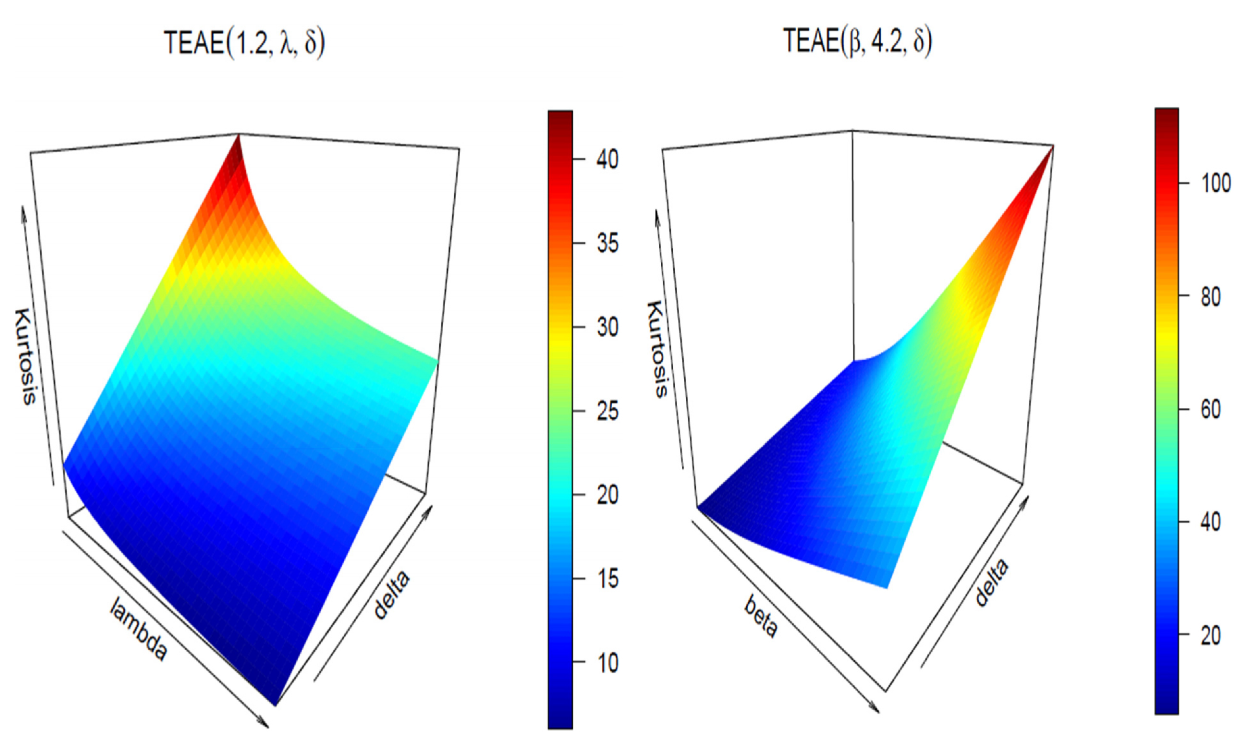


Fig. 3. Three-dimensional graphs illustrate the kurtosis values for different parametric variables.

Incomplete moments

The r^{th} (r > 0) incomplete moments for the [0, 1]TEAE distribution follow from Eq. (15) as:

$$\dot{\mu}_r(u) = \int_0^u x^r ¥ e^{-\delta(k+1)x} dx$$

Let $t = \delta(k+1)x \Rightarrow x = \frac{t}{\delta(k+1)}$
When $x = 0 \Rightarrow t = 0$, *and if* $x = u \Rightarrow t = \delta u(k+1)$ *then* $du = \frac{dt}{\delta(k+1)}$
Then

$$\dot{\mu}_r(u) = \frac{¥}{(\delta(k+1))^{r+1}} \int_0^{\delta u(k+1)} t^r e^{-t} dt$$

Using the lower incomplete gamma function, we obtain:

$$\dot{\mu}_r(u) = \frac{¥}{(\delta(k+1))^{r+1}} \lambda(r+1,\ \delta u(k+1)) \tag{24}$$

Rényi entropy

The rényi entropy, a measure of the unpredictability or uncertainty of a probability distribution, is introduced in this subsection. Here is how rényi entropy is defined:

$$0T_R(\eta)_{[0,1]TEAE} = \frac{1}{1-\eta} log \int_0^\infty f^\eta(x)_{[0,1]TEAE} dx,\ \eta > 0,\ \ \eta > 1$$

To derive a linear representation and simplify as follows:

$$f(x)^\eta{}_{[0,1]TEA-G} = (\emptyset 4\beta\lambda^2)^\eta g(x;\xi)^\eta G(x;\xi)^\eta e^{-2\lambda\eta G(x;\xi)} \left(1-(1+2\lambda G(x;\xi)) e^{-2\lambda G(x;\xi)}\right)^{\eta(\beta-1)}$$

By using binomial expansion and exponential expansion, then

$$f(x)^\eta{}_{[0,1]TEA-G} = \sum_{p=v=i=0}^{\infty} \frac{(-1)^{i+p} 2^{i+v} \lambda^{i+v} (\eta+P)^i}{i!} \binom{\eta(\beta-1)}{w} \binom{p}{v} (\emptyset 4\beta\lambda^2)^\eta g(x;\xi)^\eta G(x;\xi)^{\eta+v+i} \tag{25}$$

Substituting Eqs. (11) and (12) into Eq. (25), we get

$$f(x)^\eta{}_{[0,1]TEAE} = \sum_{p=v=i=0}^{\infty} \frac{(-1)^{i+p} 2^{i+v} \lambda^{i+v} (\eta+P)^i}{i!} \binom{\eta(\beta-1)}{w} \binom{p}{v} (\emptyset 4\beta\lambda^2 a)^\eta e^{-\eta\delta x} \left(1-e^{-\delta x}\right)^{\eta+v+i}$$

Using binomial expansion $(1-e^{-\delta x})^{\eta+v+i} = \sum_{s=0}^{\infty} (-1)^s \binom{\eta+v+i}{s} e^{-\delta s x}$
Then

$$f(x)^\eta{}_{[0,1]TEAE} = ₩ e^{-\delta(\eta+s)x} \tag{26}$$

Where $₩ = \sum_{p=v=i=s=0}^{\infty} \frac{(-1)^{i+p+s} 2^{i+v} \lambda^{i+v} (\eta+P)^i}{i!} \binom{\eta(\beta-1)}{w} \binom{p}{v} \binom{\eta+v+i}{s} (\emptyset 4\beta\lambda^2 a)^\eta$
Now substituting Eq. (26) into $T_R(\eta)_{[0,1]TEAE}$,

$$T_R(\eta)_{[0,1]TEAE} = \frac{1}{1-\eta} log \int_0^\infty ₩ e^{-\delta(\eta+s)x} dx$$

By integral, we get:

$$T_R(\eta)_{[0,1]TEAE} = \frac{1}{1-\eta}\log\left(\frac{₩}{\delta\,(\eta+s)}\right) \tag{27}$$

Quantile function

The quantile function may be derived by taking the inverse of the Eq. (11):

$$\emptyset\left(1-\left(1+2\lambda\left(1-e^{-\delta x}\right)\right)e^{-2\lambda\left(1-e^{-\delta x}\right)}\right)^{\beta} = u \tag{28}$$

By definition, the quantile function Q(u, φ) satisfies the equation F[Q(u, φ); φ] = u, where u is in the range (0, 1) and φ =(β, λ , δ)$^{\mathrm{T}}$. By the property of the Lambert function, which states that $W(x)e^{W(x)} = x$, we can solve Eq. (28) as in,[12] to obtain

$$Q\left(u,\varphi\right) = -\frac{1}{\delta}\left(\ln\left(1+\frac{1}{2\lambda}\left(W\left(e^{-1}\left(\left(\frac{u}{\emptyset}\right)^{\frac{1}{\beta}}-1\right)\right)+1\right)\right)\right) \tag{29}$$

Probability weighted moments

The expression for the $(\omega,\eta)^{th}$ probability weighted moment for the [0, 1]TEAE distributions can be given as follows:[13]

$$\rho_{(\omega,\eta)} = E\left(X^{\omega}\left(F^{\eta}\left(X\right)\right)\right) = \int_{-\infty}^{\infty} x^{\omega}F^{\eta}\left(x\right)f\left(x\right)dx \tag{30}$$

By substitution Eqs. (11) and (15) into Eq. (30),

$$\rho_{(\omega,\eta)} = \int_{-\infty}^{\infty} x^{\omega}\left(\emptyset^{\eta}\left(1-\left(1+2\lambda\left(1-e^{-ax}\right)\right)e^{-2\lambda\left(1-e^{-ax}\right)}\right)^{\eta\beta}\right)\left(¥e^{-\delta(k+1)x}\right)dx$$

Using binomial expansion

$$\left(1-\left(1+2\lambda\left(1-e^{-\delta x}\right)\right)e^{-2\lambda\left(1-e^{-\delta x}\right)}\right)^{\eta\beta} = \sum_{j=0}^{\infty}(-1)^j\binom{\eta\beta}{j}\left(1+2\lambda\left(1-e^{-\delta x}\right)\right)^j e^{-2\lambda j\left(1-e^{-\delta x}\right)}$$

And

$$\left(1+2\lambda\left(1-e^{-\delta x}\right)\right)^j = \sum_{s=0}^{\infty}\binom{j}{s}2^s\lambda^s\left(1-e^{-\delta x}\right)^s, e^{-2\lambda j\left(1-e^{-\delta x}\right)} = \sum_{z=0}^{\infty}\frac{(-1)^z}{z!}2^z\lambda^z j^z\left(1-e^{-\delta x}\right)^z$$

Then

$$\left(1-\left(1+2\lambda\left(1-e^{-\delta x}\right)\right)e^{-2\lambda\left(1-e^{-\delta x}\right)}\right)^{\eta\beta} = \sum_{j=s=z=0}^{\infty}\frac{\emptyset^{\eta}(-1)^{j+z}2^{s+z}\lambda^{s+z}j^z}{z!}\binom{\eta\beta}{j}\binom{j}{s}\left(1-e^{-\delta x}\right)^{s+z}$$

Again using binomial expansion $\left(1-e^{-\delta x}\right)^{s+z} = \sum_{q=0}^{\infty}(-1)^q\binom{s+z}{q}e^{-\delta qx}$, hence

$$\rho_{(\omega,\eta)} = \int_{-\infty}^{\infty} x^{\omega}\left(₭e^{-\delta qx}\right)\left(¥e^{-\delta(k+1)x}\right)dx$$

Where $K = \sum_{j=s=z=q=0}^{\infty} \frac{\emptyset^{\eta}(-1)^{j+z+q}2^{s+z}\lambda^{s+z}j^{z}}{z!}\binom{\eta\beta}{j}\binom{j}{s}\binom{s+z}{q}$

$$\rho_{(\omega,\eta)} = K¥\int_0^{\infty} x^{\omega}e^{-\delta(q+k+1)x}dx$$

Let $u = \delta(q+k+1)x \Rightarrow \mathrm{x} = \frac{u}{\delta(q+k+1)}$, then $\frac{\mathrm{dy}}{\mathrm{dx}} = \delta(q+k+1) \Rightarrow \mathrm{d}x = \frac{dy}{\delta(q+k+1)}$
Then

$$\rho_{(\omega,\eta)} = E\left(X^{\omega}\left(F^{\eta}(X)\right)\right) = \frac{¥\lambda(\omega+1)}{\left(\delta\left(q+k+1\right)\right)^{\omega+1}} \tag{31}$$

Maximum likelihood estimation (MLE)

Consider a random sample $x_1, x_2, \ldots, x_n$ drawn from the [0, 1]TEAE distribution with a parameter vector $\varphi = (\beta, \lambda, \delta)^{T}$. We can express the log-likelihood function of φ for n observations as follows[14–16]

$$l(\varphi) = nlog\emptyset + nlog(4) + nlog(\beta) + 2nlog(\lambda) + nlog(\delta) - \sum_{i=1}^{n}\delta x_i + log\sum_{i=1}^{n}\left(1-e^{-a\delta}\right)$$
$$-2\lambda\sum_{i=1}^{n}\left(1-e^{-\delta x_i}\right) + (\beta-1)\,log\sum_{i=1}^{n}\left(1-\left(1+2\lambda\left(1-e^{-\delta x_i}\right)\right)e^{-2\lambda\left(1-e^{-\delta x_i}\right)}\right) \tag{32}$$

In order to determine how sensitive the log-likelihood function is to each parameter, we need to compute its first partial derivative with respect to the parameter (β, λ, δ). The expressions for these partial derivatives are:

$$\frac{\partial(l)}{\partial\beta} = \frac{n}{\beta} + log\sum_{i=1}^{n}\left(1-\left(1+2\lambda\left(1-e^{-\delta x_i}\right)\right)e^{-2\lambda\left(1-e^{-\delta x_i}\right)}\right) \tag{33}$$

$$\frac{\partial(l)}{\partial\lambda} = \frac{2n}{\lambda} - 2\sum_{i=1}^{n}\left(1-e^{-\delta x_i}\right) + (\beta-1)\sum_{i=1}^{n}\frac{2\left(1-e^{-\delta x_i}\right)e^{-2\lambda\left(1-e^{-\delta x_i}\right)}\left(1-\left(1+2\lambda\left(1-e^{-\delta x_i}\right)\right)\right)}{1-\left(1+2\lambda\left(1-e^{-\delta x_i}\right)\right)e^{-2\lambda\left(1-e^{-\delta x_i}\right)}} \tag{34}$$

$$\frac{\partial(l)}{\partial\delta} = \frac{n}{\delta} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{x_ie^{-\delta x_i}}{1-e^{-\delta x_i}} - \sum_{i=1}^{n}\frac{2\lambda x_ie^{-\delta x_i}}{1-e^{-\delta x_i}} + (\beta-1)\sum_{i=1}^{n}\frac{2\lambda x_ie^{-\delta x_i}e^{-2\lambda\left(1-e^{-\delta x_i}\right)}\left(1-\left(1+2\lambda\left(1-e^{-\delta x_i}\right)\right)\right)}{1-\left(1+2\lambda\left(1-e^{-\delta x_i}\right)\right)e^{-2\lambda\left(1-e^{-\delta x_i}\right)}} \tag{35}$$

The MLE for the parameters $\varphi = (\beta, \lambda, \delta)^{T}$, denoted as $\hat{\varphi} = (\hat{\beta}, \hat{\lambda}, \hat{\delta})^{T}$ are obtained by setting the partial derivatives of the log-likelihood function Eqs. (33) to (35) equal to zero. This forms a system of non-linear equations, which can be solved simultaneously to find the desired parameter estimates.

Least squares estimation (LSE)

The LSE method is used for the [0, 1]TEAE parameter estimate. Reducing the value of the following equation is the primary goal of this estimation method:

$$L(\beta, \lambda, \delta) = \sum_{i=1}^{n}\left(F\left(x_{i;n}, \beta, \lambda, \delta\right) - \frac{i}{n+1}\right)^2$$

Table 2. Mean, RMSE, and Bias for the [0, 1]TEAE model using MLE, LSE, WLSE, and ADE with parameters $\beta = 1.6$, $\lambda = 1.9$, $\delta = 1.1$.

n		Est.Par.	MLE	LSE	WLSE	MPSE	ADE
30	Mean	$\hat{\beta}$	1.80984	1.37146	1.39775	2.43828	1.48717
		$\hat{\lambda}$	1.43688	1.69815	1.52496	1.20572	1.70513
		$\hat{\delta}$	1.44217	1.16710	1.17225	1.75510	1.23147
	RMSE	$\hat{\beta}$	1.35672	1.15834	1.14125	2.35560	1.00863
		$\hat{\lambda}$	2.07611	3.78219	3.62882	1.59031	3.45363
		$\hat{\delta}$	0.71070	0.75789	0.72879	0.89858	0.71198
	Bias	$\hat{\beta}$	0.20984	0.22853	0.20224	0.83828	0.11282
		$\hat{\lambda}$	0.46311	0.20184	0.37503	0.69427	0.19486
		$\hat{\delta}$	0.34217	0.06710	0.07225	0.65510	0.13147
60	Mean	$\hat{\beta}$	1.65207	1.38998	1.44428	1.92493	1.49766
		$\hat{\lambda}$	1.76839	1.62020	1.72613	1.49466	1.81055
		$\hat{\delta}$	1.26411	1.18871	1.16583	1.49318	1.17169
	RMSE	$\hat{\beta}$	0.60774	0.67694	0.59409	0.84134	0.55975
		$\hat{\lambda}$	1.28299	2.03411	1.75854	1.04645	1.66286
		$\hat{\delta}$	0.52070	0.61070	0.57829	0.60305	0.54790
	Bias	$\hat{\beta}$	0.05207	0.21001	0.15571	0.32493	0.10233
		$\hat{\lambda}$	0.13160	0.27979	0.17386	0.40533	0.08944
		$\hat{\delta}$	0.16411	0.08871	0.06583	0.39318	0.07169
150	Mean	$\hat{\beta}$	1.58577	1.45297	1.50162	1.70669	1.51689
		$\hat{\lambda}$	1.91166	1.84391	1.88516	1.65744	1.92422
		$\hat{\delta}$	1.14897	1.13249	1.12279	1.30456	1.11626
	RMSE	$\hat{\beta}$	0.31739	0.38420	0.33110	0.37281	0.32258
		$\hat{\lambda}$	0.89798	1.24867	0.96588	0.70718	0.97803
		$\hat{\delta}$	0.36141	0.44823	0.40517	0.37129	0.39578
	Bias	$\hat{\beta}$	0.01422	0.14702	0.09837	0.10669	0.08310
		$\hat{\lambda}$	0.01266	0.05608	0.01483	0.24255	0.02422
		$\hat{\delta}$	0.08422	0.03249	0.02279	0.20456	0.01626
200	Mean	$\hat{\beta}$	0.64201	1.47496	1.51398	1.67623	1.52613
		$\hat{\lambda}$	0.64201	1.85734	1.92676	1.70199	1.94298
		$\hat{\delta}$	0.10896	1.12892	1.10593	1.26630	1.10586
	RMSE	$\hat{\beta}$	0.25365	0.31915	0.26840	0.28319	0.26178
		$\hat{\lambda}$	0.80126	1.03226	0.86616	0.62634	0.85900
		$\hat{\delta}$	0.33011	0.40902	0.37227	0.32673	0.36410
	Bias	$\hat{\beta}$	0.01157	0.12503	0.08601	0.07623	0.07386
		$\hat{\lambda}$	0.01154	0.04265	0.01276	0.19800	0.01298
		$\hat{\delta}$	0.03987	0.02892	0.00593	0.16630	0.00586

Weighted least squares estimation (WLSE)

The WLSE method is used for the [0, 1]TEAE parameter estimate. Minimizing the value of the following equation is the main objective of this estimation method:

$$W(\beta,\ \lambda,\ \delta) = \sum_{i=1}^{n} \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left(F\left(x_{i;n}, \beta,\ \lambda,\ \delta\right) - \frac{i}{n+1}\right)^2$$

Maximum product space estimators (MPSE)

Parameter estimation for the [0, 1]TEAE requires the MPSE method. Minimizing the value of the following equation is the primary objective of this estimation method:

$$M_s(\beta,\ \lambda,\ \delta) = \frac{1}{n+1} \sum_{i=1}^{n} \ln\left(F\left(x_{i;n}, \beta,\ \lambda,\ \delta\right) - F\left(x_{i-1;n}, \beta,\ \lambda,\ \delta\right)\right)$$

Table 3. Mean, RMSE, and Bias for the [0, 1]TEAE model using MLE, LSE, WLSE, and ADE with parameters $\beta = 1.8$, $\lambda = 2$, $\delta = 1.2$.

n		Est.Par.	MLE	LSE	WLSE	MPSE	ADE
30	Mean	$\hat{\beta}$	2.07107	1.54425	1.59010	2.82644	1.67496
		$\hat{\lambda}$	1.50982	1.73627	1.65546	1.36405	1.70280
		$\hat{\delta}$	1.58371	1.28336	1.27281	1.91047	1.35516
	RMSE	$\hat{\beta}$	1.71635	1.39527	1.34641	2.88680	1.22412
		$\hat{\lambda}$	1.98580	3.59616	3.56163	1.79693	3.22420
		$\hat{\delta}$	0.78586	0.83487	0.77604	0.98558	0.78863
	Bias	$\hat{\beta}$	0.27107	0.25574	0.20989	1.02644	0.12503
		$\hat{\lambda}$	0.49013	0.26372	0.34453	0.63594	0.29719
		$\hat{\delta}$	0.38371	0.08336	0.07281	0.71047	0.15516
60	Mean	$\hat{\beta}$	1.86983	1.56590	1.62631	2.19836	1.67989
		$\hat{\lambda}$	1.89693	1.78988	1.89865	1.58377	1.97447
		$\hat{\delta}$	1.38439	1.28068	1.26045	1.64176	1.26264
	RMSE	$\hat{\beta}$	0.72472	0.79503	0.69506	1.01953	0.65513
		$\hat{\lambda}$	1.67019	2.08982	1.80749	1.06022	1.82573
		$\hat{\delta}$	0.57180	0.66221	0.62275	0.67093	0.60618
	Bias	$\hat{\beta}$	0.06983	0.23409	0.17368	0.39836	0.12010
		$\hat{\lambda}$	0.10306	0.21011	0.10134	0.41622	0.02552
		$\hat{\delta}$	0.18439	0.08068	0.06045	0.44176	0.06264
150	Mean	$\hat{\beta}$	1.78484	1.63032	1.68525	1.93173	1.70181
		$\hat{\lambda}$	2.00849	1.97418	1.97597	1.74734	2.04276
		$\hat{\delta}$	1.25767	1.23492	1.23158	1.43279	1.21796
	RMSE	$\hat{\beta}$	0.37058	0.44322	0.38329	0.44061	0.37697
		$\hat{\lambda}$	0.89841	1.31807	0.96943	0.70965	1.04854
		$\hat{\delta}$	0.39951	0.49404	0.44663	0.41446	0.44135
	Bias	$\hat{\beta}$	0.01515	0.16967	0.11474	0.13173	0.09818
		$\hat{\lambda}$	0.08149	0.02581	0.02402	0.25265	0.04276
		$\hat{\delta}$	0.05767	0.07492	0.03158	0.23279	0.01796
200	Mean	$\hat{\beta}$	1.78081	1.65490	1.69830	1.89393	1.71460
		$\hat{\lambda}$	2.01067	1.94165	2.02623	1.79624	2.02220
		$\hat{\delta}$	1.24770	1.24332	1.21000	1.38874	1.21634
	RMSE	$\hat{\beta}$	0.29666	0.37133	0.31322	0.33297	0.30399
		$\hat{\lambda}$	0.79989	1.02693	0.87782	0.63541	0.84162
		$\hat{\delta}$	0.36510	0.45767	0.41225	0.36574	0.39927
	Bias	$\hat{\beta}$	0.01918	0.14509	0.10169	0.09393	0.08539
		$\hat{\lambda}$	0.01067	0.01834	0.02238	0.20375	0.02220
		$\hat{\delta}$	0.04770	0.04332	0.01000	0.18874	0.01634

Anderson-darling estimation (ADE)

The ADE method is used for the [0, 1]TEAE parameter estimate. With this method of estimation, minimizing the following equation is the primary objective:

$$A(\beta,\ \lambda,\ \delta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2! - 1)\left(\ln\left(F\left(x_{i;n}, \beta,\ \lambda,\ \delta\right)\right) + \ln\left(S\left(x_{i;n}, \beta,\ \lambda,\ \delta\right)\right)\right)$$

Simulation study

In this part, we use a Monte Carlo experiment to investigate the asymptotic behavior of MLE, LSE, WLSE, MPSE, and ADE for [0, 1]TEAE distribution parameters. The experiment focuses on two sets of parameter values: $\beta = (1.6, \lambda = 1.9, \delta = 1.1)$, and $(\beta = 1.8, \lambda = 2, \delta = 1.2)$. We explore sample sizes of 30, 60, 150, and 200, with the experiment repeated 1000 times.

$$bias(\hat{\lambda}) = \frac{\sum_{i=1}^{N}\hat{\lambda}_i}{N} - \lambda,\ \text{and}\ RMSE(\hat{\lambda}) = \sqrt{\frac{\sum_{i=1}^{N}\left(\hat{\lambda}_i - \lambda\right)^2}{N}}.$$

Tables 2 and 3 include the estimations for the mean, root mean square error (RMSE), and bias. To evaluate the different estimating approaches. These tables illustrate:

Table 4. Comparative distributions.

distribution	CDF
Beta Exponential distribution (BeE) [17]	$p\delta(1-e^{-\delta x},\ \beta,\ \lambda)$
Kumaraswamy Exponential distribution (KuE) [18]	$1\ -\ (1\ -(1-e^{-\delta x})^{\beta})^{\lambda}$
Exponential Generalized Exponential distribution (EGE) (New)	$(1\ -\ (\ 1-e^{-\delta x})^{\beta})^{\lambda}$
Weibull Exponential distribution (WeE) [19]	$(1\ -\ exp(-\lambda^{-\beta}\ (-log(e^{-\delta x}))^{\beta})$
Gompertz Exponential distribution (GoE)	$(1\ -\ (e^{-\delta x})^{\beta})^{\lambda}$ $*(1\ -\ exp(\ (\frac{\beta}{\lambda})\ (\ 1-(e^{-\delta x})^{-\lambda}\ )\ ))$
Rayleigh Exponential distribution (RE) [New]	$e^{\frac{-\delta}{2}(-\ln(1-e^{-\delta x}))^{2}}$
Rayleigh distribution (R) [20]	$1\ -\ exp(-\delta x^{2})$

Table 5. Descriptive analysis of the three data.

Dataset	N	Min.	Median	Mean	SD	Max.	SK	KU
Data I	34	0.1	1.15	1.88	1.95	8	1.53	1.72
Data II	72	0.1	1.5	1.77	1.04	5.55	1.3	1.83
Data III	108	1.04	5.19	5.76	3.25	16.5	0.97	0.61

Table 6. Estimations and statistics measuring the accuracy of the fit for Data I.

Dist.	MLEs	–2L	AIC	CAIC	BIC	HQIC	W	A	K-S	P-V
[0, 1]TEAE	$\hat{\beta}$:1.1775 $\hat{\lambda}$:0.6839 $\hat{\delta}$:0.4247	54.94	115.88	116.68	120.46	117.44	0.0312	0.2052	0.0850	0.9666
BeE	$\hat{\beta}$:1.1126 $\hat{\lambda}$:0.4651 $\hat{\delta}$:1.2012	55.36	116.73	117.53	121.31	118.29	0.0442	0.2871	0.0984	0.8967
KuE	$\hat{\beta}$:1.1498 $\hat{\lambda}$:0.4661 $\hat{\delta}$:1.2029	55.33	116.67	117.47	121.25	118.23	0.0434	0.2820	0.0998	0.8869
EGE	$\hat{\beta}$:1.0399 $\hat{\lambda}$:1.0764 $\hat{\delta}$:0.5366	55.40	116.80	117.60	121.38	118.36	0.0455	0.2949	0.0977	0.9011
WeE	$\hat{\beta}$:1.0102 $\hat{\lambda}$:0.9712 $\hat{\delta}$:0.5144	55.44	116.89	117.69	121.47	118.46	0.0462	0.3000	0.0918	0.9366
GoE	$\hat{\beta}$:1.0360 $\hat{\lambda}$:0.0269 $\hat{\delta}$:0.5276	55.44	116.88	117.68	121.46	118.44	0.0437	0.2830	0.0857	0.9539

- The calculated parameters β, λ, and δ converge to their original values as n increases.
- The values of RMSE and Bias decrease as n increases.
- Among all the methodologies, Maximum Likelihood Estimation yields the least favorable outcome. Consequently, in the assessment of β, λ, and δ, Maximum Likelihood Estimation surpasses all other methodologies considered.

Application

In this section, we showcase the practical use of the [0, 1]TEAE model in three real-world scenarios. We directly compare the goodness-of-fit statistics and maximum likelihood estimates (MLEs) of model parameters with the [0, 1]TEAE model and other competing models. The analyzed models that are in direct competition are listed in Table 4.

To compare the various models, we evaluate them based on eight well-established criteria for measuring their level of accuracy. The measurements consist of the Kolmogorov-Smirnov (KS) statistic, Anderson-Darling (A) statistic, Cramér-von Mises (W) statistic, p-value (P-V) for the KS test, Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC), Akaike information criterion (AIC), and consistent AIC (CAIC).

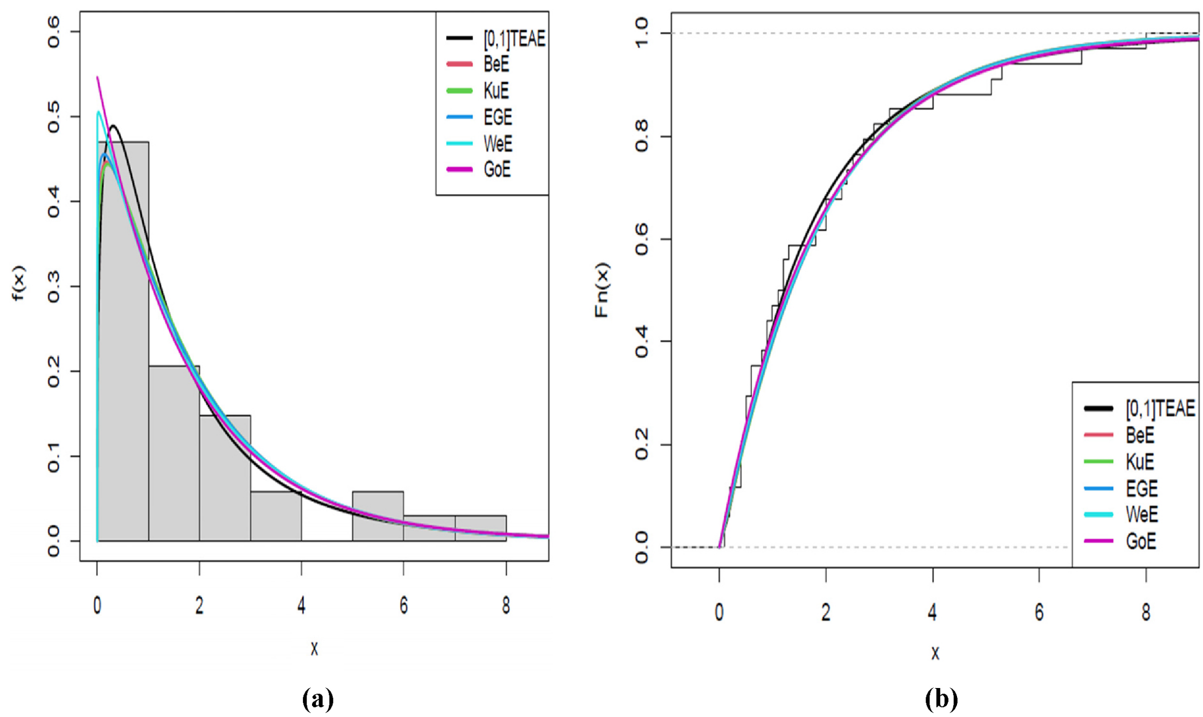


Fig. 4. (a) Histogram PDFs for Data I. (b) Estimated CDFs plots for Data I.

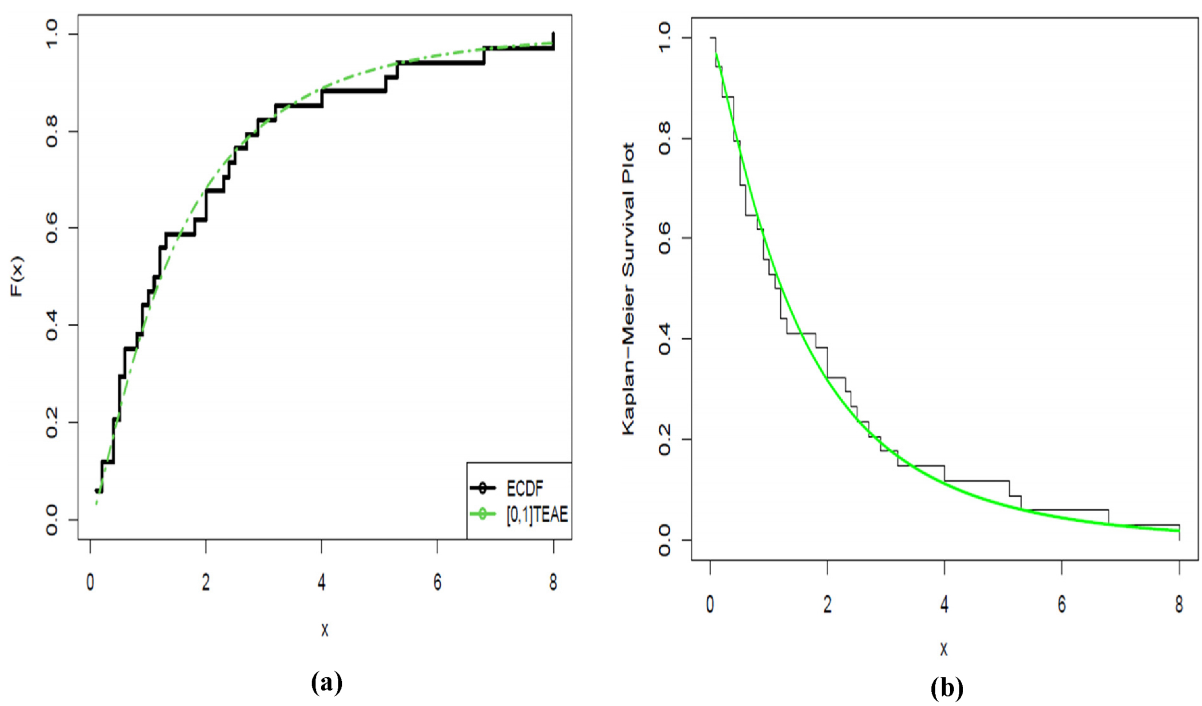


Fig. 5. (a) Empirical CDF for Data I. (b) Kaplan-Meier plots for Data I.

Analyzing these statistics and measures enables us to assess the degree of conformity between each model and the facts. The model that has the lowest values for KS, A, W, HQIC, BIC, AIC, CAIC, and the greatest p-value is considered the most optimal.

Table 5 shows the minimum, median, mean, standard deviation, maximum, skewness, and kurtosis for three datasets.

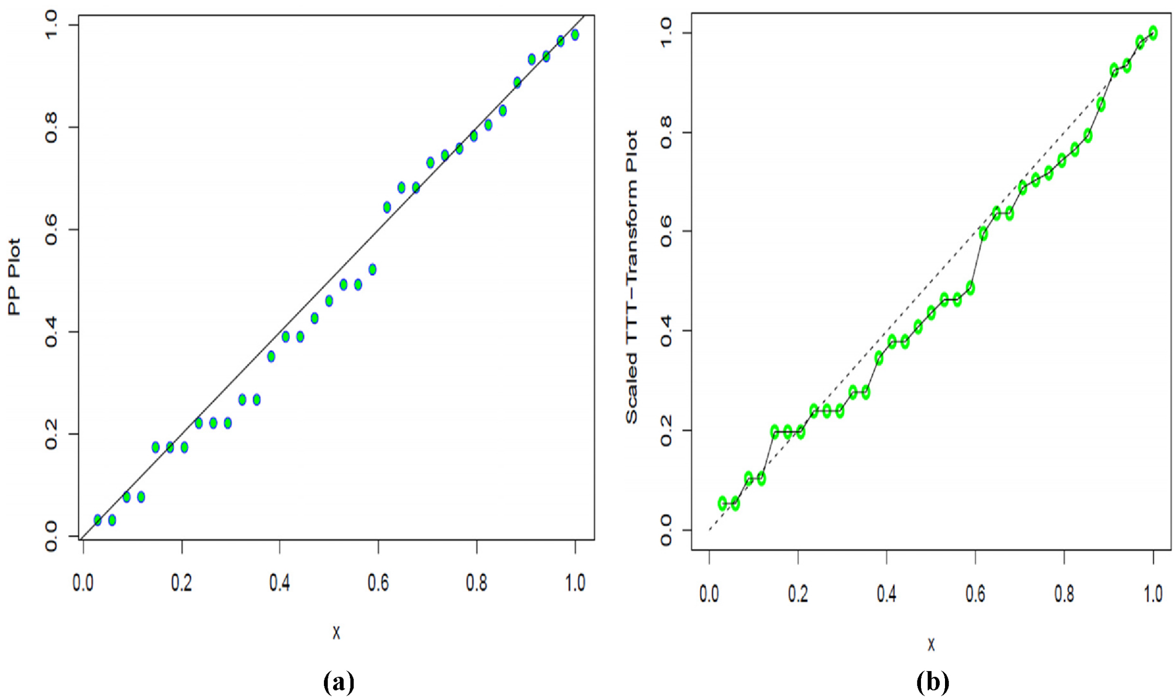


Fig. 6. (a) PP plot for Data I. (b) TTT plots for Data I.

Table 7. Estimations and statistics measuring the accuracy of the fit for Data II.

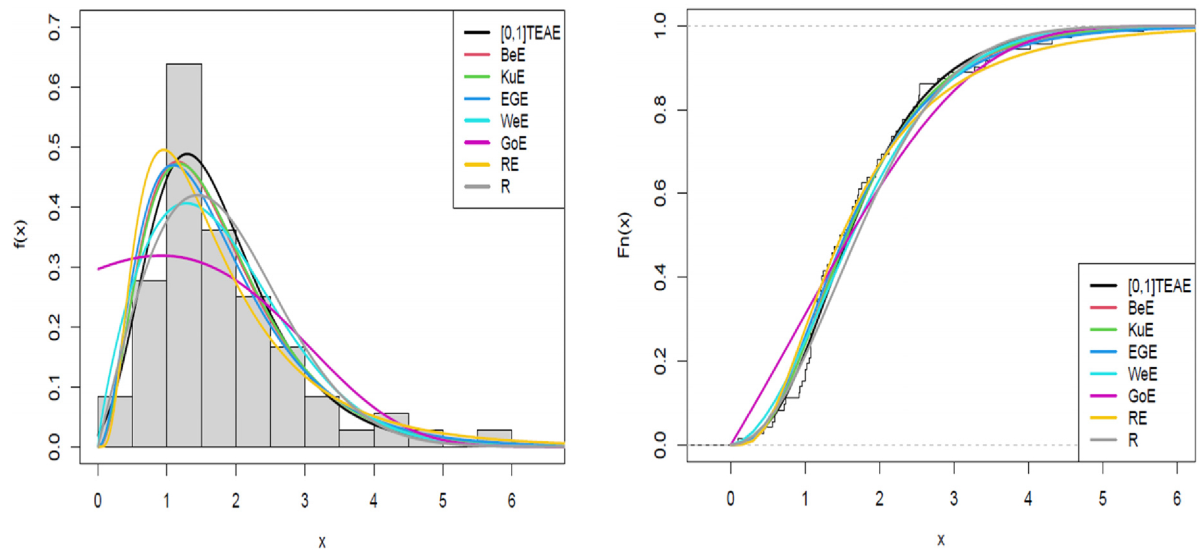
Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	P-V
[0, 1]TEAE	$\hat{\beta}$:4.6978 $\hat{\lambda}$:0.5209 $\hat{\delta}$:1.3084	93.61	193.22	193.57	200.05	195.93	0.0877	0.5020	0.0825	0.7107
BeE	$\hat{\beta}$:3.3629 $\hat{\lambda}$:1.4159 $\hat{\delta}$:0.8500	94.42	194.85	195.20	201.68	197.57	0.0832	0.5186	0.0949	0.5354
KuE	$\hat{\beta}$:3.1070 $\hat{\lambda}$:1.5504 $\hat{\delta}$:0.8133	94.32	194.64	194.99	201.47	197.36	0.0856	0.5257	0.0924	0.5695
EGE	$\hat{\beta}$:1.2167 $\hat{\lambda}$:3.2351 $\hat{\delta}$:0.8827	94.61	195.23	195.58	202.06	197.95	0.0807	0.5087	0.1105	0.3420
WeE	$\hat{\beta}$:1.8173 $\hat{\lambda}$:1.0106 $\hat{\delta}$:0.5071	95.94	197.88	198.23	204.71	200.60	0.1572	0.9203	0.1065	0.3871
GoE	$\hat{\beta}$:0.7730 $\hat{\lambda}$:1.1631 $\hat{\delta}$:0.3833	102.82	211.65	212.35	218.48	214.37	0.2938	1.7091	0.1749	0.0243
RE	$\hat{\lambda}$:1.7908 $\hat{\delta}$:0.3598	97.30	198.61	198.79	203.17	200.43	0.1048	0.7175	0.1267	0.1977
R	$\hat{\delta}$:0.2393	96.58	195.16	195.21	200.43	196.06	0.1760	1.0289	0.1087	0.3615

The first dataset I

In clean-up gradient ground-water monitoring wells, the offered dataset includes vinyl chloride values reported in micrograms per liter (μg/L). It has been used before by.[21]

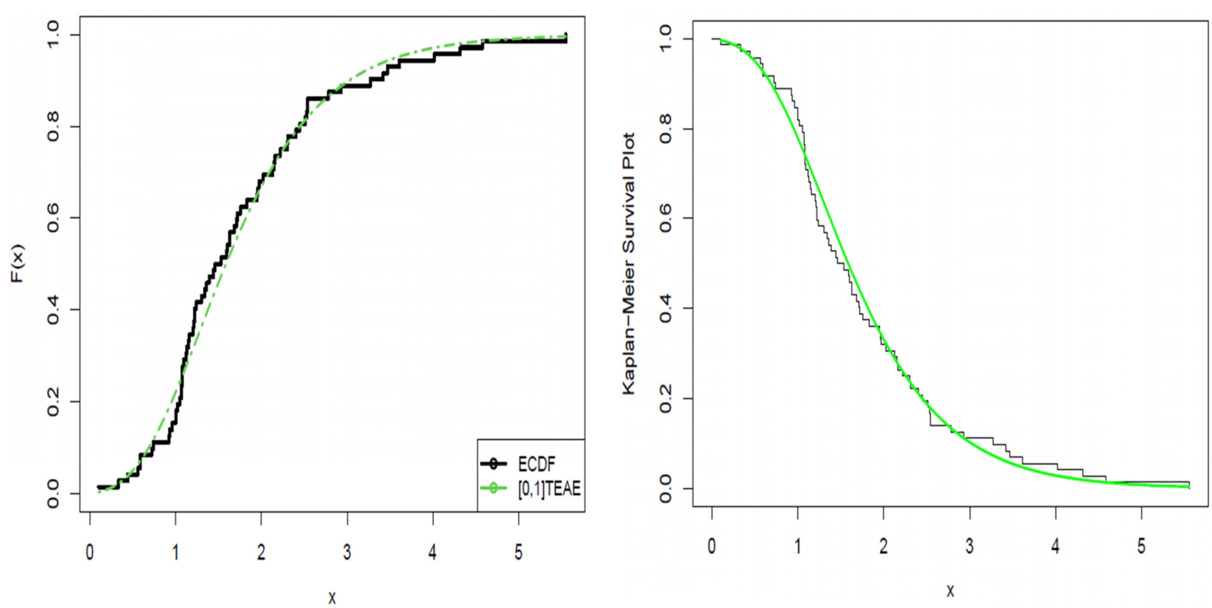
Table 6 shows that the [0, 1]TEAE model, with the lowest selection criterion and greatest p-value, illustrates how adaptable, resilient, and flexible the new model is with data I.

Figs. 4 to 6 demonstrate an outstanding fit for the [0, 1]TEAE model. All sex figs. show that the curves closely match the relevant empirical ones, proving the [0, 1]TEAE model's applicability for Data I.



(a) (b)

Fig. 7. (a) Histogram PDFs for Data II. (b) Estimated CDFs plots for Data II.



(a) (b)

Fig. 8. (a) Empirical CDF for Data II. (b) Kaplan-Meier plots for Data II.

The second dataset II

The information used in this investigation includes the duration of infection with dangerous tubercle bacilli in 72 guinea pigs. The durations of survival are quantified in days. Bjerkedal [22] first noticed and published this dataset, and Sherpieny et al. [23] studied it later.

Table 7 shows that the [0, 1]TEAE model, with the lowest selection criterion and greatest p-value, illustrates how adaptable, resilient, and flexible the new model is with data II.

Figs. 7 to 9 demonstrate an outstanding fit for the [0, 1]TEAE model. All sex figs. show that the curves closely match the relevant empirical ones, proving the [0, 1]TEAE model's applicability for Data II.

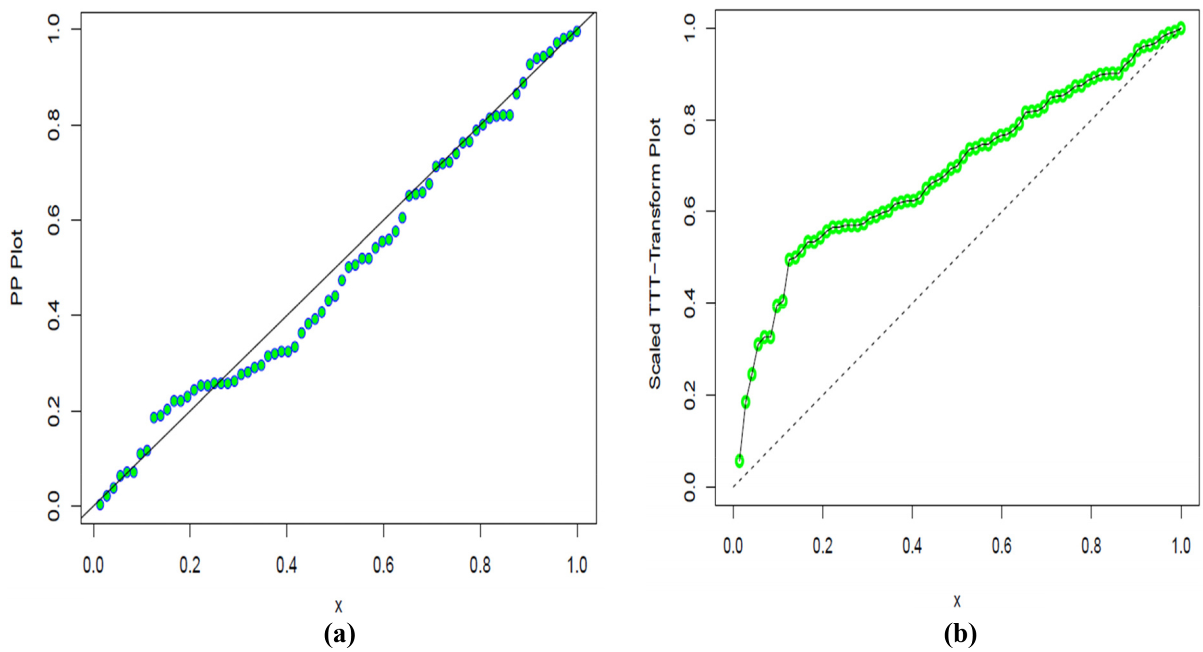


Fig. 9. (a) PP plot for Data II. (b) TTT plots for Data II.

Table 8. Estimations and statistics measuring the accuracy of the fit for Data III.

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	P-V
[0, 1]TEAE	$\hat{\beta}$:0.7707 $\hat{\lambda}$:2.8091 $\hat{\delta}$:0.3352	265.95	537.90	538.13	545.94	541.16	0.0560	0.3045	0.0615	0.8073
BeE	$\hat{\beta}$:3.7669 $\hat{\lambda}$:1.4468 $\hat{\delta}$:0.2725	266.25	538.50	538.73	546.55	541.76	0.0602	0.3378	0.0754	0.5695
KuE	$\hat{\beta}$:3.1492 $\hat{\lambda}$:1.4108 $\hat{\delta}$:0.2639	266.55	539.12	539.35	547.16	542.38	0.0624	0.3539	0.0629	0.7854
EGE	$\hat{\beta}$:1.0302 $\hat{\lambda}$:3.9422 $\hat{\delta}$:0.3490	266.18	538.36	538.59	546.41	541.62	0.0595	0.3291	0.0691	0.6796
WeE	$\hat{\beta}$:1.7832 $\hat{\lambda}$:0.9992 $\hat{\delta}$:0.1560	269.30	544.61	544.84	552.66	547.87	0.1025	0.6524	0.0710	0.6471
GoE	$\hat{\beta}$:0.7390 $\hat{\lambda}$:0.7125 $\hat{\delta}$:0.1553	280.63	567.31	567.55	575.36	570.58	0.1572	1.0308	0.1766	0.0023
RE	$\hat{\lambda}$:1.3367 $\hat{\delta}$:0.0920	271.17	546.39	546.50	551.75	548.56	0.0594	0.3259	0.0858	0.4033
R	$\hat{\delta}$:0.0229	269.23	540.46	540.49	547.14	541.54	0.1185	0.7627	0.0933	0.3036

The third dataset III

The dataset presented here represents the COVID-19 mortality rate for Mexico, covering a span of 108 days from March 4th to July 20, 2020. This dataset provides valuable information regarding the approximate mortality rate, indicating the number of deaths attributed to COVID-19 during this specific time frame. The data has been used by[24,25] previously.

Table 8 shows that the [0, 1]TEAE model, with the lowest selection criterion and greatest p-value, illustrates how adaptable, resilient, and flexible the new model is with data III.

Figs. 10 to 12 demonstrate an outstanding fit for the [0, 1]TEAE model. All sex figs. show that the curves closely match the relevant empirical ones, proving the [0, 1]TEAE model's applicability for Data III.

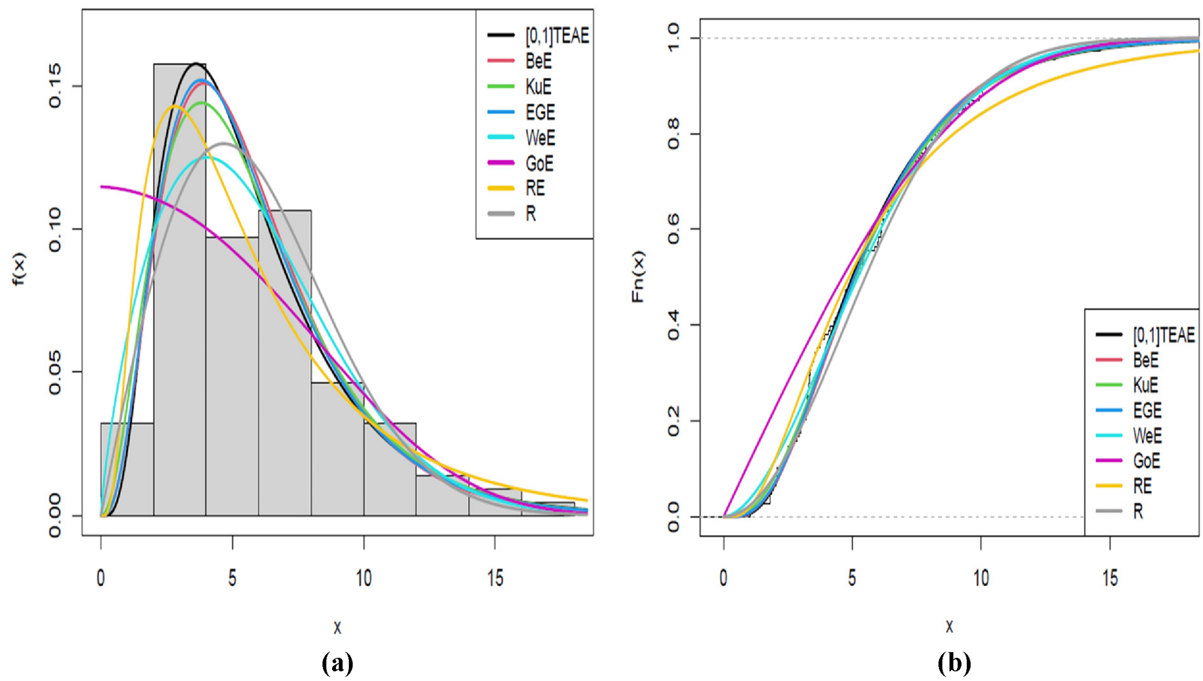


Fig. 10. (a) Histogram PDFs for Data III. (b) Estimated CDFs plots for Data III.

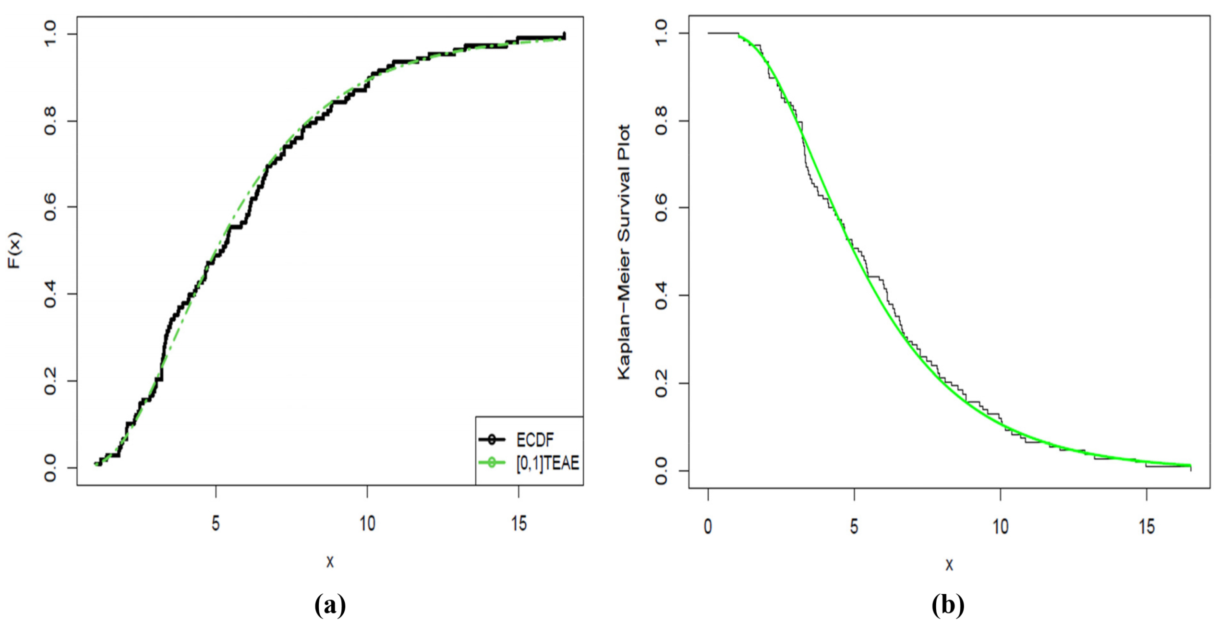


Fig. 11. (a) Empirical CDF for Data III. (b) Kaplan-Meier plots for Data III.

Results and discussion

Comparing the information criteria acquired for the [0, 1]TEAE model to those of the seven competing models (BeE, KuE, EGE, WeE, GoE, RE, and R), it is clear that the [0, 1]TEAE model surpasses the others across all three datasets. The information criteria values for the [0, 1]TEAE model are consistently lower, indicating a better fit with the data. Figs. 4, 7 and 10 show the Histogram PDFs and Estimated CDFs plots for Data I, II, and III, respectively, The [0, 1]TEAE model appears to closely reflect datasets I, II, II when compared to the BeE, KuE, EGE, WeE, GoE, RE and R models.

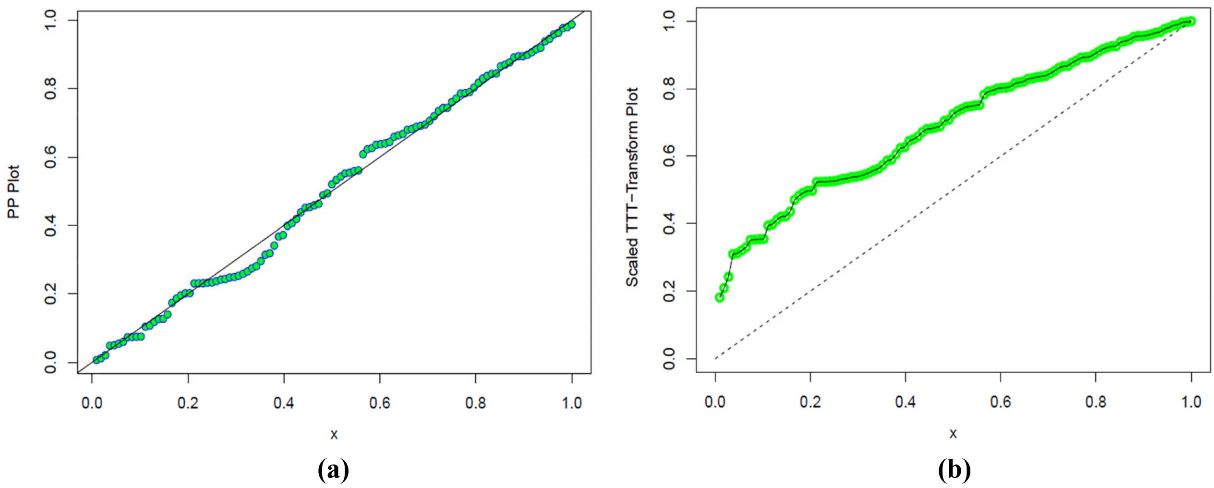


Fig. 12. (a) PP plot for Data III. (b) TTT plots for Data III.

Conclusion

This study introduces the [0, 1] Truncated Exponentiated Ailamujia Exponential ([0, 1]TEAE) Distribution by combining the Exponential distribution with the [0, 1]Truncated Exponentiated Ailamujia-G Family. Extensive analysis is conducted to explore the mathematical properties of the [0, 1]TEAE distribution. The parameters of this newly developed distribution are estimated using the MLE, LSE, WLSE, MPSE, and ADE ensuring optimal parameter estimation. The investigation of these properties and parameter estimation techniques. The simulation study provided compelling evidence that the MLE for the [0, 1]TEAE distribution is both accurate and consistent.

The relevance of the new [0, 1]TEAE distribution is proven by applying it to three real-world datasets, showing its advantages over existing statistical models like as BeE, KuE, EGE, WeE, GoE, RE, and R. The examination of these datasets shows that the [0, 1]TEAE model surpasses the others in terms of fit and accuracy. This shows that the [0, 1]TEAE distribution is a better description of the data than the other models.

Author's declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images that are not ours have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at Diyala Education Directorate.

Authors' contribution statement

S. A. S., D. A. M., H. K. K., and A. A. K. contributed to the design and implementation of the research, the analysis of the results, and the writing of the manuscript.

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توزيع Ailamujia المعمم الأسي المبتور: الخصائص والتطبيقات

سامر عبد القادر صالح[1]، ضبياء محمود علي[1]، هند خالد كليب[1]، علاء عبدالرحمن خلف[2]

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الملخص

لقد أظهر التوزيع الاحتمالي قابليته للتطبيق العملي في كل مجال من مجالات الجهد البشري تقريبًا. يقدم هذا البحث عائلة جديدة من التوزيعات تسمى عائلة [0,1] المعممة المبتورة Ailamujia-G. تم إنشاء التوزيع [0,1] المعمم الأسي المبتور TEAE[0,1]) Ailamujia)، والذي يعد نموذجًا فرعيًا للعائلة التي تم إنشاؤها مؤخرًا، بالكامل. يتم إنشاء توزيع TEAE [0,1] عن طريق دمج توزيع المعمم Ailamujia مع المبتور [0,1]. يتم فحص السمات الرياضية لتوزيع TEAE [0،1]، بما في ذلك العزوم ، والانحراف، والتفرطح، والعزوم غير المكتملة، وإنتروبيا ريني، ودالة الكمية، والعزوم المرجحة بالاحتمالات، على نطاق واسع. تم تحديد الكميات لقيم المعلمات المختارة بوضوح. تتضمن التقنيات المستخدمة لتقدير المعلمة الإمكان الاعظم، والمربعات الصغرى، والمربعات الصغرى الموزونة، وأندرسون دارلينج. تقوم هذه الدراسة بتقييم فعالية العديد من المقدرات باستخدام محاكاة مونت كارلو. بالإضافة إلى ذلك، تم استخدام المقدرات على ثلاث مجموعات من البيانات الفعلية، وتم توثيق إحصائيات كولوموغوروف-سيميرنوف لكل منها. تم استخدام التوزيع [0،1] المعمم الأسي المبتور لـ Ailamujia على ثلاث مجموعات بيانات في العالم الحقيقي. تم تقييم فعاليته من خلال مقارنته بامتدادات أخرى معروفة للتوزيع الأسي، باستخدام معايير مثل معيار معلومات حنان-كوين، ومعيار المعلومات بايزي، ومعيار معلومات أكايكي، واختبار أكايكي المتسق واختبارات جودة الملاءمة مثل إحصائية أندرسون-دارلينج، وكريمر فون ميزس، والقيمة P لاختبار KS.

الكلمات المفتاحية: توزيع Ailamujia، طرق التقدير، العزوم، المبتور [0,1]، ريني انتروبي.